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The Local Structure of Algebraic K-Theory

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The Local Structure of Algebraic K-Theory

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Preface

Algebraic K-theory draws its importance from its effective codification of a mathematical phenomenon which occurs in as separate parts of mathematics as number theory, geometric topology, operator algebras, homotopy theory and algebraic geometry. In reductionistic language the phenomenon can be phrased as

> *there is no canonical choice of coordinates*,

or, as so elegantly expressed by Hermann Weyl [312, p. 49]:

> *The introduction of numbers as coordinates … is an act of violence whose only practical vindication is the special calculatory manageability of the ordinary number continuum with its four basic operations.*

As such, algebraic K-theory is a meta-theme for mathematics, but the successful codification of this phenomenon in homotopy-theoretic terms is what has made algebraic K-theory a valuable part of mathematics. For a further discussion of algebraic K-theory we refer the reader to Chap. 1 below.

Calculations of algebraic K-theory are very rare and hard to come by. So any device that allows you to obtain new results is exciting. These notes describe one way to produce such results.

Assume for the moment that we know what algebraic K-theory is; how does it vary with its input?

The idea is that algebraic K-theory is like an analytic function, and we have this other analytic function called *topological cyclic homology* (*TC*) invented by Bökstedt, Hsiang and Madsen [27], and

> *the difference between K and TC is locally constant*.

This statement will be proven below, and in its integral form it has not appeared elsewhere before.

The good thing about this, is that TC is occasionally possible to calculate. So whenever you have a calculation of K-theory you have the possibility of calculating all the K-values of input "close" to your original calculation.

So, for instance, if somebody (please) can calculate K-theory of the integers, many "nearby" applications in geometric topology (simply connected spaces) are

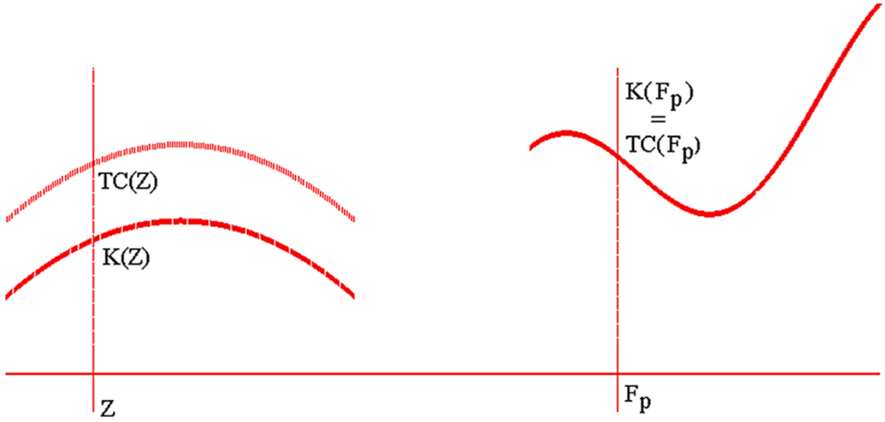

Fig. 0.1 The difference between K and TC is locally constant. The *left part of the figure* illustrates the difference between $K(\mathbf{Z})$ and $TC(\mathbf{Z})$ is quite substantial, but once you know this difference you know that it does not change in a "neighborhood" of $\mathbf{Z}$. In this neighborhood lies for instance all applications of algebraic K-theory of simply connected spaces, so here TC-calculations ultimately should lead to results in geometric topology as demonstrated by Rognes. On the *right hand side of the figure* you see that close to the finite field with p elements, K-theory and TC agree (this is a connective and p-adic statement: away from the characteristic there are other methods that are more convenient). In this neighborhood you find many interesting rings, ultimately resulting in Hesselholt and Madsen's calculations of the K-theory of local fields

available through TC-calculations (see e.g., [242, 243]). This means that calculations in motivic cohomology (giving K-groups of e.g., the integers) will actually have bearing on our understanding of diffeomorphisms of manifolds!

On a different end of the scale, Quillen's calculation of the K-theory of finite fields gives us access to "nearby" rings, ultimately leading to calculations of the K-theory of local fields [131]. One should notice that the illustration offered by Fig. 0.1 is not totally misleading: the difference between $K(\mathbf{Z})$ and $TC(\mathbf{Z})$ is substantial (though locally constant), whereas around the field $\mathbf{F}_p$ with p elements it is negligible.

Taking K-theory for granted (we'll spend quite some time developing it later), we should say some words about TC. Since K-theory and TC differ only by some locally constant term, they must have the same differential: $D_1K = D_1TC$ (see Fig. 0.2). For ordinary rings A this differential is quite easy to describe: it is the *homology* of the category $\mathcal{P}_A$ of finitely generated projective modules.

The homology of a category is like Hochschild homology, and as Connes observed, certain models of Hochschild homology carry a circle action which is useful when comparing with K-theory. Only, in the case of the homology of categories it turns out that the ground ring over which to take Hochschild homology is not an ordinary ring, but the so-called sphere spectrum. Taking this idea seriously, we end up with Bökstedt's *topological Hochschild homology THH*.

One way to motivate the construction of TC from *THH* is as follows. There is a transformation $K \to THH$ which we will call the *Dennis trace map*, and there is

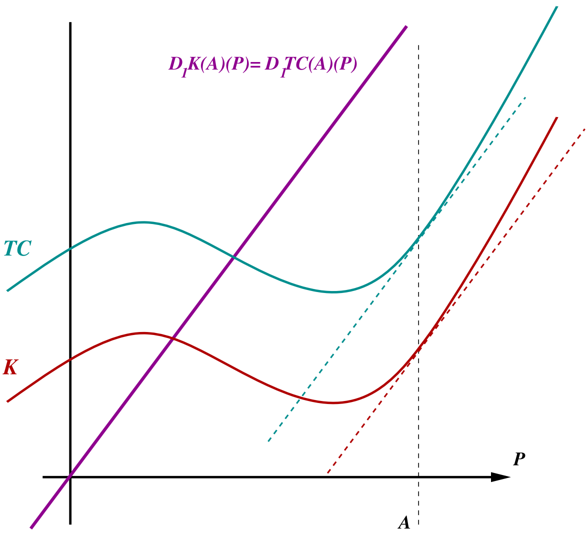

Fig. 0.2 The differentials "at an **S**-algebra A in the direction of the A-bimodule P" of K and TC are equal. For discrete rings the differential is the homology of the category of finitely generated projective modules. In this illustration the differential is the *magenta straight line* through the origin, K-theory is the *red curve* and TC is the shifted curve in *cyan* (Colon figure online)

a model for *THH* for which the Dennis trace map is just the *inclusion of the fixed points under the circle action*. That is, the Dennis trace can be viewed as a composite

$$K \cong THH^{\mathbb{T}} \subseteq THH$$

where $\mathbb{T}$ is the circle group.

The unfortunate thing about this statement is that it is *model dependent* in that fixed points do not preserve weak equivalences: if $X \to Y$ is a map of $\mathbb{T}$-spaces which is a weak equivalence of underlying spaces, normally the induced map $X^{\mathbb{T}} \to Y^{\mathbb{T}}$ will not be a weak equivalence. So, TC is an attempt to construct the $\mathbb{T}$-fixed points through techniques that **do** preserve weak equivalences.

It turns out that there is more to the story than this: *THH* possesses something called an *epicyclic structure* (which is not the case for all $\mathbb{T}$-spaces), and this allows us to approximate the $\mathbb{T}$-fixed points even better.

So in the end, the *cyclotomic trace* is a factorization

$$K \to TC \to THH$$

of the Dennis trace map.

The cyclotomic trace is the theme for this book. There is another paper devoted to this transformation, namely Madsen's eminent survey [192]. If you can get hold of a copy it is a great supplement to the current text.

It was originally an intention that readers who were only interested in discrete rings would have a path leading far into the material with minimal contact with ring spectra. This idea has to a great extent been abandoned since ring spectra and the techniques around them have become much more mainstream while these notes have matured. Some traces of this earlier approach can still be seen in that Chap. 1 does not depend at all on ring spectra, leading to the proof that stable K-theory of rings corresponds to homology of the category of finitely generated projective modules. Topological Hochschild homology is, however, interpreted as a functor of ring spectra, so the statement that stable K-theory is *THH* requires some background on ring spectra.

General Plan The general plan of the book is as follows.

In Sect. 1.1 we give some general background on algebraic K-theory. The length of this introductory section is justified by the fact that this book is primarily concerned with algebraic K-theory; the theories that fill the last chapters are just there in order to shed light on K-theory, we are not really interested in them for any other reason. In Sect. 1.2 we give Waldhausen's interpretation of algebraic K-theory and study in particular the case of radical extensions of rings. Finally, Sect. 1.3 compares stable K-theory and homology.

Chapter 2 aims at giving a crash course on ring spectra. In order to keep the presentation short we have limited our presentation only the simplest version: Segal's Γ-spaces. This only gives us connective spectra and the behavior with respect to commutativity issues leaves something to be desired. However, for our purposes Γ-spaces suffice and also fit well with Segal's version of algebraic K-theory, which we are using heavily later in the book.

Chapter 3 can (and perhaps should) be skipped on a first reading. It only asserts that various reductions are possible. In particular, K-theory of simplicial rings can be calculated degreewise "locally" (i.e., in terms of the K-theory of the rings appearing in each degree), simplicial rings are "dense" in the category of (connective) ring spectra, and all definitions of algebraic K-theory we encounter give the same result.

In Chap. 4, topological Hochschild homology is at long last introduced, first for ring spectra, and then in a generality suitable for studying the correspondence with algebraic K-theory. The equivalence between the topological Hochschild homology of a ring and the homology of the category of finitely generated projective modules is established in Sect. 4.3, which together with the results in Sect. 1.3 settle the equivalence between stable K-theory and topological Hochschild homology of rings.

In order to push the theory further we need an effective comparison between K-theory and *THH*, and this is provided by the Dennis trace map $K \to THH$ in the following chapter. We have here chosen a model which "localizes at the weak equivalences", and so conforms nicely with the algebraic case. For our purposes this works very well, but the reader should be aware that other models are more appropriate for proving structural theorems about the trace. The comparison between stable K-theory and topological Hochschild homology is finalized Sect. 5.3, using the trace. As a more streamlined alternative, we also offer a new and more direct trace construction in Sect. 5.4.

In Chap. 6 topological cyclic homology is introduced. This is the most involved of the chapters in the book, since there are so many different aspects of the theory that have to be set in order. However, when the machinery is set up properly, and the trace has been lifted to topological cyclic homology, the local correspondence between K-theory and topological cyclic homology is proved in a couple of pages in Chap. 7.

Chapter 7 ends with a quick and inadequate review of the various calculations of algebraic K-theory that have resulted from trace methods. We first review the general framework set up by Bökstedt and Madsen for calculating topological cyclic homology, and follow this through for three important examples: the prime field $\mathbf{F}_p$, the (p-adic) integers $\mathbf{Z}_p$ and the Adams summand ℓ_p. These are all close enough to $\mathbf{F}_p$ so that the local correspondence between K-theory and topological cyclic homology make these calculations into actual calculations of algebraic K-theory. We also discuss very briefly the Lichtenbaum-Quillen conjecture as seen from a homotopy theoretical viewpoint, which is made especially attractive through the comparison with topological cyclic homology. The inner equivariant workings of topological Hochschild homology display a rich and beautiful algebraic structure, with deep intersections with log geometry through the de Rham-Witt complex. This is prominent in Hesselholt and Madsen's calculation of the K-theory of local fields, but facets are found in almost all the calculations discussed in Sect. 7.3. We also briefly touch upon the first problem tackled through trace methods: the algebraic K-theory Novikov conjecture.

The Appendix collects some material that is used freely throughout the notes. Much of the material is available elsewhere in the literature, but for the convenience of the reader we have given the precise formulations we actually need and set them in a common framework. The reason for pushing this material to an appendix, and not working it into the text, is that an integration would have produced a serious eddy in the flow of ideas when only the most diligent readers will need the extra details. In addition, some of the results are used at places that are meant to be fairly independent of each other.

The rather detailed index is meant as an aid through the plethora of symbols and complex terminology, and we have allowed ourselves to make the unorthodox twist of adding hopefully helpful hints in the index itself, where this has not taken too much space, so that in many cases a brief glance at the index makes checking up the item itself unnecessary.

Displayed diagrams commute, unless otherwise noted. The ending of proofs that are just sketched or referred away and of statements whose verification is embedded in the preceding text are marked with a ☺.

Acknowledgments This book owes a lot to many people. The first author especially wants to thank Marcel Bökstedt, Bjørn Jahren, Ib Madsen and Friedhelm Waldhausen for their early and decisive influence on his view on mathematics. The third author would like to thank Marcel Bökstedt, Dan Grayson, John Klein, Jean-Louis Loday and Friedhelm Waldhausen whose support has made all the difference.

These notes have existed for quite a while on the net, and we are grateful for the helpful comments we have received from a number of people, in particular from Morten Brun, Lars Hesselholt, Harald Kittang, Birgit Richter, John Rognes, Stefan Schwede and Paul Arne Østvær.

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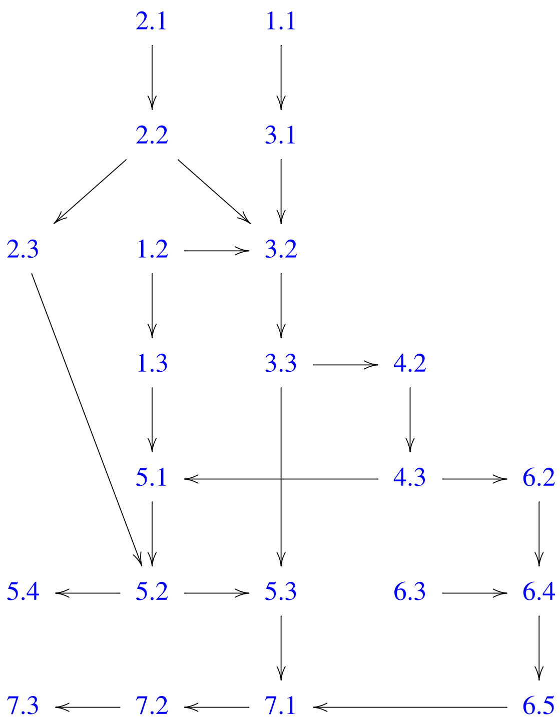
Finally, the first author wants to thank his wife Siv and daughters Karen and Vår for their patience with him and apologize for all the time spent thinking, writing and generally not paying attention to the important things.

Leitfaden For the convenience of the reader we provide the following Leitfaden. It should not be taken too seriously, some minor dependencies are not shown, and many sections that are noted to depend on previous chapters are actually manageable if one is willing to retrace some cross referencing. In particular, Chap. 3 should be postponed upon a first reading.



Contents

Chapter 1
Algebraic K-Theory

In this chapter we define and discuss the algebraic K-theory functor. This chapter will mainly be concerned with the algebraic K-theory of rings, but we will extend this notion at the end of the chapter. There are various possible extensions, but we will mostly focus on a class of objects that are close to rings. In later chapters this will be extended again to include ring spectra and even more exotic objects.

In the first section we give a quick nontechnical overview of K-theory. Many of the examples are but lightly touched upon and not needed later, but are included to give an idea of the scope of the theory. Some of the examples in the introduction may refer to concepts or ideas that are unfamiliar to the reader. If this is the case, the reader may consult the index to check whether this is a topic that will be touched upon again (and perhaps even explained), or if it is something that can be left for later investigations. In any case, the reader is encouraged to ignore such problems at a first reading. Although it only treats the first three groups, Milnor's book [213] is still one of the best elementary introductions to algebraic K-theory with Bass' book [13] providing the necessary support for more involved questions. For a more modern exposition one may consult Rosenberg's book [244]. For a fuller historical account, the reader may want to consult for instance [310] or [14].

In the second section we introduce Waldhausen's S-construction of algebraic K-theory and prove some of its basic properties.

The third section concerns itself with comparisons between K-theory and various homology theories, giving our first identification of the differential of algebraic K-theory, as discussed in the preface.

1.1 Introduction

The first appearance of what we now would call truly K-theoretic questions are the investigations of J.H.C. Whitehead (for instance [314, 315] or the later [316]), and Higman [133]. The name "K-theory" is much younger (said to be derived from the German word "Klassen"), and first appears in Grothendieck's work [1] in 1957 on

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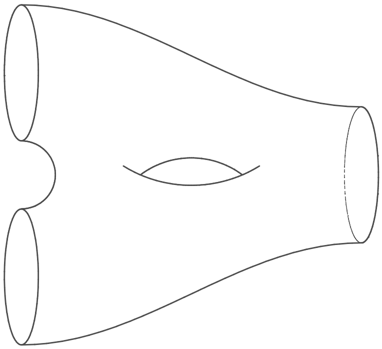


Fig. 1.1 A cobordism W between a disjoint union M of two circles and a single circle N

the Riemann-Roch theorem, see also [35]. But, even though it was not called K-theory, we can get some motivation by studying the early examples.

1.1.1 Motivating Example from Geometry: Whitehead Torsion

The "Hauptvermutung" states that two homeomorphic finite simplicial complexes have isomorphic subdivisions. The conjecture was formulated by Steinitz and Tietze in 1908, see [236] for references and a deeper discussion.

Unfortunately, the Hauptvermutung is not true: already in 1961 Milnor [212] gave concrete counterexamples built from lens spaces in all dimensions greater than six. To distinguish the simplicial structures he used an invariant of the associated chain complexes in what he called the *Whitehead group*. In the decade that followed, the Whitehead group proved to be an essential tool in topology, and especially in connection with problems related to "cobordisms". For a more thorough treatment of the following example, see Milnor's very readable article [210].

Let M and N be two smooth n-dimensional closed manifolds. A *cobordism* between M and N is an $n+1$-dimensional smooth compact manifold W with boundary the disjoint union of M and N, see Fig. 1.1 (in the oriented case we assume that M and N are oriented, and W is an oriented cobordism from M to N if it is oriented so that the orientation agrees with that on N and is the opposite of that on M).

Here we are interested in a situation where M and N are deformation retracts of W. Obvious examples are cylinders $M \times I$, where $I = [0, 1]$ is the closed unit interval.

More precisely: Let M be a closed, connected, smooth manifold of dimension $n > 5$. Suppose we are given an *h-cobordism* $(W; M, N)$, that is, a compact smooth $n+1$ dimensional manifold W, with boundary the disjoint union of M and N, such that both the inclusions $M \subset W$ and $N \subset W$ are homotopy equivalences. The obvious examples are cylinders, i.e. diffeomorphic to $M \times I$ (see Fig. 1.2).

Question 1.1.1.1 *Is W diffeomorphic to $M \times I$?*

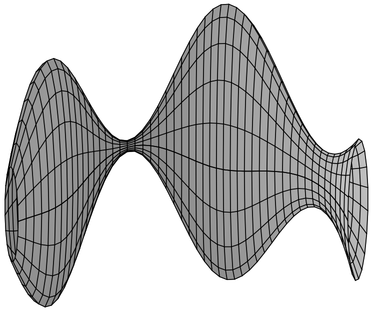


Fig. 1.2 An h-cobordism $(W; M, N)$. This one is a cylinder

It requires some imagination to realize that the answer to this question can be "no". In particular, in the low dimensions of the illustrations all h-cobordisms **are** cylinders.

However, this is not true in high dimensions, and the h-cobordism Theorem 1.1.1.2 below gives a precise answer to the question.

To fix ideas, let $M = L$ be a *lens space* of dimension, say, $n = 7$. That is, the cyclic group of order l, $\pi = \mu_l = \{1, e^{2\pi i/l}, \ldots, e^{2\pi i(l-1)/l}\} \subseteq \mathbf{C}$, acts on the seven-dimensional sphere $S^7 = \{\mathbf{x} \in \mathbf{C}^4 \text{ s.t. } |\mathbf{x}| = 1\}$ by complex multiplication

$$\pi \times S^7 \to S^7 \quad (t, \mathbf{x}) \mapsto (t \cdot \mathbf{x})$$

and we let the lens space M be the quotient space $S^7/\pi = S^7/(\mathbf{x} \sim t \cdot \mathbf{x})$. Then M is a smooth manifold with fundamental group π.

Let

$$\cdots \xrightarrow{\partial} C_{i+1} \xrightarrow{\partial} C_i \xrightarrow{\partial} \ldots \longrightarrow C_0 \longrightarrow 0$$

be the relative cellular complex of the universal cover, calculating the homology $H_* = H_*(\tilde{W}, \tilde{M})$ (see Sects. 7 and 9 in [210] for details). Each C_i is a finitely generated free $\mathbf{Z}[\pi]$-module, and, up to orientation and translation by elements in π, has a preferred basis over $\mathbf{Z}[\pi]$ coming from the i-simplices added to get from M to W in some triangulation of the universal covering spaces. As always, the groups Z_i and B_i of i-cycles and i-boundaries are the kernel of $\partial\colon C_i \to C_{i-1}$ and image of $\partial\colon C_{i+1} \to C_i$. Since $M \subset W$ is a deformation retract, we have by homotopy invariance of homology that $H_* = 0$, and so $B_* = Z_*$.

By induction on i, we see that the exact sequence

$$0 \longrightarrow B_i \longrightarrow C_i \longrightarrow B_{i-1} \longrightarrow 0$$

is split. For each i we choose a splitting and consider the resulting isomorphism

$$C_i \xrightarrow[\cong]{\alpha_i} B_i \oplus B_{i-1}.$$

This leads us to the following isomorphism

$$\begin{array}{ccc} \bigoplus_{i \text{ even}} C_i & \xrightarrow{\bigoplus_{i \text{ even}} \alpha_i} & \bigoplus_{i \text{ even}} B_i \oplus B_{i-1} \\ & & \cong \big\downarrow \text{can. rearrangement} \\ \bigoplus_{i \text{ odd}} C_i & \xrightarrow{\bigoplus_{i \text{ odd}} \alpha_i} & \bigoplus_{i \text{ odd}} B_i \oplus B_{i-1}. \end{array} \tag{1.1}$$

We will return to this isomorphism shortly in order to define the obstruction to the answer to Question 1.1.1.1 being "yes" (see Sect. 1.1.1.2), but first we need some basic definitions from linear algebra.

1.1.1.1 K_1 and the Whitehead Group

For any ring A (all the rings we consider are associative and unital) we may consider the ring $M_k(A)$ of $k \times k$ matrices with entries in A, as a monoid under multiplication (recall that a *monoid* satisfies all the axioms of a group except for the requirement that inverses must exist). The *general linear group* is the subgroup of invertible elements $GL_k(A)$. Take the colimit (or more concretely, the union) $GL(A) = \lim_{k\to\infty} GL_k(A) = \bigcup_{k\to\infty} GL_k(A)$ with respect to the stabilization

$$GL_k(A) \xrightarrow{g \mapsto g \oplus 1} GL_{k+1}(A)$$

(thus every element $g \in GL(A)$ can be thought of as an infinite matrix

$$\begin{bmatrix} g' & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

with $g' \in GL_k(A)$ for some $k < \infty$). Let $E(A)$ be the subgroup of *elementary matrices* (i.e., $E_k(A) \subset GL_k(A)$ is the subgroup generated by the matrices e_{ij}^a with ones on the diagonal and a single nontrivial off-diagonal entry $a \in A$ in the ij position). The "Whitehead lemma" (see Lemma 1.1.2.2 below) implies that the quotient

$$K_1(A) = GL(A)/E(A)$$

is an abelian group. In the particular case where A is an integral group ring $\mathbf{Z}[\pi]$ we define the *Whitehead group* as the quotient

$$Wh(\pi) = K_1\big(\mathbf{Z}[\pi]\big)/\{\pm\pi\}$$

via $\{\pm\pi\} \subseteq GL_1(\mathbf{Z}[\pi]) \to K_1(\mathbf{Z}[\pi])$.

1.1.1.2 Classifying Cobordisms

Let $(W; M, N)$ be an h-cobordism, and consider the isomorphism $\bigoplus_{i \text{ even}} C_i \to \bigoplus_{i \text{ odd}} C_i$ given in Eq. (1.1) for the lens spaces, and similarly in general. This depended on several choices and in the preferred basis for the C_i it gives a matrix with coefficients in $\mathbf{Z}[\pi_1(M)]$. Stabilizing we get an element $\tau(W, M)_{\text{choices}} \in GL(\mathbf{Z}[\pi_1(M)])$ and a class $\tau(W, M) = [\tau(W, M)_{\text{choices}}] \in Wh(\pi_1(M))$.

The class $\tau(W, M)$ is independent of our preferred basis and choices of splittings and is called the *Whitehead torsion*.

The Whitehead torsion turns out to be a vital ingredient in Barden (Thesis, 1963), Mazur [202] and Stallings' [272] extension of the famous results of Smale [264] (where he proves the high dimensional Poincaré conjecture) beyond the simply connected case (for a proof, see also [163]):

Theorem 1.1.1.2 (Barden, Mazur, Stallings) *Let M be a compact, connected, smooth manifold of dimension ≥ 5 and let $(W; M, N)$ be an h-cobordism. The Whitehead torsion $\tau(W, M) \in Wh(\pi_1(M))$ is well defined, and τ induces a bijection*

$$\left\{\begin{matrix} \textit{diffeomorphism classes (rel. } M) \\ \textit{of } h\textit{-cobordisms } (W; M, N) \end{matrix}\right\} \longleftrightarrow Wh\big(\pi_1(M)\big)$$

In particular, $(W; M, N) \cong (M \times I; M, M)$ if and only if $\tau(W, M) = 0$.

Example 1.1.1.3 The Whitehead group, $Wh(\pi)$, has been calculated for only a very limited set of groups π. We list a few of them; for a detailed study of Wh of finite groups, see [220]. The first three refer to the lens spaces discussed above (see p. 375 in [210] for references).

1. $l = 1$, $M = S^7$. "Exercise": show that $K_1\mathbf{Z} = \{\pm 1\}$, and so $Wh(0) = 0$. Thus any h-cobordism of S^7 is diffeomorphic to $S^7 \times I$.
2. $l = 2$. $M = P^7$, the real projective 7-space. "Exercise": show that $K_1\mathbf{Z}[\mu_2] = \{\pm\mu_2\}$, and so $Wh(\mu_2) = 0$. Thus any h-cobordism of P^7 is diffeomorphic to $P^7 \times I$.
3. $l = 5$. $Wh(\mu_5) \cong \mathbf{Z}$ generated by the invertible element $t + t^{-1} - 1 \in \mathbf{Z}[\mu_5]$ (where t is a chosen fifth root of unity)—the inverse is $t^2 + t^{-2} - 1$. That is, there exist countably infinitely many non-diffeomorphic h-cobordisms with incoming boundary component S^7/μ_5.
4. Waldhausen [297]: If π is a free group, free abelian group, or the fundamental group of a submanifold of the three-sphere, then $Wh(\pi) = 0$.
5. Farrell and Jones [81]: If M is a closed Riemannian manifold with non-positive sectional curvature, then $Wh(\pi_1 M) = 0$.

Remark 1.1.1.4 The presentation of the Whitehead torsion differs slightly from that of [210]. It is easy to see that they are the same in the case where the B_i are free $\mathbf{Z}[\pi]$-modules (the splittings ensure that each B_i is "stably free" which is sufficient, but the argument is slightly more involved). Choosing bases we get matrices

$M_i \in GL(\mathbf{Z}[\pi])$ representing the isomorphisms $\alpha_i : C_i \cong B_i \oplus B_{i-1}$, and from the definition of $\tau(W, M)_{\text{choices}}$ we see that

$$\tau(W, M) = \Big(\sum_{i \text{ even}} [M_i]\Big) - \Big(\sum_{i \text{ odd}} [M_i]\Big) = \sum (-1)^i [M_i] \in Wh\big(\pi_1(M)\big).$$

1.1.2 K_1 of Other Rings

1. Commutative rings: The map from the units in A

$$A^* = GL_1(A) \to GL(A)/E(A) = K_1(A)$$

is split by the determinant map, and so the units of A is a split summand in $K_1(A)$. In certain cases (e.g., if A is *local* (A has a unique maximal ideal), or the integers in a number field, see next example) this is all of $K_1(A)$. We may say that the rest of $K_1(A)$ measures to what extent we can do Gauss elimination, in that $\ker\{\det\colon K_1(A) \to A^*\}$ is the group of equivalence classes of matrices up to stabilization in the number of variables and elementary row operations (i.e., multiplication by elementary matrices and multiplication of a row by an invertible element).
2. Let F be a number field (i.e., a finite extension of the rational numbers), and let $A \subseteq F$ be the ring of integers in F (i.e., the integral closure of $\mathbf{Z}$ in F). A result of Dirichlet asserts that A^* is finitely generated of rank $r_1 + r_2 - 1$ where r_1 (resp. $2r_2$) is the number of distinct real (resp. complex) embeddings of F, and in this case $K_1(A) \cong A^*$, see [213, Corollary 18.3] or the arguments on pp. 160–163.
3. Let $B \to A$ be an epimorphism of rings with kernel $I \subseteq rad(B)$—the Jacobson radical of B (that is, if $x \in I$, then $1 + x$ is invertible in B). Then

$$(1+I)^\times \longrightarrow K_1(B) \longrightarrow K_1(A) \longrightarrow 0$$

is exact, where $(1+I)^\times \subset GL_1(B)$ is the group $\{1 + x | x \in I\}$ under multiplication (see e.g., p. 449 in [13]). Moreover, if B is commutative and $B \to A$ is split, then

$$0 \longrightarrow (1+I)^\times \longrightarrow K_1(B) \longrightarrow K_1(A) \longrightarrow 0$$

is exact.

For later reference, we record the Whitehead lemma mentioned above. For this we need some definitions.

Definition 1.1.2.1 The *commutator* $[G, G]$ of a group G is the (normal) subgroup generated by all commutators $[g, h] = ghg^{-1}h^{-1}$. A group G is called *perfect* if it is equal to its commutator, or in other words, if its first homology group $H_1(G) = G/[G, G]$ vanishes. Any group G has a *maximal perfect subgroup*, which we call PG, and which is automatically normal. We say that G is *quasi-perfect* if $PG = [G, G]$.

The symmetric group Σ_n on $n \geq 5$ letters is quasi-perfect, since its commutator subgroup is the alternating group A_n, which in turn is a simple group. Further examples are provided by the

Lemma 1.1.2.2 (The Whitehead Lemma) *Let A be a unital ring. Then $GL(A)$ is quasi-perfect with maximal perfect subgroup $E(A)$, i.e.,*

$$\big[GL(A), GL(A)\big] = \big[E(A), GL(A)\big] = \big[E(A), E(A)\big] = E(A).$$

Proof See e.g., p. 226 in [13]. □

1.1.3 The Grothendieck Group K_0

Definition 1.1.3.1 Let $\mathfrak{C}$ be a small category and let $\mathcal{E}$ be a collection of diagrams $c' \to c \to c''$ in $\mathfrak{C}$. Then the Grothendieck group $K_0(\mathfrak{C}, \mathcal{E})$ is the abelian group, defined (up to unique isomorphism) by the following universal property. Any function f from the set of isomorphism classes of objects in $\mathfrak{C}$ to an abelian group A such that $f(c) = f(c') + f(c'')$ for all sequences $c' \to c \to c''$ in $\mathcal{E}$, factors uniquely through $K_0(\mathfrak{C})$.

If there is a final object $0 \in ob\mathfrak{C}$ such that for any isomorphism $c' \cong c \in \mathfrak{C}$ the sequence $c' \cong c \to 0$ is in $\mathcal{E}$, then $K_0(\mathfrak{C}, \mathcal{E})$ can be given as the free abelian group on the set of isomorphism classes $[c]$, of $\mathfrak{C}$, modulo the relations $[c] = [c'] + [c'']$ for $c' \to c \to c''$ in $\mathcal{E}$. Notice that $[0] = [0] + [0]$, so that $[0] = 0$.

Most often the pair $(\mathfrak{C}, \mathcal{E})$ will be an *exact category* in the sense that $\mathfrak{C}$ is an *additive category* (i.e., a category with all finite coproducts where the morphism sets are abelian groups and where composition is bilinear) such that there exists a full embedding of $\mathfrak{C}$ in an abelian category $\mathfrak{A}$, such that $\mathfrak{C}$ is closed under extensions in $\mathfrak{A}$ and $\mathcal{E}$ consists of the sequences in $\mathfrak{C}$ that are short exact in $\mathfrak{A}$.

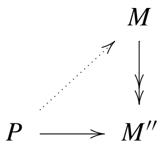
Any additive category is an exact category if we choose the exact sequences to be the split exact sequences, but there may be other exact categories with the same underlying additive category. For instance, the category of abelian groups is an abelian category, and hence an exact category in the natural way, choosing $\mathcal{E}$ to consist of the short exact sequences. These are not necessary split, e.g., $\mathbf{Z} \xrightarrow{2} \mathbf{Z} \longrightarrow \mathbf{Z}/2\mathbf{Z}$ is a short exact sequence which does not split.

The definition of K_0 is a case of "additivity": K_0 is a (or perhaps, *the*) functor to abelian groups insensitive to extension issues. We will dwell more on this issue later, when we introduce the higher K-theories. Higher K-theory plays exactly the same rôle as K_0, except that the receiving category has a much richer structure than the category of abelian groups.

The choice of $\mathcal{E}$ will always be clear from the context, and we drop it from the notation and write $K_0(\mathfrak{C})$.

Example 1.1.3.2

1. Let A be a unital ring. An *A-module* is an abelian group M, together with a homomorphism $A \to \mathrm{End}(M)$ of rings, or otherwise said, a homomorphism $A \otimes M \to M$ of abelian groups, sending $a \otimes m$ to am with the property that $1m = m$ and $a(bm) = (ab)m$. Recall that an A-module M is *finitely generated* if there is a surjective homomorphism $A^n = A \oplus \cdots \oplus A \twoheadrightarrow M$ (n summands) of A-modules. An A-module P is *projective* if for all (solid) diagrams



   of A-modules where the vertical homomorphism is a surjection, there is a (dotted) homomorphism $P \to M$ making the resulting diagram commute. It is a consequence that an A-module P is finitely generated and projective precisely when there is an n and an A-module Q such that $A^n \cong P \oplus Q$. Note that Q is automatically finitely generated and projective.

   If, in a given subcategory of the category of A-modules we say that a certain sequence is exact, we usually mean that the sequence is exact when considered as a sequence of A-modules.

   If $\mathfrak{C} = \mathcal{P}_A$, the category of finitely generated projective A-modules, with the usual notion of (short) exact sequences, we often write $K_0(A)$ for $K_0(\mathcal{P}_A)$. Note that $\mathcal{P}_A$ is *split exact*, that is, all short exact sequences in $\mathcal{P}_A$ split. Thus we see that we could have defined $K_0(A)$ as the quotient of the free abelian group on the isomorphism classes in $\mathcal{P}_A$ by the relation $[P \oplus Q] \sim [P] + [Q]$. It follows that all elements in $K_0(A)$ can be represented as a difference $[P] - [F]$ where F is a finitely generated free A-module.
2. Inside $\mathcal{P}_A$ sits the category $\mathcal{F}_A$ of finitely generated free A-modules, and we let $K_0^f(A) = K_0(\mathcal{F}_A)$. If A is a principal ideal domain, then every submodule of a free module is free, and so $\mathcal{F}_A = \mathcal{P}_A$. This is so, e.g., for the integers, and we see that $K_0(\mathbf{Z}) = K_0^f(\mathbf{Z}) \cong \mathbf{Z}$, generated by the module of rank one. Generally, $K_0^f(A) \to K_0(A)$ is an isomorphism if and only if every finitely generated projective module is *stably free* (P and P' are said to be *stably isomorphic* if there is a finitely generated free A-module Q such that $P \oplus Q \cong P' \oplus Q$, and P is stably free if it is stably isomorphic to a free module). Whereas $K_0(A \times B) \cong K_0(A) \times K_0(B)$, the functor K_0^f does not preserve products: e.g., $\mathbf{Z} \cong K_0^f(\mathbf{Z} \times \mathbf{Z})$, while $K_0(\mathbf{Z} \times \mathbf{Z}) \cong \mathbf{Z} \times \mathbf{Z}$ giving an easy example of a ring where not all projectives are free.

3. Note that K_0 does not distinguish between stably isomorphic modules. This is not important in some special cases. For instance, if A is a commutative Noetherian ring of Krull dimension d, then every stably free module of rank $> d$ is free ([13, p. 239]).
4. The initial map $\mathbf{Z} \to A$ defines a map $\mathbf{Z} \cong K_0^f(\mathbf{Z}) \to K_0^f(A)$ which is always surjective, and in most practical circumstances, an isomorphism. If A has the *invariance of basis property*, that is, if $A^m \cong A^n$ if and only if $m = n$, then $K_0^f(A) \cong \mathbf{Z}$. Otherwise, $A = 0$, or there is an $h > 0$ and a $k > 0$ such that $A^m \cong A^n$ if and only if either $m = n$ or $m, n > h$ and $m \equiv n \mod k$. There are examples of rings with such h and k for all $h, k > 0$ (see [171] or [54]): let $A_{h,k}$ be the quotient of the free ring on the set $\{x_{ij}, y_{ji} | 1 \le i \le h, 1 \le j \le h+k\}$ by the matrix relations

$$[x_{ij}] \cdot [y_{ji}] = I_h, \quad \text{and} \quad [y_{ji}] \cdot [x_{ij}] = I_{h+k}.$$

   Commutative (non-trivial) rings always have the invariance of basis property.
5. Let X be a compact Hausdorff topological space, and let $\mathfrak{C} = \mathrm{Vect}(X)$ be the category of finite rank complex vector bundles on X, with exact sequences meaning the usual thing. Then $K_0(\mathrm{Vect}(X))$ is the complex K-theory $K(X)$ of Atiyah and Hirzebruch [9]. Note that the possibility of constructing normal complements assures that $\mathrm{Vect}(X)$ is a split exact category. Swan's theorem [280] states that the category $\mathrm{Vect}(X)$ is equivalent to the category of finitely generated projective modules over the ring $C(X)$ of complex valued continuous functions on X. The equivalence is given by sending a bundle to its $C(X)$-module of sections. Furthermore, Bott periodicity (see the survey [36] or the neat proof [119]) states that there is a canonical isomorphism $K(S^2) \otimes K(X) \cong K(S^2 \times X)$. A direct calculation shows that $K(S^2) \simeq \mathbf{Z} \oplus \mathbf{Z}$ where it is customary to let the first factor be generated by the trivial bundle 1 and the second by $\xi - 1$ where ξ is the tautological line bundle on $S^2 = \mathbf{CP}^1$.
6. Let X be a scheme, and let $\mathfrak{C} = \mathrm{Vect}(X)$ be the category of finite rank vector bundles on X. Then $K_0(\mathrm{Vect}(X))$ is the $K(X)$ of Grothendieck. This is an example of K_0 of an exact category which is not split exact. The analogous statement to Swan's theorem above is that of Serre [258].

1.1.3.1 Example of Applications to Homotopy Theory

As an illustration we review Loday's [178] early application of the functors K_0 and K_1 to establishing a result about polynomial functions.

Let $T^n = \{(x_1, x_2, \dots, x_{2n-1}, x_{2n}) \in \mathbf{R}^{2n} | x_{2i-1}^2 + x_{2i}^2 = 1, i = 1, \dots, n\}$ be the n-dimensional torus and $S^n = \{(y_0, \dots, y_n) \in \mathbf{R}^{n+1} | y_0^2 + \dots + y_n^2 = 1\}$ the n-dimensional sphere. A polynomial function $T^n \to S^n$ is a polynomial function $f : \mathbf{R}^{2n} \to \mathbf{R}^{n+1}$ such that $f(T^n) \subseteq S^n$.

Proposition 1.1.3.3 (Loday [178]) *Let $n > 1$. Any polynomial function $f : T^n \to S^n$ is homotopic to a constant map.*

Sketch proof We only sketch the case $n=2$. The other even dimensional cases are similar, whereas the odd cases uses K_1 instead of K_0. The heart of the matter is the following commutative diagram

$$\begin{array}{ccc} \mathbf{C}[y_0,y_1,y_2]/\big(y_0^2+y_1^2+y_2^2-1\big) & \longrightarrow & C\big(S^2\big) \\ {\scriptstyle f^*}\downarrow & & {\scriptstyle f^*}\downarrow \\ \mathbf{C}[x_1,x_2,x_3,x_4]/\big(x_1^2+x_2^2-1,x_3^2+x_4^2-1\big) & \longrightarrow & C\big(T^2\big) \end{array}$$

of $\mathbf{C}$-algebras, where the vertical maps are induced by the polynomial function f and the horizontal maps are defined as follows. If $X\subseteq\mathbf{R}^m$ is the zero set of some polynomial function $p=(p_1,\dots,p_k)\colon\mathbf{R}^m\to\mathbf{R}^k$ there is a preferred map of $\mathbf{C}$-algebras $\mathbf{C}[x_1,\dots,x_m]/(p_1,\dots,p_k)\to C(X)$ given by sending the generator x_l to the composite function $X\subseteq\mathbf{R}^m\subseteq\mathbf{C}^m\twoheadrightarrow\mathbf{C}$ where the last map is projection onto the lth factor.

Let $\tilde{K}_0$ be the functor from rings to abelian groups whose value at A is the cokernel of the canonical map $K_0(\mathbf{Z})\to K_0(A)$. Considering the resulting diagram

$$\begin{array}{ccc} \tilde{K}_0\big(\mathbf{C}[y_0,y_1,y_2]/\big(y_0^2+y_1^2+y_2^2-1\big)\big) & \longrightarrow & \tilde{K}_0\big(C(S^2)\big) \\ {\scriptstyle f^*}\downarrow & & {\scriptstyle f^*}\downarrow \\ \tilde{K}_0\big(\mathbf{C}[x_1,x_2,x_3,x_4]/\big(x_1^2+x_2^2-1,x_3^2+x_4^2-1\big)\big) & \longrightarrow & \tilde{K}_0\big(C\big(T^2\big)\big) \end{array}.$$

By Swan's theorem Example 1.1.3.2(5) we may identify the right hand vertical map with $f^*\colon\tilde{K}(S^2)\to\tilde{K}(T^2)$ (where $\tilde{K}(X)$ is the cokernel of the canonical map $K(*)\to K(X)$). Hence we are done if we can show

1. The top horizontal map is a surjection,
2. the lower left hand group is trivial and
3. a polynomial function $T^2\to S^2$ is homotopic to a constant map if it induces the trivial map $\tilde{K}(S^2)\to\tilde{K}(T^2)$.

By the statements about complex K-theory in Example 1.1.3.2(5), $\tilde{K}(S^2)$ is a copy of the integers (generated by $\xi-1$), so to see that the top horizontal map is a surjection it is enough to see that a generator is hit (i.e., the canonical line bundle is algebraic), and this is done explicitly in [178, Lemme 2].

The substitution $t_k=x_{2k-1}+ix_{2k}$ induces an isomorphism

$$\mathbf{C}[x_1,x_2,x_3,x_4]/\big(x_1^2+x_2^2-1,x_3^2+x_4^2-1\big)\cong\mathbf{C}\big[t_1,t_1^{-1},t_2,t_2^{-1}\big],$$

and by [13, p. 636] $\tilde{K}_0(\mathbf{C}[t_1,t_1^{-1},t_2,t_2^{-1}])=0$. This vanishing of a K-group is part of a more general statement about algebraic K-theory's behavior with respect to localizations and about polynomial rings over regular rings.

To see the last statement, one has to know that the Chern class is natural: the diagram

$$\begin{array}{ccc} \tilde{K}(S^2) & \xrightarrow{c_1} & H^2(S^2;\mathbf{Q}) \cong \mathbf{Q} \\ {\scriptstyle f^*}\big\downarrow & & {\scriptstyle f^*}\big\downarrow \\ \tilde{K}(T^2) & \xrightarrow{c_1} & H^2(T^2;\mathbf{Q}) \cong \mathbf{Q} \end{array}$$

commutes. Since $c_1(\xi_1 - 1) \neq 0$ we get that if the left vertical map is trivial, so is the right vertical map (which is multiplication by the degree). However, a map $f\colon T^2 \to S^2$ is homotopic to a constant map exactly if its degree is trivial. ☺

1.1.3.2 Geometric Example: Wall's Finiteness Obstruction

Let A be a space which is dominated by a finite CW-complex X (dominated means that there are maps $A \xrightarrow{i} X \xrightarrow{r} A$ such that $ri \simeq id_A$).

Question *is A homotopy equivalent to a finite CW-complex?*

The answer is *yes* if and only if a certain finiteness obstruction in the abelian group $\tilde{K}_0(\mathbf{Z}[\pi_1 A]) = \ker\{K_0(\mathbf{Z}[\pi_1 A]) \to K_0(\mathbf{Z})\}$ vanishes. So, for instance, if we know that $\tilde{K}_0(\mathbf{Z}[\pi_1 A])$ vanishes for algebraic reasons, we can always conclude that A is homotopy equivalent to a finite CW-complex. As for K_1, calculations of $K_0(\mathbf{Z}[\pi])$ are very hard, but we give a short list.

1.1.3.3 K_0 of Group Rings

1. If C_p is a cyclic group of prime order p less than 23, then $\tilde{K}_0(\mathbf{Z}[\pi])$ vanishes. The first nontrivial group is $\tilde{K}_0(\mathbf{Z}[C_{23}]) \cong \mathbf{Z}/3\mathbf{Z}$ (Kummer, see [213, p. 30]).
2. Waldhausen [297]: If π is a free group, free abelian group, or the fundamental group of a submanifold of the three-sphere, then $\tilde{K}_0(\mathbf{Z}[\pi]) = 0$.
3. Farrell and Jones [81]: If M is a closed Riemannian manifold with non-positive sectional curvature, then $\tilde{K}_0(\mathbf{Z}[\pi_1 M]) = 0$.

1.1.3.4 Facts About K_0 of Rings

1. If A is a commutative ring, then $K_0(A)$ has a ring structure. The additive structure comes from the direct sum of modules, and the multiplication from the tensor product.
2. If A is local, then $K_0(A) = \mathbf{Z}$.

3. Let A be a commutative ring. Define $rk_0(A)$ to be the split summand of $K_0(A)$ of classes of rank 0, cf. [13, p. 459]. The modules P for which there exists a Q such that $P \otimes_A Q \cong A$ form a category. The isomorphism classes form a group under tensor product. This group is called the Picard group, and is denoted $Pic(A)$. There is a "determinant" map $rk_0(A) \to Pic(A)$ which is always surjective. If A is a Dedekind domain (see [13, pp. 458–468]) the determinant map is an isomorphism, and $Pic(A)$ is isomorphic to the ideal class group $Cl(A)$.
4. Let A be the integers in a number field. Then Dirichlet tells us that $rk_0(A) \cong Pic(A) \cong Cl(A)$ is finite. For instance, if $A = \mathbf{Z}[e^{2\pi i/p}] = \mathbf{Z}[t]/\sum_{i=0}^{p-1} t^i$, the integers in the cyclotomic field $\mathbf{Q}(e^{2\pi i/p})$, then $K_0(A) \cong K_0(\mathbf{Z}[C_p])$ (Sect. 1.1.3.31).
5. If $f\colon B \to A$ is a surjection of rings with kernel I contained in the Jacobson radical, $rad(B)$, then $K_0(B) \to K_0(A)$ is injective ([13, p. 449]). It is an isomorphism if
   (a) B is complete in the I-adic topology ([13]),
   (b) (B, I) is a Hensel pair ([88]) or
   (c) f is split (as K_0 is a functor).

   That (B, I) is a *Hensel pair* means that if $f \in B[t]$ has image $\bar{f} \in A[t]$ and $a \in A = B/I$ satisfies $\bar{f}(a) = 0$ and $f'(a)$ is a unit in B/I, then there is a $b \in B$ mapping to a, and such that $f(b) = 0$. It implies that $I \subseteq rad(B)$.

1.1.3.5 An Example from Algebraic Geometry

Algebraic K-theory appears in Grothendieck's proof of the Riemann–Roch theorem, see Borel and Serre [35], where Bott's entry in Mathematical Reviews can serve as the missing introduction. Let X be a non-singular quasi-projective variety (i.e., a locally closed subvariety of some projective variety) over an algebraically closed field. Let $CH(X)$ be the Chow ring of cycles under linear equivalence (called $A(X)$ in [35, Sect. 6]) with product defined by intersection. Tensor product gives a ring structure on $K_0(X)$, and Grothendieck defines a natural ring homomorphism

$$ch\colon K_0(X) \to CH(X) \otimes \mathbf{Q},$$

similar to the Chern character for vector bundles, cf. [214]. This map has good functoriality properties with respect to pullback, i.e., if $f\colon X \to Y$, then

$$\begin{array}{ccc} K_0(X) & \xrightarrow{ch} & CH(X) \otimes \mathbf{Q} \\ {\scriptstyle f^!}\uparrow & & \uparrow{\scriptstyle f^*} \\ K_0(Y) & \xrightarrow{ch} & CH(Y) \otimes \mathbf{Q} \end{array}$$

commutes, where $f^!$ and f^* are given by pulling back along f. For proper morphisms $f\colon X \to Y$ [35, p. 100] there are "transfer maps" (defined as a sort of Euler characteristic) $f_!\colon K_0(X) \to K_0(Y)$ [35, p. 110] and direct image maps

$f_*\colon CH(X) \to CH(Y)$. The Riemann–Roch theorem is nothing but a quantitative measure of the fact that

$$\begin{array}{ccc} K_0(X) & \xrightarrow{ch} & CH(X)\otimes \mathbf{Q} \\ {\scriptstyle f_!}\big\downarrow & & \big\downarrow{\scriptstyle f_*} \\ K_0(Y) & \xrightarrow{ch} & CH(Y)\otimes \mathbf{Q} \end{array}$$

fails to commute: $ch(f_!(x)) \cdot Td(Y) = f_*(ch(x) \cdot Td(X))$ where $Td(X)$ is the value of the "Todd class" [35, p. 112] on the tangent bundle of X.

1.1.3.6 A Number-Theoretic Example

Let F be a number field and A its ring of integers. Then there is an exact sequence connecting K_1 and K_0:

$$0 \longrightarrow K_1(A) \longrightarrow K_1(F) \longrightarrow \bigoplus_{\mathfrak{m}\in Max(A)} K_0(A/\mathfrak{m}) \longrightarrow K_0(A) \longrightarrow K_0(F) \longrightarrow 0$$

(cf. [13, pp. 323, 702], or better [232, Corollary to Theorem 5] plus the fact that $K_1(A) \to K_1(F)$ is injective). The *zeta function* $\zeta_F(s)$ of F is defined as the meromorphic function on the complex plane $\mathbf{C}$ we get as the analytic continuation of

$$\zeta_F(s) = \sum_{I \text{ non-zero ideal in } A} |A/I|^{-s}.$$

This series converges for $Re(s) > 1$. The zeta function has a zero of order $r = \operatorname{rank}(K_1(A))$ (see Sect. 1.1.2.(2)) at $s = 0$, and the class number formula says that

$$\lim_{s\to 0} \frac{\zeta_F(s)}{s^r} = -\frac{R|K_0(A)_{tor}|}{|K_1(A)_{tor}|},$$

where $|-_{tor}|$ denotes the cardinality of the torsion subgroup, and the regulator R is a number that depends on the map δ above, see [175].

This is related to the Lichtenbaum-Quillen conjecture, which is now confirmed due to work of among many others Voevodsky, Suslin, Rost, Grayson (see Sect. 1.1.7 and Sect. 7.3.2 for references and a deeper discussion).

1.1.4 The Mayer–Vietoris Sequence

The reader may wonder why one chooses to regard the functors K_0 and K_1 as related. Example 1.1.3.6 provides one motivation, but that is cheating. Historically, it was an insight of Bass that K_1 could be obtained from K_0 in analogy with the definition of $K^1(X)$ as $K^0(S^1 \wedge X)$ (cf. Example 1.1.3.2(5)). This manifests itself in exact sequences connecting the two theories. As an example: if

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow f \\ C & \xrightarrow{g} & D \end{array}$$

is a cartesian square of rings and g (or f) is surjective, then we have a long exact "Mayer–Vietoris" sequence

$$K_1(A) \longrightarrow K_1(B) \oplus K_1(C) \longrightarrow K_1(D) \longrightarrow K_0(A) \longrightarrow K_0(B) \oplus K_0(C) \longrightarrow K_0(D).$$

However, it is not true that this continues to the left. For one thing there is no simple analogy to the Bott periodicity $K^0(S^2 \wedge X) \cong K^0(X)$. Milnor proposed in [213] a definition of K_2 (see below) which would extend the Mayer–Vietoris sequence if **both** f and g are surjective, i.e., we have a long exact sequence

$$K_2(A) \longrightarrow K_2(B) \oplus K_2(C) \longrightarrow K_2(D) \longrightarrow K_1(A) \longrightarrow K_1(B) \oplus K_1(C) \longrightarrow K_1(D) \longrightarrow \cdots.$$

However, this was the best one could hope for:

Example 1.1.4.1 Swan [281] gave the following example showing that there exists no functor K_2 giving such a sequence if only g is surjective. Let A be commutative, and consider the pullback diagram

$$\begin{array}{ccc} A[t]/t^2 & \xrightarrow{t \mapsto 0} & A \\ {\scriptstyle a+bt \mapsto \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}} \downarrow & & \downarrow \Delta \\ T_2(A) & \xrightarrow{g} & A \times A \end{array}$$

where $T_2(A)$ is the ring of upper triangular 2×2 matrices, g is the projection onto the diagonal, while Δ is the diagonal inclusion. As g splits $K_2(T_2(A)) \oplus K_2(A) \to K_2(A \times A)$ must be surjective, but, as we shall see below, $K_1(A[t]/t^2) \to K_1(T_2(A)) \oplus K_1(A)$ is not injective.