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Yaakov Friedman
with the assistance of Tzvi Scarr

Physical Applications of Homogeneous Balls

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To my wife Rachel

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Preface

Based on this century of experience, it is generally supposed that a final theory will rest on principles of symmetry.

Dreams of a Final Theory
Steven Weinberg
1979 Nobel Prize winner

This book introduces *homogeneous balls* as a new mathematical model for several areas of physics. It is widely known that the set of all relativistically admissible velocities is a ball in R^3 of radius c, the speed of light. It is also well known that the state space of a quantum system can be represented by positive trace-class operators on a Hilbert space that belong to the unit ball in the trace norm. In relativistic quantum mechanics, the Dirac bispinors belong to a ball in the space C^4. Is there something in common among these balls? At first glance, they look very different. Certainly, they do not represent commutative objects, for which the unit ball is a simplex. They cannot represent a binary algebraic operation, since for such an operation, we need an order on the space, and there is no order for the first and third examples. But as we will show, in all of the above situations, either the ball in question or its dual is *homogeneous*. Moreover, there is a triple structure which is uniquely constructed from either the homogeneity of the domain or the geometry of the dual ball.

Homogeneous balls could serve as a unifying language for different areas in physics. For instance, both the ball of relativistically admissible velocities in Special Relativity and the unit ball of operators on a Hilbert space, which is the dual of the state space in Quantum Mechanics, are homogeneous balls. In Special Relativity, the homogeneity of the velocity ball is an expression of the principle of Special Relativity and not artificially imposed. The surprising fact that the unit ball of the space of operators is a homogeneous ball was first discovered and utilized in solving the engineering problems involved in transatlantic telephone communication. But not much has been done in physics to take advantage of this structure.

In addition, some aspects of General Relativity may be described by the methods presented in the book. But in order to describe General Relativity efficiently, our model must be generalized. This generalization is obtained by weakening slightly one of the axioms of the triple product algebraic structure. At this stage, the generalized model needs to be developed further before it will be ready for applications.

Recall the definitions of a *homogeneous ball* and a *symmetric domain.* Let D be a domain in a real or complex Banach space. We denote by $Aut(D)$ the collection of all automorphisms (one-to-one smooth maps) of D. The exact meaning of "smooth" will vary with the context, but it will always mean either projective (preserving linear segments), conformal (preserving angles) or complex analytic. The unit ball D in a Banach space is one example of a bounded domain. It is called *homogeneous* if for any two points $z, w \in D$, there is an automorphism $\varphi \in Aut(D)$ such that $\varphi(z) = w$. A domain D is called *symmetric* if for any element $a \in D$, there is a symmetry $s_a \in Aut(D)$ which fixes only the point a. Any bounded symmetric domain can be realized as a homogeneous ball in a Banach space.

The theory of bounded symmetric domains as mathematical objects in their own right is highly developed (see [52], [62], [68] and [69]). However, these works are written at a high mathematical level and contain no physical applications.

The current text completely changes this situation. Not only do we develop the theory of homogeneous balls and bounded symmetric domains and their algebraic structure *informally*, but also our primary goal is to show how to construct an appropriate domain to model a given law of physics. The research physicist, and even the graduate student, can walk away with both an understanding of these domains *and* the ability to construct his own homogeneous balls. After seeing our new methodology applied to Special Relativity and Quantum Mechanics, the reader should be able to extrapolate our techniques to his own areas of interest.

In Chapter 1, we show how the principle of relativity leads to a symmetry on the space-time continuum. From this symmetry alone, we derive the Lorentz transformations and show that the set D_v of all relativistically admissible velocities is a homogeneous ball and a bounded symmetric domain with respect to the group $Aut_p(D_v)$ of projective automorphisms. We derive the formula for Einstein velocity addition and explore its geometric properties. We study the Lie algebra $aut_p(D_v)$ and show that relativistic dynamics is described by elements of this algebra. This observation provides an efficient tool for solving relativistic dynamic equations, regardless of initial conditions. As an example, we obtain explicit solutions for the relativistic evolution equation for a charged particle in an electric field E, a magnetic field B and an electromagnetic field E, B in which E and B are parallel.

In Chapter 2, we show that the ball D_s of all relativistically admissible *symmetric* velocities is a bounded symmetric domain with respect to the group $Aut_c(D_s)$ of conformal automorphisms and is a Cartan factor of type 4, called the *spin factor.* This enables us to express the non-commutativity and the non-associativity of Einstein velocity addition as well as the non-transitivity of parallelism among inertial frames in Special Relativity. The

Lie algebra $aut_c(D_s)$ is described in terms of the spin triple product. We describe relativistic evolution using elements of $aut_c(D_s)$. Utilizing the fact that the evolution equation for symmetric velocities of a charged particle in a constant, uniform electromagnetic field E, B, with $E \cdot B = 0$, becomes a one-dimensional complex analytic differential equation, we obtain *explicit* solutions for this evolution.

In Chapter 3, we study the complex spin factor, which is the complex extension of the conformal ball from the previous chapter. The natural basis in this space satisfies a triple product analog of the Canonical Anticommutation Relations. We derive a spectral decomposition for elements of this factor and then represent it geometrically. The two types of tripotents (building blocks of the triple product) determine a duality on this object. This duality is crucial in obtaining different representations of the Lorentz group on the spin factor. The three-dimensional complex spin factor efficiently represents the electromagnetic field, and the Lorentz group acts on it by linear operators defined directly by the triple product. We show that the properties of the field are related to the algebraic structure of its representation.

The four-dimensional complex spin factor admits several representations of the Lorentz group. The operators representing the generators of this group belong to a spin factor of dimension 6. If we use the representation provided by one type of tripotents, we obtain the usual representation of this group on four-vectors. These four-vectors form the invariant subspaces of the spin factor under this representation. If we switch the representation to the second type, the invariant subspaces are the Dirac bispinors with the proper action of the Lorentz group on them. This reveals the connection between the spin 1 and the spin 1/2 representations.

In Chapter 4, we study classical homogeneous unit balls of subspaces of operators on a Hilbert space. Since these operators are not necessarily self-adjoint, we first study some relevant results for non–self-adjoint operators. Based on ideas from transmission line theory, we show that such a ball is a symmetric domain with respect to the analytic automorphisms. Here we study the connection between the geometric properties of such domains and their JC^*-triple structure.

Chapter 5 consists of general results about homogeneous unit balls, bounded symmetric domains, and the Jordan triple product associated with them. Since these domains are homogeneous with respect to the analytic maps on a complex Banach space, we introduce and study some properties of such maps. From the study of the Lie group of analytic automorphisms of a bounded domain and its Lie algebra, we derive the Jordan triple product associated to the domain. We study the Peirce decomposition (which occurs also in earlier chapters) on JB^*-triples and their duals. We explore how the geometry inherited by the state space from the measuring process allows one to define grid bases on the set of observables. This justifies the use of homogeneous balls and bounded symmetric domains in modeling quantum mechanical phenomena.

Chapter 6 includes a complete classification of atomic JB^*-triples. We show how to build convenient bases, called grids, for such spaces. These grids are constructed from basic elements of the triple structure which may be interpreted as compatible observables. These grids span the full non-commutative object. Our methodology reveals why there are six different fundamental domains (called factors) for the same algebraic structure. This explains how apparently unrelated models in physics, corresponding to different types of factors, can have common roots. Furthermore, the mystery of the occurrence of two exceptional factors of dimensions 16 and 27 is explained.

In this book, the reader will find the answer to the following questions:

1. Does the principle of relativity imply the existence of an invariant speed and the preservation of an interval? (Answer: Section 1.2)
2. Why is there time contraction in the transformations from inertial system K to K', while space contraction is obtained in the transformations from K' to K? (Answer: Section 1.1.4)
3. The relative velocity between two inertial systems can be considered as a linear map between time displacement and space displacement. What is the adjoint of this map? (Answer: end of Section 1.2)
4. What geometry is preserved in the transformation of the ball of relativistically admissible velocities from one inertial system to another? (Answer: Section 1.4)
5. What is the connection between the relativistic dynamic equation and the Lie algebra of the velocity transformations? (Answer: Section 1.5)
6. If one has found a solution to the relativistic dynamic equation with a given initial condition, how can one obtain a solution which satisfies a different initial condition? (Answer: Section 1.5.5)
7. The relativistic evolution equation in the plane is not analytic. How can it be made analytic? What are the analytic solutions for a constant field in this case? (Answer: Sections 2.5 and 2.6)
8. How are the Canonical Anticommutation Relations related to the basis in a spin factor? (Answer: Section 3.1.2)
9. How can n Canonical Anticommutation Relations be represented in a space of dimension $2n$ (and not the usual space of dimension 2^n)? (Answer: Section 3.1.2)
10. What is the group of automorphisms of the spin factor? (Answer: Section 3.1.3)
11. Why, in quantum mechanics, do we use expressions like $a = \hat{x} + i\hat{p}_x$ and $J_+ = J_x + iJ_y$? (Answer: Section 3.3.7)
12. How can one represent the transformations of the electromagnetic field strength as operators of the triple product in the spin factor? (Answer: Section 3.5.4)
13. How can one represent four-vectors and Dirac bispinors on the same object? What is the relationship between these two representations? (Answer: Sections 3.5 and 3.6)

14. How can non–self-adjoint operators produce real numbers (similar to the eigenvalues of self-adjoint operators) and filtering projections? (Answer: Section 4.1)
15. Most balls of spaces of operators on a Hilbert space are homogeneous with respect to analytic maps. How can signal transformations in a lossless transmission line be used to demonstrate this homogeneity? (Answer: Section 4.2)
16. How can one derive an algebraic product from the geometry of a bounded homogeneous domain? (Answer: Section 5.3.5)
17. What is the algebraic non-commutative structure built on geometry alone? (Answer: Sections 5.3.5 and 5.4)
18. Can the homogeneity of the ball of observables be derived from the geometry of the state space induced by the measuring process? (Answer: section 5.7)
19. Why are there exactly six different types of bounded symmetric domains, or equivalently, JB^*-triple factors? (Answer: Section 6.3.1)
20. What is the principle difference between the spin domain and the domains in spaces of operators? (Answer: Section 6.3)
21. What is the bridge between the classical and the exceptional domains? (Answer: Sections 6.2.3 and 6.3.7)

For the most part, this book represents the results of more than 30 years of the author's research. During this period, the theory of homogeneous, bounded and unbounded symmetric domains has progressed significantly. We do not cover here all major topics of this area, but concentrate more on the aspects that seem currently ripe for physical applications.

I want to thank my research collaborators: Jonathan Arazy, with whom we started this project during our Ph.D. program, Bernard Russo with whom we worked together for more than 20 years, Thomas Barton, Truong Dang, Ari Naimark and Yuriy Gofman. Tzvi Scarr assisted me in writing this book. I want to thank Alexander Friedman and Hadar Crown for technical assistance. I want to thank Uziel Sandler, Mark Semon and Alex Gelman for helpful comments. This work was supported in part by a research grant from the Jerusalem College of Technology.

The book is dedicated to my wife Rachel, for without her encouragement over the last 30 years, I would not have been able to achieve the results presented in the book.

Jerusalem College of Technology

Yaakov Friedman
June 2004

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1 Relativity based on symmetry

In this chapter, we will derive the Lorentz transformations without assuming the constancy of the speed of light. We will use only the principle of special relativity and the symmetry associated with it. We will see that this principle allows only Galilean or Lorentz space-time transformations between two inertial systems. In the case of the Lorentz transformations, we obtain the conservation of an interval and a certain speed. From known experiments, this speed is c, the speed of light in a vacuum.

The Einstein velocity-addition formula is also obtained. From this, it follows that the ball of all relativistically admissible velocities is a bounded symmetric domain. The Lie algebra of the automorphism group of this domain consists of the generators of boosts and rotations. The relativistic dynamics and the dynamics of a charged particle in an electromagnetic field are given by elements of this Lie algebra.

Our methodology in special relativity is outlined in the following steps, which we apply to two inertial systems:

Step 1 Choice of the parameters for the purpose of obtaining simpler (ideally, *linear*) transformations between the two systems
Step 2 Identification of *symmetry* in the basic principle of the area of application (in this case, the principle of special relativity)
Step 3 Choice of reference frames which preserve the symmetries
Step 4 Choice of inputs and outputs which reflect the description of the system
Step 5 Derivation of the explicit form of the symmetry operator
Step 6 Identification of invariants
Step 7 Construction of an appropriate (bounded) symmetric domain for the area of application
Step 8 Derivation of the equation of evolution based on the algebraic structure of the Lie algebra of the domain

1.1 Space-time transformation based on relativity

In this section, we derive the space-time transformation between two inertial systems, using only the isotropy of space and symmetry, both of which follow from the principle of relativity. The transformation will be defined uniquely up to a constant e, which depends only on the process of synchronization of clocks inside each system. If $e = 0$, the transformations reduce to Galilean.

1.1.1 Step 1 - Choice of the Parameters

We begin with two "systems", for example, an airplane flying at 30,000 feet and an observer standing on the ground. We assume that the airplane is flying in uniform motion with constant velocity, that there is no turbulence, etc. Passengers in the airplane feel themselves at rest. When they put their cup of coffee on the fold-down tray in front of them, it doesn't slide or move. When they drop a penny, it falls "straight" to the floor. This is a manifestation of the principle of special relativity, which states that (Pauli [59], page 4) "there exists a triply infinite set of reference systems moving rectilinearly and uniformly relative to one another, in which the phenomena occur in an identical manner."

Typically, there are events which are observable from both systems. Lake Michigan can be observed by both Observer A who is standing on its edge and also by Observer B who is flying over it (Figure 1.1). Certainly, Observers A and B will not observe Lake Michigan in the same way. To Observer A, Lake Michigan is next to him and standing still, while to Observer B, it is below him and moving. Each observer sets up a system of axes and scales in order to measure the position in space and time of each event. Imagine a kingfisher flying above the lake. It swoops down, snatches a fish from the lake, and takes off again. Let's take the snatch as our event. Each observer has a different set of four numbers to describe the location of this event in space-time. The connection between these two sets of four numbers is the *space-time transformation* between the two systems.

Why have we chosen space and time as the parameters with which to describe events? Why not velocity and time? Why not position and momentum? Why is the space-time description more convenient for transformations between inertial systems?

The advantage is *linearity*. Newton's First Law states that an object moves with constant velocity if there are no forces acting on it or if the sum of all forces on it is zero. Such a motion is called *free motion* and is described by straight lines in the space-time continuum, as shown in Figure 1.2. Conversely, any line (except lines with constant t) in the space-time continuum represents free motion. A system is called an *inertial system* if an object moves with constant velocity when there are no forces acting on it. By the definition of an inertial system, free motion will be observed as free motion in *any* inertial system. This means that the space-time transformations will map lines to lines. Thus the space-time description of events leads to *linear* transformations. For an example of how the choice of parameters with which to describe events affects the linearity of the transformations, see Figure 1.3.

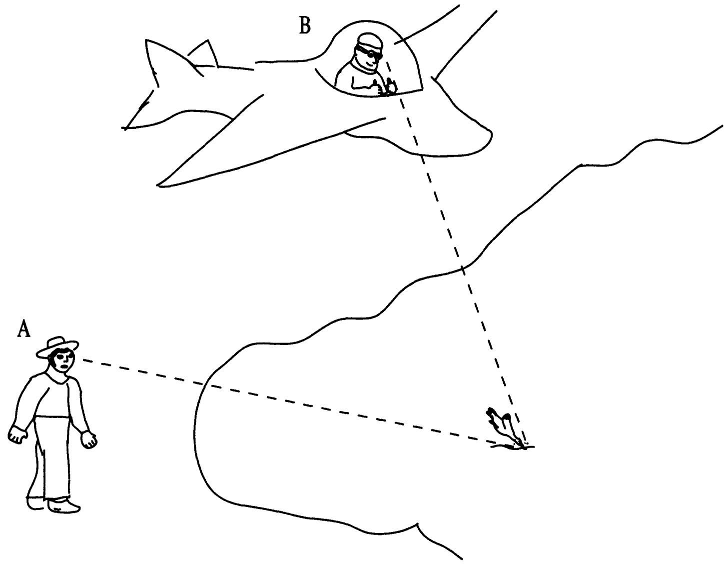

Fig. 1.1. The space-time transformation between two systems is the connection between the space-time coordinates of the same event (snatches of a fish) observed and described by two observers in the two systems. Above, Observer A is standing (at rest) on the edge of Lake Michigan, and Observer B is flying over it with constant velocity.

We restrict ourself to inertial systems with the same space origin at time $t = 0$. By a well-known theorem in mathematics, a transformation between

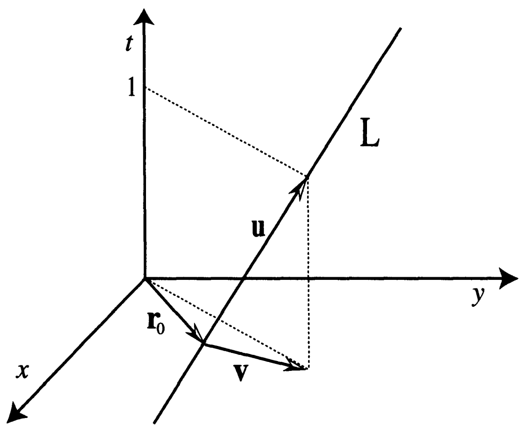

Fig. 1.2. Lines in space-time and free motion. Two space coordinates, x and y, and time are displayed. The line L intersects the plane $t = 0$ at $\mathbf{r}_0$, the position of an object at time $t = 0$. The direction of L is given by a vector $\mathbf{u} = (1, \mathbf{v})$, where $\mathbf{v}$ is the constant velocity of the object. The line $L = \{(t, \mathbf{r}_0 + \mathbf{v}t) : t \in R\}$ is the world-line of this free motion.

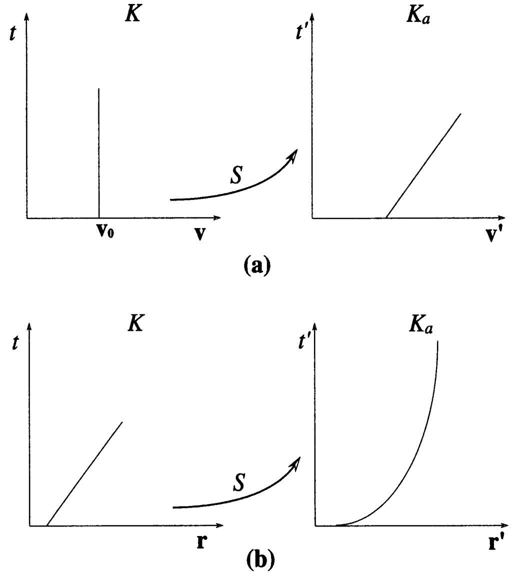

Fig. 1.3. Two descriptions of free motion for an inertial system K and a system K_a whose acceleration with respect to K is $\mathbf{a}$. (a) In the velocity-time description, the constant velocity $\mathbf{v}_0$ in K is represented by a line $L = \{(t, \mathbf{v}_0) : t \in R\}$ in K and also by a line $L = \{(t, \mathbf{v}_0 - \mathbf{a}t) : t \in R\}$ in K_a. (b) In the space-time description, the constant velocity $\mathbf{v}_0$ in K is represented by a line $L = \{(t, \mathbf{r}_0 + \mathbf{v}_0 t) : t \in R\}$ in K, while in K_a, it is represented by a parabola $(t, \mathbf{r}_0 + \mathbf{v}_0 t - 0.5\mathbf{a}t^2) : t \in R\}$. Hence, the space-time transformation between K and K_a cannot be linear.

two vector spaces which maps lines to lines and the origin to the origin is linear. Thus, the space-time transformation between our two systems is a *linear* map. After choosing space axes in each system, we can represent this transformation by a matrix.

1.1.2 Step 2 - Identification of symmetry inherent in principle of special relativity

Albert Einstein formulated the *principle of special relativity* ([21], p.25): "If K is an inertial system, then every other system K' which moves uniformly and without rotation relatively to K, is also an inertial system; the laws of nature are in concordance for all inertial systems." Observation of the same event from these two systems defines the space-time transformation between the systems. By the principle of special relativity, this transformation will depend only on the choice of the space axes, the measuring devices (consisting of rods and clocks) and the relative position in time between these systems. The relative position in time between two inertial systems is described by their *relative velocity.* We denote by $\mathbf{v}$ the relative velocity of K' with respect to K and by $\mathbf{v}'$ the relative velocity of K with respect to K'. If we choose the measuring devices in each system to be the same and choose the axes in such a way that the coordinates of $\mathbf{v}$ are equal to the coordinates of $\mathbf{v}'$, then the space-time transformation S from K to K' will be equal to the space-time transformation S' from K' to K. Since, in general, $S' = S^{-1}$, in this case we will have $S^2 = I$. Such an operator S is called a *symmetry.* Thus, the principle of special relativity implies that with an appropriate choice of axes and measuring devices, the space-time transformation S between two inertial systems is a symmetry.

1.1.3 Step 3 - Choice of reference frames

Following Einstein, the space axes in special relativity are chosen as in Figure 1.4. If we assume that the interval $ds^2 = (cdt)^2 - d\mathbf{r}^2$ is conserved, the

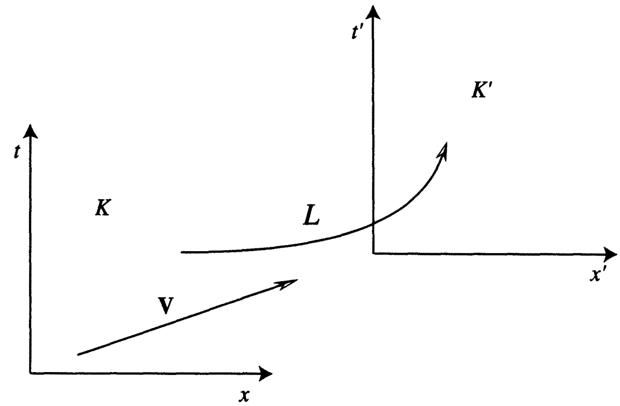

Fig. 1.4. The usual Lorentz space-time transformations between two inertial systems K and K', moving with relative velocity $\mathbf{v}$. The space axes are chosen to be parallel. The Lorentz transformation L transforms the space-time coordinates $(t, \mathbf{r})$ in K of an event to the space-time coordinates $(t', \mathbf{r}')$ in K' of the same event.

resulting space-time transformation between systems is called the *Lorentz transformation.* In the case $\mathbf{v} = (v, 0, 0)$, the Lorentz transformation L is given by

$$\begin{aligned} t' &= \tfrac{1}{\sqrt{1-v^2/c^2}}\left(t - \tfrac{vx}{c^2}\right), \\ x' &= \tfrac{1}{\sqrt{1-v^2/c^2}}(x - vt), \\ y' &= y, \\ z' &= z. \end{aligned} \tag{1.1}$$

Note that the assumption that the system K' is moving with velocity $\mathbf{v}$ with respect to the system K implies that system K is moving with velocity $-\mathbf{v}$ with respect to K'. And this apparently minor lack of symmetry means that the Lorentz transformation L' from system K' to system K will be different from L. In fact, we have

$$\begin{aligned} t &= \tfrac{1}{\sqrt{1-v^2/c^2}}(t' + \tfrac{vx'}{c^2}), \\ x &= \tfrac{1}{\sqrt{1-v^2/c^2}}(x' + vt'), \\ y &= y', \\ z &= z'. \end{aligned} \tag{1.2}$$

We would like to arrange things so that the two transformations L and L' are the *same*! It certainly would help if K were *also* moving with velocity $\mathbf{v}$ with respect to system K'.

We will synchronize the two systems by observing events from each system and comparing the results. System 1 begins with the following configuration. There is a set of three mutually orthogonal space axes and a system of rods. In this way, each point in space is associated with a unique vector in R^3. In addition, there is a clock at each point in space, and all of the clocks are synchronized to each other by some synchronization procedure. System 2 has the same setup, only we do not assume that the rods of system 1 are identical to the rods of system 2, nor do we assume that the clock synchronization procedure in system 2 is the same as that of system 1.

First, we synchronize the origins of the frames. Produce an event E_0 at the origin O of system 1 at time $t = 0$ on the system 1 clock positioned at O. This event is observed at some point O' in system 2, and the system 2 clock at O' shows some value $t' = t'_0$. Translate the origin of system 2 to the point O' (without rotating). Subtract t'_0 from the system 2 clock at O'. Synchronize all of the system 2 clocks to this clock. This completes the synchronization of the origins.

Next, we will adjust the x-axis of each system. Note that system 2 is moving with some (perhaps unknown) constant velocity $\mathbf{v}$ with respect to system 1 and that the origin O' of system 2 was at the point O of system 1 at time $t = 0$. Therefore, the point O' will always be on the line $\mathbf{v}t$ in system 1. Rotate the axes in system 1 so that the new negative x-axis coincides with the ray $\{\mathbf{v}t : t > 0\}$. Similarly, system 1 is moving with some constant velocity $\mathbf{w}$ with respect to system 2, and the origin O of system 1 was at the point O' of system 2 at time $t' = 0$. Therefore, the point O will always be on the line $\mathbf{w}t$ in system 2. Rotate the axes in system 2 so that the new negative x'-axis coincides with the ray $\{\mathbf{w}t : t > 0\}$. The two x-axes now coincide as lines and point in opposite directions. We are finished manipulating the axes and clocks of system 1 and will henceforth refer to system 1 as the inertial frame K. However, it still remains to manipulate system 2, as we must adjust the y'- and z'-axes of system 2 to be parallel and oppositely oriented to the corresponding axes of K.

To adjust the y'-axis of system 2, produce an event E_1 at the point $\mathbf{r} = (0, 1, 0)$ of K. This event is observed in system 2 at some point $\mathbf{r}'$. Rotate the space axes of system 2 around the x'-axis so that $\mathbf{r}'$ will lie in the new

x'-y' plane and have a negative y' coordinate y'_1. After this rotation, the z-axis of K and the z'-axis of system 2 will be parallel. We need to make sure that they have opposite orientations. Produce an event E_2 at the point $\mathbf{r} = (0,0,1)$ of K. This event is observed in system 2 at some point $\mathbf{r}'$. If the z' coordinate of $\mathbf{r}'$ is positive, reverse the direction of the z'-axis. This completes the adjustment of the space axes of the two systems. See Figure 1.5.

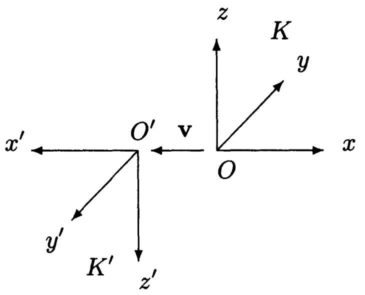

Fig. 1.5. Two symmetric space reference frames. The relative velocity of the inertial system K' with respect to K is $\mathbf{v}$. The coordinates of $\mathbf{v}$ in K are equal to the coordinates (in K') of the relative velocity of the system K with respect to K'.

It remains to redefine the space and time units of system 2 to match those of K. The new space unit of system 2 is defined to be y'_1 times the previous space unit. In order to adjust the time unit of system 2, we will measure the speed $|\mathbf{v}|$ of system 2 with respect to K and the speed $|\mathbf{v}'|$ of K with respect to system 2. To calculate $|\mathbf{v}|$, produce an event E_3 at O' at any time $t' > 0$. This event is observed in K at some point $\mathbf{r} = (x_0, 0, 0)$, and the clock at this point shows time t_0. The relative speed of system 2 with respect to K is $|\mathbf{v}| = |x_0|/t_0$. The calculation of $|\mathbf{v}'|$ is symmetric. Produce an event E_4 at O at any time $t > 0$. This event is observed in system 2 at some point $\mathbf{r}' = (x'_0, 0, 0)$, and the clock at this point shows time t'_0. The relative speed of K with respect to system 2 is $|\mathbf{v}'| = |x'_0|/t'_0$. Finally, the time unit in system 2 is chosen as $|\mathbf{v}'|/|\mathbf{v}|$ times the previous unit. With this choice of units, the speeds $|\mathbf{v}|$ and $|\mathbf{v}'|$ are equal. System 2 will henceforth be called K'.

The transformations from system K to system K' will now be mathematically *identical* to the transformations from system K' to system K. In other words, the space-time transformation S from system K to system K' will be a symmetry operator.

The space-time coordinates of K and K' will be denoted $\begin{pmatrix} t \\ \mathbf{r} \end{pmatrix}$ and $\begin{pmatrix} t' \\ \mathbf{r}' \end{pmatrix}$, respectively. These coordinates will be considered as a 4×1 matrix. By the above synchronization procedure, the frames have the same origin and the two clocks at each origin are synchronized at time $t = 0$. Moreover, the space axes are reversed as in Figure 1.5 . Note that with this choice of axes, the velocity coordinates of O' in K are equal to the velocity coordinates of O in K'. Thus, the transformation is fully symmetric with respect to K and K' (see Figure 1.6). We will denote the space-time transformation from K to K'

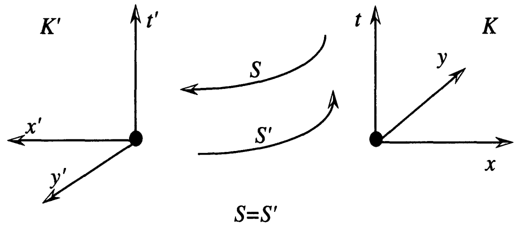

Fig. 1.6. The time t and two space axes x and y of systems K and K' are displayed. With our choice of the axes, the space-time map S from K to K' is identical to the space-time map S' from K' to K, and, thus, S is a symmetry.

by $S_{\mathbf{v}}$, since it is a symmetry and depends only on the velocity $\mathbf{v}$ between the systems.

1.1.4 Step 4 - Choice of inputs and outputs

The space-time transformation between two inertial systems can be considered as a "two-port linear black box" transformation with two inputs and two outputs. There are two ways to define the inputs and outputs for such a transformation.

Cascade connection

The first one, called the *cascade connection*, takes time and space of one of the systems, say $\begin{pmatrix} t \\ \mathbf{r} \end{pmatrix}$ of K, as input, and gives time and space of the second system, say $\begin{pmatrix} t' \\ \mathbf{r}' \end{pmatrix}$ of K', as output (see Figure 1.7) [1]

The cascade connection is the one usually used in special relativity.

We represent the linear transformation induced by the cascade connection by a 4×4 matrix E, which we decompose into four block matrix components E_{ij}, as follows:

$$\begin{pmatrix} t' \\ \mathbf{r}' \end{pmatrix} = E \begin{pmatrix} t \\ \mathbf{r} \end{pmatrix} = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} \begin{pmatrix} t \\ \mathbf{r} \end{pmatrix}. \tag{1.3}$$

[1] We use a circle instead of the usual box notation in order that the connection between any two ports will be displayed *inside* the box (see Figure 1.8).

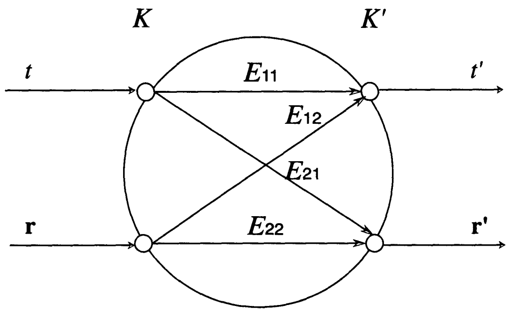

Fig. 1.7. The cascade connection for space-time transformations. The circle represents a black box. One side has two input ports: the time t and the space $\mathbf{r}$ coordinates of an event in system K. The other side has two output ports: the time t' and the space $\mathbf{r}'$ coordinates of the same event in system K'. The linear operators E_{ij} represent the functional connections between the corresponding ports.

To understand the meaning of the blocks, assume that the system K is the airplane. Let t be the time between two events (say crossing two lighthouses) measured by a clock at rest at $\mathbf{r} = 0$ on the airplane. The time difference t' of the same two events measured by synchronized clocks at the two lighthouses (in system K', the earth) will be equal to $t' = E_{11}t$. If we denote the distance between the lighthouses by $\mathbf{r}'$, then $\mathbf{r}' = E_{21}t$, and E_{21} is the so-called *proper velocity* of the plane. Generally, the proper velocity $\mathbf{u}$ of an object (the airplane) in an inertial system is the ratio of the space displacement $d\mathbf{r}$ in this system (the earth) divided by the time interval, called the proper time interval $d\tau$, measured by the clock of the object (on the plane). Thus,

$$\mathbf{u} = \frac{d\mathbf{r}}{d\tau}. \tag{1.4}$$

Hybrid connection

The second type of connection, called the *hybrid connection*, uses time of one of the systems, say t of K, and the space coordinates $\mathbf{r}'$ of the second system K', as input, and gives $\begin{pmatrix} t' \\ \mathbf{r} \end{pmatrix}$ as output (see Figure 1.8). Usually we use relative *velocity* (not relative proper velocity) to describe the relative position between inertial systems. To define the relative position of system K' with respect to K, we consider an event that occurs at O', corresponding to $\mathbf{r}' = 0$, at time t, and express its position $\mathbf{r}$ in K. If we denote by $\mathbf{v}$ the uniform velocity of system K' with respect to K, then

$$\mathbf{r} = \mathbf{v}t. \tag{1.5}$$

Note our use of the hybrid connection. In this section we will use the hybrid connection in order to be consistent with the description of relative position