Applied and Numerical Harmonic Analysis

$$\widehat{f}(\gamma) = \int f(x) e^{-2\pi i x \gamma} dx$$

Travis D. Andrews · Radu Balan
John J. Benedetto · Wojciech Czaja
Kasso A. Okoudjou
Editors

Excursions in Harmonic Analysis, Volume 2

The February Fourier Talks at the Norbert Wiener Center

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Dedicated to
Tom Grasso,
Friend and Editor Extraordinaire

ANHA Series Preface

The *Applied and Numerical Harmonic Analysis (ANHA)* book series aims to provide the engineering, mathematical, and scientific communities with significant developments in harmonic analysis, ranging from abstract harmonic analysis to basic applications. The title of the series reflects the importance of applications and numerical implementation, but richness and relevance of applications and implementation depend fundamentally on the structure and depth of theoretical underpinnings. Thus, from our point of view, the interleaving of theory and applications and their creative symbiotic evolution is axiomatic.

Harmonic analysis is a wellspring of ideas and applicability that has flourished, developed, and deepened over time within many disciplines and by means of creative cross-fertilization with diverse areas. The intricate and fundamental relationship between harmonic analysis and fields such as signal processing, partial differential equations (PDEs), and image processing is reflected in our state-of-theart *ANHA* series.

Our vision of modern harmonic analysis includes mathematical areas such as wavelet theory, Banach algebras, classical Fourier analysis, time-frequency analysis, and fractal geometry, as well as the diverse topics that impinge on them.

For example, wavelet theory can be considered an appropriate tool to deal with some basic problems in digital signal processing, speech and image processing, geophysics, pattern recognition, biomedical engineering, and turbulence. These areas implement the latest technology from sampling methods on surfaces to fast algorithms and computer vision methods. The underlying mathematics of wavelet theory depends not only on classical Fourier analysis, but also on ideas from abstract harmonic analysis, including von Neumann algebras and the affine group. This leads to a study of the Heisenberg group and its relationship to Gabor systems, and of the metaplectic group for a meaningful interaction of signal decomposition methods. The unifying influence of wavelet theory in the aforementioned topics illustrates the justification for providing a means for centralizing and disseminating information from the broader, but still focused, area of harmonic analysis. This will be a key role of *ANHA*. We intend to publish the scope and interaction that such a host of issues demands.

Along with our commitment to publish mathematically significant works at the frontiers of harmonic analysis, we have a comparably strong commitment to publish major advances in the following applicable topics in which harmonic analysis plays a substantial role:

Biomedical signal processing
Compressive sensing
Communications applications
Data mining/machine learning
Digital signal processing
Fast algorithms
Gabor theory and applications
Image processing
Numerical partial differential equations
Prediction theory
Radar applications
Sampling theory
Spectral estimation
Speech processing
Time-frequency and time-scale analysis
Wavelet theory

The above point of view for the *ANHA* book series is inspired by the history of Fourier analysis itself, whose tentacles reach into so many fields.

In the last two centuries, Fourier analysis has had a major impact on the development of mathematics, on the understanding of many engineering and scientific phenomena, and on the solution of some of the most important problems in mathematics and the sciences. Historically, Fourier series were developed in the analysis of some of the classical PDEs of mathematical physics; these series were used to solve such equations. In order to understand Fourier series and the kinds of solutions they could represent, some of the most basic notions of analysis were defined, e.g., the concept of "function". Since the coefficients of Fourier series are integrals, it is no surprise that Riemann integrals were conceived to deal with uniqueness properties of trigonometric series. Cantor's set theory was also developed because of such uniqueness questions.

A basic problem in Fourier analysis is to show how complicated phenomena, such as sound waves, can be described in terms of elementary harmonics. There are two aspects of this problem: first, to find, or even define properly, the harmonics or spectrum of a given phenomenon, e.g., the spectroscopy problem in optics; second, to determine which phenomena can be constructed from given classes of harmonics, as done, e.g., by the mechanical synthesizers in tidal analysis.

Fourier analysis is also the natural setting for many other problems in engineering, mathematics, and the sciences. For example, Wiener's Tauberian theorem in Fourier analysis not only characterizes the behavior of the prime numbers, but also provides the proper notion of spectrum for phenomena such as white light; this latter process leads to the Fourier analysis associated with correlation functions in filtering and prediction problems, and these problems, in turn, deal naturally with Hardy spaces in the theory of complex variables.

Nowadays, some of the theory of PDEs has given way to the study of Fourier integral operators. Problems in antenna theory are studied in terms of unimodular trigonometric polynomials. Applications of Fourier analysis abound in signal processing, whether with the fast Fourier transform (FFT), or filter design, or the adaptive modeling inherent in time-frequency-scale methods such as wavelet theory. The coherent states of mathematical physics are translated and modulated Fourier transforms, and these are used, in conjunction with the uncertainty principle, for dealing with signal reconstruction in communications theory. We are back to the raison d'être of the *ANHA* series!

University of Maryland
College Park

John J. Benedetto
Series Editor

Preface

The chapters in these two volumes have at least one (co)author who spoke at the February Fourier Talks during the period 2006–2011.

The February Fourier Talks

The February Fourier Talks (*FFT*) were initiated in 2002 as a small meeting on harmonic analysis and applications, held at the University of Maryland, College Park. Since 2006, the *FFT* has been organized by the Norbert Wiener Center in the Department of Mathematics, and it has become a major annual conference. The *FFT* brings together applied and pure harmonic analysts along with scientists and engineers from industry and government for an intense and enriching two-day meeting. The goals of the *FFT* are the following:

- To offer a forum for applied and pure harmonic analysts to present their latest cutting-edge research to scientists working not only in the academic community but also in industry and government agencies,
- To give harmonic analysts the opportunity to hear from government and industry scientists about the latest problems in need of mathematical formulation and solution,
- To provide government and industry scientists with exposure to the latest research in harmonic analysis,
- To introduce young mathematicians and scientists to applied and pure harmonic analysis,
- To build bridges between pure harmonic analysis and applications thereof.

These goals stem from our belief that many of the problems arising in engineering today are directly related to the process of making pure mathematics applicable. The Norbert Wiener Center sees the *FFT* as the ideal venue to enhance this process in a constructive and creative way. Furthermore, we believe that our vision is shared by the scientific community, as shown by the steady growth of the *FFT* over the years.

The *FFT* is formatted as a two-day single-track meeting consisting of thirty-minute talks as well as the following:

- Norbert Wiener Distinguished Lecturer series
- General interest keynote address
- Norbert Wiener Colloquium
- Graduate and postdoctoral poster session

The talks are given by experts in applied and pure harmonic analysis, including academic researchers and invited scientists from industry and government agencies.

The Norbert Wiener Distinguished Lecture caps the technical talks of the first day. It is given by a senior harmonic analyst, whose vision and depth through the years have had profound impact on our field. In contrast to the highly technical day sessions, the keynote address is aimed at a general public audience and highlights the role of mathematics, in general, and harmonic analysis, in particular. Furthermore, this address can be seen as an opportunity for practitioners in a specific area to present mathematical problems that they encounter in their work. The concluding lecture of each *FFT*, our Norbert Wiener Colloquium, features a mathematical talk by a renowned applied or pure harmonic analyst. The objective of the Norbert Wiener Colloquium is to give an overview of a particular problem or a new challenge in the field. We include here a list of speakers for these three lectures:

Distinguished lecturer

- Peter Lax
- Richard Kadison
- Elias Stein
- Ronald Coifman
- Gilbert Strang

Keynote address

- Frederick Williams
- Steven Schiff
- Peter Carr
- Barry Cipra
- William Noel
- James Coddington
- Mario Livio

Colloquium

- Christopher Heil
- Margaret Cheney
- Victor Wickerhauser
- Robert Fefferman
- Charles Fefferman
- Peter Jones

The Norbert Wiener Center

The Norbert Wiener Center for Harmonic Analysis and Applications provides a national focus for the broad area of mathematical engineering. Applied harmonic analysis and its theoretical underpinnings form the technological basis for this area. It can be confidently asserted that mathematical engineering will be to today's mathematics departments what mathematical physics was to those of a century ago. At that time, mathematical physics provided the impetus for tremendous advances within mathematics departments, with particular impact in fields such as differential equations, operator theory, and numerical analysis. Tools developed in these fields were essential in the advances of applied physics, e.g., the development of the solid-state devices which now enable our information economy.

Mathematical engineering impels the study of fundamental harmonic analysis issues in the theories and applications of topics such as signal and image processing, machine learning, data mining, waveform design, and dimension reduction into mathematics departments. The results will advance the technologies of this millennium.

The golden age of mathematical engineering is upon us. The Norbert Wiener Center reflects the importance of integrating new mathematical technologies and algorithms in the context of current industrial and academic needs and problems. The Norbert Wiener Center has three goals:

- Research activities in harmonic analysis and applications
- Education—undergraduate to postdoctoral
- Interaction within the international harmonic analysis community

We believe that educating the next generation of harmonic analysts, with a strong understanding of the foundations of the field and a grasp of the problems arising in applications, is important for a high-level and productive industrial, government, and academic workforce.

The Norbert Wiener Center web site: www.norbertwiener.umd.edu

The Structure of the Volumes

To some extent the eight parts of these two volumes are artificial placeholders for all the diverse chapters. It is an organizational convenience that reflects major areas in harmonic analysis and its applications, and it is also a means to highlight significant modern thrusts in harmonic analysis. Each of the following parts includes an introduction that describes the chapters therein:

Volume 1		Volume 2	
I	Sampling Theory	V	Measure Theory
II	Remote Sensing	VI	Filtering
III	Mathematics of Data Processing	VII	Operator Theory
IV	Applications of Data Processing	VIII	Biomathematics

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The Norbert Wiener Center gratefully acknowledges the indispensable support of the following groups: Australian Academy of Science, Birkhäuser the IEEE Baltimore Section, MiMoCloud, Inc., Patton Electronics Co., Radyn, Inc., the SIAM Washington–Baltimore Section, and SR2 Group, LLC. One of the successes of the February Fourier Talks has been the dynamic participation of graduate student and postdoctoral engineers, mathematicians, and scientists. We have been fortunate to be able to provide travel and living expenses to this group due to continuing, significant grants from the National Science Foundation, which, along with the aforementioned organizations and companies, believes in and supports our vision of the FFT.

Contents

Part V
Measure Theory

The four chapters in this part treat old themes as well as some modern applications of measure theory. Measure theory was developed in the late nineteenth and early twentieth centuries, and its founders include E. Borel, H. Lebesgue, and J. Radon. It remains a central field in mathematics offering rigorous tools to tackle problems arising in mathematical analysis. Measure theory is also one of the foundational courses that many aspiring mathematicians are required to take. Even more, modern probability theory owes its very existence to measure theory, which also plays essential roles in areas such dynamical systems, harmonic analysis, and partial differential equations. The breadth and the depth of the chapters in this part partially illustrate how these two volumes are intended for various audiences ranging from graduate students, researchers, to practitioners in harmonic analysis and its applications.

The first chapter of this part is by ROBERT B. BURCKEL, who provides new proofs of two classical results about measures on the circle. The first result characterizing nonzero analytic measures on the unit circle is due to the Riesz brothers (1916). The second result, due to Szegö (1920), gives a condition under which the closed linear span of the monomials on the unit circle is dense in the Hilbert space associated with a measure on the circle. Burckel's beautiful proofs are based largely on Hilbert space arguments rather than being purely measure theoretical.

In the second chapter of this part, CARLOS CABRELLI, UDAYAN B. DARJI, AND URSULA MOLTER give a nontrivial extension of a construction of strongly invisible sets due to R. O. Davies to measures on Polish spaces (complete separable metric spaces). The setting of this type of result is measure theoretic dimension theory which offers methods to classify measurable sets. In this short but elegant chapter, Cabrelli, Darji, and Molter introduce tools that allow them to construct a large class of visible Cantor sets in Polish spaces. Visible Cantor sets are sets of finite positive measure for some Hausdorff measure or some translation invariant Borel measure. They also investigated strongly invisible sets in the setting of Polish groups. The new results they obtain include the density of certain of these classes of measurable sets within the set of all compact sets in the underlying space.

JEAN-PIERRE GABARDO'S chapter offers new insights to some applications of measure theory in time-frequency analysis. In particular, he establishes relationships between convolution inequalities for positive Borel measures in Euclidean space and the notion of upper and lower Beurling density for these measures. Such relationships arise in the study of the packing and tiling properties of translates of sets in Euclidean spaces, in which case the Borel measures considered are sums of Dirac masses. The connection to time-frequency analysis lies in the fact that the quantitative behavior of Beurling density is important in the theory of uniform and nonuniform Gabor systems. A Gabor system consists of the time-frequency shifts of a fixed window function along a discrete set of points in Euclidean space. If this system is a frame, then the upper Beurling density of the corresponding discrete set must be finite, while its lower Beurling density must be at least one. In this case, an appropriate choice of the window function can lead to the decomposition and reconstruction of any function from samples of its short-time Fourier transform.

In the final chapter of this part, BILL MORAN, STEPHEN HOWARD, AND DOUG COCHRAN use the parallel between frame theory and the theory of positive operator-valued measures (POVMs) on a Hilbert space, $\mathscr{H}$, to introduce the new concept of a *framed POVM*. Frames are redundant sets of vectors that provide stable and efficient algorithms for the decomposition and reconstruction of any vector in $\mathscr{H}$. Consequently, frames are a natural tool in many signal processing applications. On the other hand, the notion of POVM was introduced and developed in quantum mechanics as a tool to represent the most general form of quantum measurement of a system. After giving an overview of the similarities between frames for $\mathscr{H}$ and POVMs associated to $\mathscr{H}$, the authors show that the class of framed POVMs is large in a sense defined in their chapter. In fact, this class includes frames as well as more recent extensions such as fusion frames and generalized frames. One of the interesting aspects of framed POVMs is that tools from quantum mechanics can now be translated to obtain new results in frame theory.

Absolute Continuity and Singularity of Measures Without Measure Theory

R.B. Burckel

Abstract This chapter presents proofs, largely by Hilbert-space arguments, of two classical results about measures on the circle associated with the Riesz Brothers (1916) and Gabor Szegö (1920).

Keywords Absolutely continuous measures • Approximation theorems of Weierstrass and Féjer • Hilbert-space methods in measure theory • M. Riesz Theorem • Shifts in Hilbert space • Singular measures • Szegö's Theorem • Wold decomposition

This chapter presents proofs of two classical results about measures on the circle associated with the Riesz Brothers [3] and G. Szegö [5]. Of course, measure theory cannot be fully eschewed (so the title is something of a come-on), but the proofs are largely by Hilbert-space arguments, and some minimal notation is needed:

$\mathbb{Z}$ is the integers, $\mathbb{N} := \{n \in \mathbb{Z} : n \geq 1\}$, $\mathbb{N}_0 := \{n \in \mathbb{Z} : n \geq 0\}$, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{T} := \partial\mathbb{D}$, $C(\mathbb{T})$ is the continuous functions on $\mathbb{T}$, λ is normalized Lebesgue measure on the Borel subsets of $\mathbb{T}$. For a complex-valued Borel measure ν on $\mathbb{T}$ the number

$$\hat{\nu}(k) := \int_{\mathbb{T}} z^{-k} \, \mathrm{d}\nu(z)$$

is called the k*-th Fourier coefficient* of ν, for each $k \in \mathbb{Z}$. The absolute continuity relation is signalled, as is customary, by $\ll$. For a subset S of a vector space span(S) will denote its (algebraic) linear span. Classical theorems of Weierstrass and Fejér assert that

$$\text{span}\,\{z^k : k \in \mathbb{Z}\} \text{ is uniformly dense in } C(\mathbb{T}). \tag{WF}$$

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A consequence of (WF) is

$$\text{span}\,\{z^k : k \in \mathbb{Z}\} \text{ is dense in every } L^2(\nu) \text{ space,} \qquad (\text{WF})_\nu$$

and so the Fourier coefficients uniquely determine ν; that is, $\hat{\nu}(k) = 0$ for all k if and only if $\nu = 0$. ν is called *analytic* if

$$\hat{\nu}(-n) := \int_{\mathbb{T}} z^n \,\mathrm{d}\nu(z) = 0 \quad \forall n \in \mathbb{N}. \qquad (\text{A})$$

This terminology derives from the fact (Cauchy's integral theorem) that if f is continuous on $\overline{\mathbb{D}}$ and analytic in $\mathbb{D}$, then the measure $f\,\mathrm{d}\lambda$ is analytic.

Theorem 1 (F. and M. Riesz Theorem). *Every non-zero analytic measure ν is mutually absolutely continuous with respect to λ.*

Proof. Let μ denote the total variation measure of ν and f the Radon–Nikodym derivative $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}$. That is, $\mathrm{d}\nu = f\,\mathrm{d}\mu$ and $|f| = 1$ μ-a.e. Then (A) can be written

$$\int_{\mathbb{T}} z^n f(z)\mathrm{d}\mu(z) = 0 \quad \forall n \in \mathbb{N}. \qquad (1)$$

Let $\langle\, ,\, \rangle$ and $\|\,\|$ denote inner product and norm in $L^2(\mu)$ and U (suggesting *unilateral* shift) the operator, evidently unitary, of multiplication by z on this Hilbert space. According to (1) the constant function 1 is orthogonal to every $\mathrm{U}^n f$ $(n \in \mathbb{N})$, so the set

$$M := L^2(\mu)\text{-closure of span } \{\mathrm{U}^n f : n \in \mathbb{N}\} \qquad (2)$$

is a *proper* subspace of $L^2(\mu)$, evidently U-invariant. Since $|f| = 1$ μ-a.e., $(\text{WF})_\mu$ entails that

$$\text{span}\{z^k f : k \in \mathbb{Z}\} \text{ is dense in } L^2(\mu). \qquad (3)$$

Let us note that

$$\mathrm{U}M \subsetneq M. \qquad (4)$$

For if $\mathrm{U}M = M$, then $\mathrm{U}^*M = \mathrm{U}^*\mathrm{U}M = M$, so M would contain $(\mathrm{U}^*)^m\mathrm{U}^n f = (\bar{z})^m z^n f = z^{n-m} f$ for all $n, m \in \mathbb{N}$ and consequently $z^k f$ for all $k \in \mathbb{Z}$. From this and (3) would follow the contradiction $L^2(\mu) \subset M$. This confirms (4).

Form the orthogonal complement $M \ominus \mathrm{U}M$, which is not $\{0\}$ by (4), and note that the (closed) subspaces $\mathrm{U}^k(M \ominus \mathrm{U}M)$ are orthogonal, which is pretty clear when they are written as $\mathrm{U}^k M \ominus \mathrm{U}^{k+1}M$: $m_1, m_2 \in M \ominus \mathrm{U}M$, $p > k \Rightarrow \langle \mathrm{U}^p m_1, \mathrm{U}^k m_2\rangle = \langle \mathrm{U}^{p-k} m_1, m_2\rangle = 0$ since $\mathrm{U}^{p-k} m_1 \in \mathrm{U}M$ and $m_2 \perp \mathrm{U}M$.

As a special case of this

$$\{\mathrm{U}^k h\}_{k\in\mathbb{Z}} \text{ is an orthonormal sequence in } L^2(\mu) \qquad (5)$$

for every unit vector $h \in M \ominus \mathrm{U}M$. Note that

$$\text{subspace } \bigcap_{n\geq 0} \mathrm{U}^n M \text{ is orthogonal to } \mathrm{U}^k(M \ominus \mathrm{U}M) \quad \forall k \in \mathbb{Z}. \tag{6}$$

For if $m_1 \in M \ominus \mathrm{U}M$ and m_0 lies in this intersection, then $m_0 = \mathrm{U}^{|k|+1} m_2$ for some $m_2 \in M$, and so

$$\langle m_0, \mathrm{U}^k m_1 \rangle = \langle \mathrm{U}^{|k|+1} m_2, \mathrm{U}^k m_1 \rangle = \langle \mathrm{U}^{|k|-k+1} m_2, m_1 \rangle = 0,$$

since $\mathrm{U}^{|k|-k+1} m_2 \in \mathrm{U}M$. The same argument shows that

$$\bigcap_{n\geq 0} \mathrm{U}^n M \text{ is orthogonal to } \mathrm{U}^k 1 = z^k \quad \forall k \in \mathbb{Z}. \tag*{$(6)_1$}$$

For $\langle m_0, \mathrm{U}^k 1 \rangle = \langle \mathrm{U}^{|k|+1} m_2, \mathrm{U}^k 1 \rangle = \langle \mathrm{U}^{|k|-k+1} m_2, 1 \rangle = 0$, since, as already noted, $1 \perp M$.

Now the well-known *Wold decomposition* (see Sz.-Nagy and Foiaş [6, p.3]) says that M is the orthogonal sum

$$M = \bigcap_{n\geq 0} \mathrm{U}^n M \oplus \bigoplus_{k\geq 0} \mathrm{U}^k (M \ominus \mathrm{U}M). \tag{7}$$

Again, this is pretty transparent when the right side is written out:

$$\bigcap_{n\geq 0} \mathrm{U}^n M \oplus (M \ominus \mathrm{U}M) \oplus (\mathrm{U}M \ominus \mathrm{U}^2 M) \oplus (\mathrm{U}^2 M \ominus \mathrm{U}^3 M) \oplus \cdots$$

From Eq. $(6)_1$ and $(\mathrm{WF})_\mu$ it follows that the space $\bigcap_{n\geq 0} \mathrm{U}^n M$ must be $\{0\}$ and (7) reads

$$M = \bigoplus_{k\geq 0} \mathrm{U}^k (M \ominus \mathrm{U}M) \tag{8}$$

Next we aim to show that

$$M \ominus \mathrm{U}M \text{ is 1-dimensional.} \tag{9}$$

To this end, consider a fixed $h \in M \ominus \mathrm{U}M$ of norm 1 [recalling (4)]. If (9) fails, there exists $g \in M \ominus \mathrm{U}M$ of norm 1 which is orthogonal to h. Then, by the familiar maneuvers, $\mathrm{U}^m h \perp \mathrm{U}^n g$ for all $m, n \in \mathbb{N}_0$.

$$0 = \langle \mathrm{U}^m h, \mathrm{U}^n g \rangle = \langle \mathrm{U}^{m-n} h, g \rangle \quad \forall m, n \geq 0 \Rightarrow$$
$$0 = \langle \mathrm{U}^k h, g \rangle = \int_{\mathbb{T}} z^k h \bar{g} \, \mathrm{d}\mu \quad \forall k \in \mathbb{Z}.$$

The function $h\bar{g}$ being in $L^1(\mu)$, these equations affirm that all Fourier coefficients of the (complex Borel) measure $h\bar{g}\,\mathrm{d}\mu$ vanish. Hence this measure is 0. That is,

$$|h\,\bar{g}| = |h||g| = 0 \quad \mu\text{-a.e.} \tag{10}$$

As noted in Eq. (5)

$$\int_{\mathbb{T}} z^k |h|^2 \,\mathrm{d}\mu(z) = \langle \mathrm{U}^k h, h\rangle = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \in \mathbb{Z} \setminus \{0\}, \end{cases}$$

that is, the measure $|h|^2\mathrm{d}\mu$ has exactly the same Fourier coefficients as λ, so

$$|h|^2\mathrm{d}\mu = \mathrm{d}\lambda. \tag{11}$$

Similarly, $|g|^2\mathrm{d}\mu = \mathrm{d}\lambda$, which with Eq. (11) gives $|h|^3\mathrm{d}\mu = |h|\mathrm{d}\lambda = |h||g|^2\mathrm{d}\mu{=}0$, by (10). That is, $|h| = 0$ μ-a.e., contrary to h being of norm 1 in $L^2(\mu)$. This contradiction confirms (9). That is, $M \ominus \mathrm{U}M = \mathbb{C}h$ and then by Eq. (8)

$$L^2(\mu)\text{-closure of span } \{\mathrm{U}^n h : n \in \mathbb{N}_0\} = M.$$

In particular, since $\mathrm{U}f = zf \in M$, we see that zf lies in the $L^2(\mu)$-closure of $\mathrm{span}\{\mathrm{U}^n h = z^n h : n \in \mathbb{N}_0\}$. It follows that for each $k \in \mathbb{Z}$, $z^k f$ lies in the $L^2(\mu)$-closure of $\mathrm{span}\{z^n h : n \in \mathbb{Z}\}$. That is,

$$L^2(\mu)\text{-closure of span } \{z^k f : k \in \mathbb{Z}\} \subset L^2(\mu)\text{-closure of span } \{z^n h : n \in \mathbb{Z}\}. \tag{12}$$

Equality (3) forces equality in (12), which then clearly entails that

$$f\,\mathrm{d}\mu \ll h\,\mathrm{d}\mu \ll f\,\mathrm{d}\mu,$$

and thanks to Eq. (11)

$$\mathrm{d}\lambda \ll |h|\,\mathrm{d}\mu \ll \mathrm{d}\lambda.$$

Thus $\mathrm{d}\lambda$ and $f\,\mathrm{d}\mu$ (which is $\mathrm{d}\nu$) are mutually absolutely continuous.

Corollary 1 (Szegö). *Let σ be a Borel probability measure on $\mathbb{T}$ that annihilates some set of positive Lebesgue measure. Then the functions z^n, $n \in \mathbb{N}$, span $L^2(\sigma)$.*

Proof. Denote by M the $L^2(\sigma)$-closure of $\mathrm{span}\{z^n : n \in \mathbb{N}\}$ and assume $M \neq L^2(\sigma)$. There is then a non-zero function $g \in L^2(\sigma)$ orthogonal to M:

$$0 = \langle g, z^n\rangle_{L^2(\sigma)} = \int_{\mathbb{T}} \overline{z^n}\, g \,\mathrm{d}\sigma \quad \forall n \in \mathbb{N}.$$

This says that $\bar{g}\mathrm{d}\sigma$ is a non-zero analytic measure, hence Lebesgue measure λ is absolutely continuous with respect to $\bar{g}\mathrm{d}\sigma$. By hypothesis, some Borel set B with

$\lambda(B) > 0$ has $\sigma(B) = 0$. The latter equality implies that $\bar{g}\mathrm{d}\sigma$ also gives measure 0 to B, contradicting $\mathrm{d}\lambda \ll \bar{g}\mathrm{d}\sigma$.

Remark 1. The usual formulation of Szegö's theorem is more quantitative and more general: If h denotes the Radon–Nikodym derivative with respect to λ of the absolutely continuous part of σ, then the distance in $L^2(\sigma)$ from 1 to span $\{z^n : n \in \mathbb{N}\}$ is $\exp(\int \log h \, \mathrm{d}\lambda)$. In the notation of the proof just given, h would be 0 on B, so $\int \log h \, \mathrm{d}\lambda = -\infty$ and the exponential of it is 0, which puts 1 into the $L^2(\sigma)$-closure of span$\{z^n : n \in \mathbb{N}\}$, from which the above corollary follows easily.

Remark 2. On the other hand, a weaker version of the corollary was proved by Holland [1]. He started with a Borel probability measure σ on $\mathbb{T}$ which is singular with respect to Lebesgue measure λ and by a very clever, explicit and elementary construction he manufactured a sequence of polynomials

$$P_n(z) = \sum_{k=1}^{n} A_k z^k \quad (n \in \mathbb{N})$$

such that

$$\sum_{k=1}^{\infty} |A_k|^2 = 1$$

and

$$\int_{\mathbb{T}} |1 - P_n|^2 \, \mathrm{d}\sigma = 1 - \sum_{k=1}^{n} |A_k|^2 \quad (n \in \mathbb{N}).$$

In fact, the A_k are the Taylor coefficients of the holomorphic function

$$\frac{F(z) - 1}{F(z) + 1}, \text{ where } F(z) := \int_{\mathbb{T}} \frac{u + z}{u - z} \, \mathrm{d}\sigma(u) \quad (z \in \mathbb{D}).$$

Mirabile dictu.

Remark 3. Also the half of the F. and M. Riesz theorem asserting that $\nu \ll \lambda$ was given a remarkable one-page function-theoretic proof by Øksendal [2]. A complex-valued Borel measure ν on $\mathbb{T}$ satisfying (A) is given and what has to be shown is that $\nu(K) = 0$ for every λ-null Borel set K. Because Borel measures on $\mathbb{T}$ are inner regular, it suffices to consider compact K. Clearly it suffices to show this for the modified measure $\nu_0 := \nu - \nu(\mathbb{T})\lambda$. This measure is also analytic but in addition annihilates 1. That is,

$$\widehat{\nu_0}(-n) = \int_{\mathbb{T}} z^n \, \mathrm{d}\nu_0(z) = 0 \quad \forall n \in \mathbb{N}_0. \tag{A$_0$}$$

For each $n \in \mathbb{N}$, an $N \in \mathbb{N}$, $z_j \in K$ and $\rho_j > 0$ are chosen appropriately and the rational functions

$$g_n(z) := 1 - \prod_{j=1}^{N} \frac{z - z_j}{z - (1 + \rho_j)z_j}$$

are introduced. They are bounded by 2 on $\mathbb{T}$ and are shown to converge there to the indicator function of K as $n \to \infty$. Since g_n is holomorphic in a neighborhood of $\overline{\mathbb{D}}$, the partial sums of its Taylor series at 0 approximate it uniformly on $\mathbb{T}$ and each sum has ν_0-integral 0 thanks to $(\mathrm{A})_0$. Consequently, $\int_{\mathbb{T}} g_n \, d\nu_0 = 0$. It follows then from the dominated convergence theorem that $\nu_0(K) = \lim \int g_n \, d\nu_0 = 0$, as wanted.

Acknowledgments In February of 2010 this proof was kindly communicated to me by Donald Sarason when I sought his insight on Holland [1]. He informed me that a version of it had been discovered by David Lowdenslager sometime between 1959 and 1963, and independently by himself while a graduate student of Paul Halmos' at the University of Michigan about 1962. Apparently neither was ever published, but some of Don's ideas appear in Sarason [4]. He has graciously agreed to my presenting his work here. It is an honor to be the purveyor of such elegant mathematics. I extend deepest thanks to John Benedetto for inviting me to speak to such a distinguished group.

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Further Reading

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