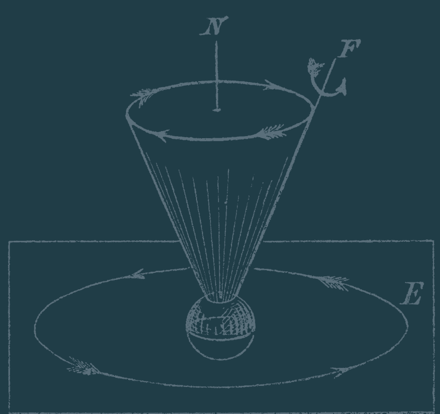
Felix Klein
Arnold Sommerfeld

The Theory of the Top
Volume III

Perturbations. Astronomical and Geophysical Applications



Translated by
Raymond J. Nagem
Guido Sandri

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Foreword to Volume III by Michael Eckert

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Translators:
Raymond J. Nagem
Department of Mechanical Engineering
Boston University
Boston, MA, USA
nagem@bu.edu

Guido Sandri
Department of Mechanical Engineering
Boston University
Boston, MA, USA
sandri@bu.edu

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Foreword

When the second volume of the *Theory of the Top* was published in 1898, the mathematical foundation of the work was complete. Sommerfeld's text had already reached the size of a treatise of 512 pages—far more than Klein had initially expected. Applications were reserved for a concluding third volume, as both Klein and Sommerfeld had originally planned. Their expectations, however, were defied by the contingencies of life. Five years lapsed before Sommerfeld finished the third volume, and seven more years would pass before the appearance of the fourth and final volume.

A year after the publication of the second volume, Klein expressed in a letter to Sommerfeld a "quiet concern" about the pace of progress with "our top" [Klein 1899], which largely became Sommerfeld's top as the projected content expanded far beyond the bounds of Klein's original lectures. But Sommerfeld's pending move from Clausthal to Aachen, where he was called as professor of mechanics, left little room for additional work. Furthermore, Klein had involved Sommerfeld by this time with the editing of the physics volumes of the *Enzyklopädie der mathematischen Wissenschaften*, a long-term effort that would occupy Sommerfeld for almost three decades. Progress with Volume III of the *Theory of the Top* was confined to the gathering of pertinent material and correspondence with colleagues who were involved with practical applications. The initial momentum was waning. Occasional meetings with Klein and the publisher, however, roused Sommerfeld's sense of duty. "The top makes progress," he assured Klein early in 1902, "even though with interruptions" [Sommerfeld 1902a]. In June of 1902, he promised more progress for the holidays after the summer semester [Sommerfeld 1902b].

The third volume was intended to confront theory with practice, to account for deviations (perturbations or "disturbing influences," as they were called in the subtitle of this volume) from the abstract theory of Volumes I and II. As professor of mechanics at the *Technische Hochschule* in Aachen, the gap between theory and practice was an obvious concern for Sommerfeld. While gathering relevant material for the theory of the top, he published, for example, an article on the theory of railway brakes [Sommerfeld 1902c]. It is not accidental, therefore, that he dedicated the first chapter of Volume III to the influence of friction on the motion of the top. He was at pains, however, to preserve the actual technical applications for later, since he first wished to discuss more general issues, such as the numerical evaluation of integrals, that were involved in the problems under consideration. The second chapter of Volume III was devoted to astronomical and geophysical applications. The two chapters turned out to be enough for one volume, so that "the technical and physical applications," as Sommerfeld wrote in the preface to Volume III, "remain for the fourth (final) volume."

In the summer of 1902, the astrophysicist Karl Schwarzschild joined the project as an advisor for the astronomical applications—presumably at the request of Klein in order to advance the pace of progress. Schwarzschild had been called to Göttingen in 1901 as the new director of the astronomical observatory. In July 1902, Sommerfeld sent him a preliminary manuscript for review. "The manuscript must, in order to be consistent, be written by me. If you feel like rewriting one part or another, I would be more than happy, but the final authority for the manuscript before publication must be left to me" [Sommerfeld 1902d]. Thus Sommerfeld prevented any aspirations of co-authorship in this effort. The part of the chapter that dealt with astronomical applications took shape during the following months. Schwarzschild did not resent that Sommerfeld had not offered him joint authorship. In August 1902, Sommerfeld sent him a new draft and asked for a "most severe critique," because he felt "not at all at home" in this matter. "The more you add, eliminate, change, the better" [Sommerfeld 1902e]. In September 1902, Sommerfeld traveled to Göttingen in order to discuss some details of this chapter with Schwarzschild. By January 1903, "the fruit of our former deliberations" was in print, as Sommerfeld informed Schwarzschild, requesting some final corrections of one or another numerical value [Sommerfeld 1903]. In the beginning, they had addressed each other in a rather formal manner as *"Lieber Herr College"*; now they changed to the more customary *"Lieber Schwarzschild"* and *"Lieber*

Sommerfeld". In the coming years they would correspond about many other scientific issues and become friends [Sommerfeld 1916].

As a consultant for the geophysical applications, Klein and Sommerfeld turned to Emil Wiechert. Wiechert, like Sommerfeld, had begun his career in Königsberg. In 1897, Wiechert was called to Göttingen as professor of geophysics. "Klein told me that you will soon turn up [*kreiseln*] again in Göttingen for discussions about the top," Schwarzschild alluded to the collaboration between Sommerfeld and Wiechert in the spring of 1903, when the astronomical work was finished and the geophysical part of Volume III was in the making [Schwarzschild 1903]. Like Schwarzschild, Wiechert was an outstanding expert. His Göttingen institute became the nursery of a world-famous school of geophysics [Mulligan 2001]. Wiechert's expert knowledge entered the book not only in the form of consulting, but also as a part of the content. In 1897 he had published a shell model for the composition of the Earth. According to this model, the Earth's interior consists of an iron core that is surrounded by a crust of much smaller density. When Sommerfeld discussed the fourteen-month periodicity in the motion of the Earth's axis (the "Chandler wobble" discovered by the American astronomer Seth Carlo Chandler in 1891) as a peculiar top-phenomenon, he referred to Wiechert's shell model as the most recent explanation (cf. Vol. III, Ch. VIII, §7).

Thus the Göttingen advisors contributed to making the theory of the top relevant for contemporary astronomical and geophysical debates—a virtue that made this volume, as Sommerfeld wrote hopefully in his preface, "useful not only for the mathematician and physicist who studies mechanics for its own sake." It is ironic that the subjects which were at this time closer to Sommerfeld's work as professor of mechanics in Aachen, such as gyroscopes for torpedo guidance and ship stabilization, had to wait for Volume IV, which appeared only in 1910, when Sommerfeld had become professor of theoretical physics in Munich and no longer counted these subjects in his major area of interest.

Michael Eckert
Deutsches Museum, Munich

Advertisement for Volume III of the Theory of the Top.

After a long interval caused by my passage to a new teaching position, a third volume now follows the second (1898) volume of the *Theory of the Top*. This third volume is not, as it was intended to be, the conclusion of the work; it happened, namely, that the subject matter expanded enormously as soon as the general mathematical schema was applied, in accordance with the original plan of the work, to particular experimental conditions or to the many problems of the various special sciences with an interest in the theory of the top. As a result, this volume presents only applications of the theory to astronomy and geophysics; the technical and physical applications remain for a fourth (final) volume.

While the beginning of the present volume depends upon the content of the preceding chapters and treats, in an appendix to Chapter VI, of the top on the horizontal plane (through approximate calculations with rigorous error estimation),* the content of Chapter VII extends essentially beyond the circle of problems that are usually addressed in the analytic mechanics of ideal mechanisms. The general empirical facts concerning the effects of friction are presented, and, in association, friction at the support point of the top and its effect of uprighting the axis of the top are discussed in detail. Since, on the one hand, the experimental foundations of the theory of friction are not very certain, and, on the other hand, the mathematical difficulties of a rigorous development of the theory would be very great, the treatment is carried out in part graphically, with auxiliary assumptions, omissions, and approximation methods, as has been repeatedly recommended at earlier stages of the theory of the top. The precision of these methods suffices completely, in so far as one keeps in mind the appropriate goal: to create a clear qualitative image of actually observed phenomena, and to supply a quantitatively accurate description within the error bounds of the observations. In addition to friction at the support point, air resistance and elasticity of the top material and the support are considered as further causes of disfigurement for the ideal motion of the top. These investigations are partly important for later applications,

*The appendix to Chapter VI appears in our translation of Vol. II. — *Trans.*

and are partly intended to serve as examples of the mechanics of actual phenomena, or, as it is occasionally expressed here, terrestrial mechanics (in contrast to celestial mechanics, in which the influences treated here do not come into consideration, or to pure analytic mechanics, in which such influences are usually neglected in favor of an elegant mathematical development). In an appendix to this chapter, the treatment of the top on the horizontal plane is supplemented by the consideration of friction, with the enlistment of experimental data.

Chapter VIII treats in a first part of astronomical applications of the theory of the top, and in a second part of geophysical applications.

In the classical problems of the precession and the forced nutation caused by the motion of the Moon, new results can hardly be produced. The subject has been treated so exhaustively that the present exposition is aimed merely at providing the nonspecialist an intuitive procedure to replace the sometimes obscure manner of presentation of the astronomers. The means for this is afforded by a method of Gaufs for perturbation calculations, which is extended here in various directions.

Some of the problems investigated in the geophysical part, in contrast, are of the most recent date. We consider, in particular, the free nutation of the axis of the Earth, whose period was established by Chandler, and, furthermore, the phenomena of pole oscillations in general. In the presentation of the objective state of affairs and in the explanation of the same, the treatment given here may offer decided advances. Because of the fundamental importance of the problem, the auxiliary theorems from hydrodynamics and elasticity that are required for the explanation of the fourteen-month Chandler period are taken up and proven in the simplest manner. In addition, the theory of meteorological transport is developed for the explanation of the yearly period of the pole oscillations, where once again the previously emphasized impulse theory and the free and intuitive conception of the dynamic differential equations prove particularly fruitful. The conclusion of the geophysical part is formed by a discussion of the famous Foucault top experiment for the proof of the rotation of the Earth. Here we eliminate the unnecessary mathematical difficulties that occupy a large space in the older presentations of the Foucault experiment, and emphasize instead the perturbing influences and the estimation of their order of magnitude.

At the wish of my highly esteemed teacher F. K l e i n, I must point out, finally, that in the writing of this volume I have exceeded the content of the original university lecture given by Mr. Klein to a still greater degree than in the previous volumes. The mechanical analysis of the perturbations in Chapter VII was only postulated in general outline in that lecture; the integration and approximation methods that are required here (and also in the appendix to Chapter VI), as well as the final results, are due to me alone. For what concerns the astronomical applications, Mr. Klein recognized the advantage of the Gaußian procedure, according to which he treated, in particular, the precession problem in his lecture; I myself, in contrast, have added the application of the same procedure to the problem of the nutation, as well as all numerical matters. The problem of the pole oscillations was not at all considered in Mr. Klein's lecture, so that the possible advances offered here (Chap. VIII, §6–8) are to be considered as my own. The foundation lines for the conception of the Foucault experiment were already drawn by Mr. Klein.

Moreover, I would like to emphasize that the constant interest that Mr. Klein has taken in the continuation of the work, as well as the encouragement that he has provided to me by many discussions and corrections, have essentially lightened my labor. Further, I must thank Mr. Schwarzschild and Mr. Wiechert for many valuable verifications and emendations in the astronomical and geophysical subjects.

May the present volume be of use not only to mathematicians and physicists who study mechanics for its own sake and will advance to a deeper and more lively understanding of the science by the availability of the thoroughly developed examples here, but rather may the representatives of astronomy, geophysics, and engineering also draw with pleasure upon this and the following volume as often as they come, in their particular fields, into contact with the theory of the top!

Aachen, July 1903. **A. Sommerfeld.**

Volume III

Perturbations. Astronomical and Geophysical Applications.

Chapter VII.

Theory and reality. The influence of friction, air resistance, and elasticity of the material and the support on the motion of the top.

§1. The contrast between rational and physical or celestial and terrestrial mechanics.

Abstract mechanics, which seeks to derive all phenomena of motion by mathematical deduction from a few fundamental principles, is traditionally designated as *rational* (= deductive) mechanics. The treatment of mechanical problems that accounts for reality in a more extensive manner by the use of empirical facts and experiments proceeds alongside, somewhat bashfully, as *physical* (= inductive) mechanics. In view of the vast differences between the results of the abstract theory and the facts of reality, however, one may raise the question whether, for the majority of applications, physical mechanics is actually rational, and the so-called rational mechanics is, in truth, most highly unphysical and irrational.

The differences between theory and reality that continually confront us are due, as everyone knows, to the occurrence of energy-absorbing or energy-dissipating forces. The abstract theory prefers to suppose that these forces have only a secondary significance, and may thus be treated as phenomena of the second order that can indeed blur but not completely disfigure the overall picture. A somewhat crassly chosen example may show us the extent to which this supposition is valid.

The German Model 88 infantry rifle[200] imparts to the bullet an initial velocity of approximately 620 $\frac{\text{m}}{\text{sec}}$. (The observation of this velocity is made about 25 m beyond the muzzle.) If we ignore air resistance and air friction, the energy-dissipating effects that come into play here, then the flight trajectory is the well-known parabola, and the greatest

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firing range is attained for an initial angle (elevation angle) of 45° with respect to the horizontal. If v is the common value of the initial horizontal and vertical velocity, H is the greatest elevation of the shot, and W is the range of the shot, then

$$v = \frac{620}{\sqrt{2}} \frac{\mathrm{m}}{\mathrm{sec}},$$

and one calculates, according to the elementary laws of projectile motion,

$$H = \frac{v^2}{2g} = \text{ca. } 10 \text{ km}, \quad W = \frac{2v^2}{g} = \text{ca. } 40 \text{ km}.$$

If, however, one opens the infantry firing regulations, which in this domain summarize a rich set of observational data, one finds on page 17 that the greatest observed firing range is approximately 4 km, and that it corresponds to an initial angle of 32°. The greatest elevation of the shot will be about ½ km for this trajectory, and lies not at the midpoint, as in the abstract projectile motion, but rather at a distance of 2,2 km from the muzzle.

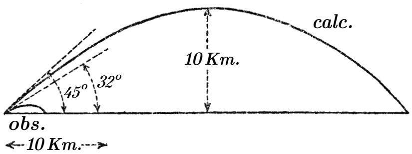

Fig. 71.

The adjacent figure speaks even more clearly than the given numbers. It shows that the trajectory calculated under the neglect of air resistance ("calc.") gives not even a remote approximation to the observed trajectory ("obs.").

It must be admitted that no one would think of neglecting air resistance in ballistic problems. It can also be conceded that the influence of air resistance will not be as significant for smaller velocities as for the tremendous velocities of modern firearms. It is known theoretically, and confirmed by observation, that air resistance must increase rapidly with increasing velocity, particularly in the neighborhood of the speed of sound. Nevertheless, our example may be an appropriate warning against the overestimation of the results of rational mechanics, and the underestimation of so-called "secondary circumstances" such as friction, which can easily, on occasion, become the primary consideration.

The aversion of mathematical writers to friction problems is already present in the great founder of analytic mechanics, Lagrange.

Nowhere in his work does he mention the friction of one rigid body upon another. This unworthy example is also followed by Kirchhoff in his *Lectures on Mechanics.* Yet friction, after gravity, is the most important force in our existence. It is usually regarded as something detrimental and undesirable. In fact, the energy losses and thus the operational cost of all our machines, vehicles, etc., depend for the most part on some kind of friction. It would be unjust, however, to fail to recognize the beneficial side of friction. It is only friction that enables us to move forward at will on the Earth, be it through the force of our limbs or with the help of a conveyance. One may recall, for example, that a streetcar is unable to move if the beneficial effect of friction between the wheels and the rail is weakened by the formation of ice. It is friction, further, that enables me to hold the pen between my fingers, if only—according to the law of friction to be stated below—I press with sufficient force against the holder. Friction prevents the books that lie on my writing table (certainly not exactly horizontal) from sliding to the Earth as a result of gravity; it prevents the mountain that lies before my window from entering the valley and burying the city.

Why is it, we ask ourselves, that in spite of its decisive importance for all earthly phenomena, friction finds such slight regard in theoretical mechanics? One reason is of historical nature.

The oldest field of application of theoretical mechanics, and the oldest branch of mathematical science in general, is *astronomy.* The founders and primary developers of mechanics—Galilei, Newton, Lagrange, Laplace—had essentially astronomical questions in mind in their investigations. Thus it occurred that theoretical mechanics took on a garb that was essentially tailored to astronomical purposes. Now the heavenly bodies move in empty space without perceptible friction, and can, for most purposes, be regarded as single mass particles. Thus in celestial mechanics—but also only here—frictional phenomena withdraw. The problem of central interest is the problem of n mass particles that move freely in space while acting upon one another with conservative forces, a problem that constitutes the principal subject of lectures and textbooks on theoretical mechanics, a problem which, however, is never realized in earthly occurrence.

The state of affairs is described more clearly if one speaks, rather than of rational and physical mechanics, of *celestial* and *terrestrial mechanics*. The form of theoretical mechanics that has been handed down to us has its origin and its appropriate domain of application in celestial mechanics. In order to be applied to the often obscure and complicated phenomena on Earth, it must be essentially supplemented by experimental material.

Another reason for the slight regard that theoreticians give to friction problems is the limited validity of the physical foundations of the theory of friction. We are aware from the outset that the usual laws on the sliding friction of solid bodies upon one another, on air resistance, on the internal friction of fluids or solid bodies, etc., are only rough approximations, and that the physical details of these processes are extremely complicated, and perhaps not at all able to be encompassed in formulas of general validity. A natural consideration of air resistance, for example, begins from the entrained motion of the surrounding air. The energy loss that the air resistance produces would be calculated, on the one hand, from the internal friction of the air, and, on the other hand, from the transfer of kinetic energy to the more remote layers of air. In very rapid motions, which are accompanied by a perceptible compression and rarefaction of the air, the energy transfer occurs at the speed of sound; thus in addition to the inertia of the air, elasticity and thermodynamics must also be taken into account. A glimpse into the multiplicity of the phenomena that are present here is granted to us by the beautiful and well-known instantaneous photograph of the airwaves generated by a bullet. Compared to this image, a formula that expresses the air resistance in terms of any power of the velocity must look very poor. It is just as hopeless, in view of the complicated play of waves in the vicinity of a steamship, to capture the ship's resistance in a simple formula. In any case, only extensive experiments carried out with the greatest means will provide any information here. Theory and calculation can, in this domain, sooner serve as guides to the judicious construction of experiments than for the prediction of the phenomena. In many cases, the theory has been able to provide rules (so-called similarity laws) for applying results from model experiments on a diminished scale to the larger scales of actual problems. The achievements of the theory are very valuable in such cases, but are still much more limited than, for example, in problems of celestial mechanics.

The magnitude of the resistance in the preceding examples will also depend on the particular form of the bullet or the ship. It can very well be that a small change of form can effect a large change in the law of resistance. It is similar for sliding friction. Apparently minor circumstances are much more influential than one would expect and wish. A surface altered by abrasion behaves differently from a freshly worked surface. Small pieces of abraded material or specks of dust between the sliding surfaces can influence the magnitude of the friction considerably; moisture condensed from the surrounding air can act as a lubricant, and the law of friction can be changed not only quantitatively, but also in a fundamental qualitative manner. We see here the operation of a principle that is most highly inconvenient and disturbing for the natural scientist: small causes, large effects. A warning against the uncritical use of numerical results for friction must be made on these grounds. Numerical values that are found for certain experimental conditions need not hold for others. To give these values to two or three decimal places, as is often done in technical handbooks and in many textbooks on experimental physics, has, in any case, no value.

One thus understands that the laboratory physicist who is in the pleasant position to choose his problems freely, and in part according to the aesthetic point of view, will prefer to pass over friction problems, since he can promise himself no pure and general laws from his studies. The technical worker stands in a different relation to the laws of friction, which for him are matters of vital importance. Thus the more recent contributions to knowledge of the laws of friction arise essentially from the technical side, as we will discuss in the following section.

It further follows from this state of affairs, however, that the mathematical treatment of friction problems must be carried out from a point of view different from that of the problems of rational mechanics. The mathematician seeks, according to his education and custom, to solve the problems before him with complete rigor, so that a calculation to arbitrarily many decimal places is theoretically possible. Considering the great accuracy of astronomical observations, this method is in fact appropriate for problems of celestial mechanics; for all problems, however, in which frictional influences are essential—that is, all problems of terrestrial mechanics—such a precise calculation would be unattractively incompatible with the precision of the physical foundation. Here it is indicated to seek not a quantitative calculation, but rather a qualitative understanding of the phenomena, and, when one does proceed quantitatively, to calculate from the beginning not with an arbitrary but with a bounded precision. The differential equations in the theory of the top that include friction terms, for example, become quite complicated, and, as one usually says, "may not be integrated." To the contrary, we emphasize the basic principle that one *should* not integrate such equations; one should interpret them and construct their solutions, as we will seek to do in the following.

The preceding discussion gives the motivation and the sense of our detailed treatment of friction problems for the top. Although we go beyond what has previously been done, we fall short of the desired goal; the lack of an adequate experimental foundation must restrict us to a schematic treatment of the problem.

§2. Report on the laws of friction.

Our knowledge of the laws of friction was founded, as is well known, by Coulomb.[201] We enter into these laws in some detail, since the relevant questions are generally unfamiliar in theoretical circles.*)

Coulomb found that a force appears at the boundary of two rigid bodies that slide upon one another, a force that opposes the motion of each body with respect to the other, and that is proportional to the total normal force with which the two bodies are pressed together. The factor of proportionality is called the *coefficient of friction* (or, more precisely, the coefficient of kinetic friction). This coefficient is to be regarded as a material constant, or, more correctly, as a characteristic

*) For further orientation, we refer to the very comprehensive treatment of friction problems in the textbook by J. Perry, Applied Mechanics, New York 1898, which we have used repeatedly for this and the previous sections.[202] An informative report on the entire literature of friction is due to F. Masi: Le nuove vedute nelle ricerche theoriche ed experimentali sull' attrito. Bologna, Zanichelli 1897.[203]

constant for the material and the surface condition of the two bodies. The friction force and the coefficient of friction should thus be independent of the sliding velocity and the size of the contact area, or, for equal total normal force, independent of the magnitude of the specific normal force, the normal force per unit area of the contact surface. The formula for the Coulomb friction law is, if one denotes the coefficient of friction by μ, the total normal force by N, and the magnitude of the frictional resistance by W,

$$W = \mu N.$$

We first wish to restrict this statement to *dry friction*; it is invalid for friction in the presence of a lubricant. Further, we must recall the well-known difference between kinetic (dynamic) friction and static friction. We explain static friction in the following manner.

If the applied force that acts to produce the motion of a test body with respect to its support, or the relative motion of the two, is not sufficient to overcome the friction, we will ascribe to the friction only the magnitude of the applied force that holds the body in equilibrium. This remains valid until the applied force exceeds a limiting value at which motion occurs, a value that is again proportional to the normal force N. The factor of proportionality may be denoted by μ_0, and is called the *coefficient of static friction.* The law of static friction can thus be expressed by the equation

$$W \leq \mu_0 N.$$

The coefficient of static friction is generally substantially larger than that of kinetic friction. This circumstance, as well as the law of friction in general, is illustrated in a beautiful experiment given by G. Herrmann,*) an experiment that anyone can repeat without any difficulty.

A stick is placed horizontally on the two index fingers. The fingers are then brought together. At which finger does the stick begin to slide? At the finger that stands farther from the center of gravity of the stick; for the normal forces N_1 and N_2 that the stick applies to the two fingers are, according to the law of the lever, proportional for each finger to

*) Der Reibungswinkel, Festschrift zum Jubiläum der Univ. Würzburg, 1882.[204]

the distance from the center of gravity to the other finger. The normal force is therefore smaller for the finger at the greater distance. According to the law of friction, the frictional force that opposes the sliding here is also smaller than for the other finger; the sliding must therefore begin here.

The stick now slides on this finger, and not only until the distance of this finger from the center of gravity is equal to that of the other, but rather somewhat farther, because $\mu_0 > \mu$. When the distances of the fingers from the center of gravity are in the proportion $\mu : \mu_0$, a change in the motion occurs; the stick now rests on the finger on which it previously slid, and begins to slide on the other. For the normal forces N on the two fingers are inversely proportional to the distances to the center of gravity, and are therefore, at this instant, in the ratio $\mu_0 : \mu$. The kinetic friction at one finger will then equal the static friction at the other, and would, if the sliding continued in the previous sense, become even greater, which is obviously absurd. The sequence of events is continuously repeated, so that the distance from the center of gravity to the finger on which the stick slides diminishes each time to the $\left(\frac{\mu}{\mu_0}\right)^{\text{th}}$ part of the distance from the center of gravity to the other finger. As a result, the distances from the center of gravity to the two fingers will be diminished alternately, and, when the fingers have come together, the stick will be supported at its center of gravity, and therefore freely oscillate!

In addition to the intuitive illustration of the friction law, the experiment also permits of a measurement of the ratio $\mu_0 : \mu$. It is enough, for this purpose, to mark a few reversal points of the motion of the stick, and to measure their distances from the center of gravity. The ratio of the distances between two successive reversal points provides the desired ratio of the coefficients of friction. The totality of the reversal points forms, on both sides of the stick, the successive points of a geometric series. If the stick is not too short, the measurement can be relatively precise.

The difference in the values of μ_0 and μ suggests the hypothesis of a continuous passage between the value μ_0 corresponding to the sliding velocity zero and the value μ corresponding to perceptibly larger sliding velocities. This conjecture has been confirmed by the experiments of Jenkin and Ewing.*) For materials with significantly different

*) London Philos. Transactions, Vol. 167 (1877), p. 509.[205]

coefficients of friction μ_0 and μ, the continuous passage can in fact be detected.

If we therefore wish to claim, with Coulomb, the independence of the coefficient of friction on the velocity, we must first exclude the domain of very small velocities. How well is this independence now confirmed for greater velocities?

The older experiments of General Morin, which were performed in the years 1831–1833 in Metz and encompassed a velocity range up to $4\,\frac{\mathrm{m}}{\mathrm{sec}}$, appear to confirm the independence. To this day, Morin's experimental results form the permanent supply of numerical specifications for coefficients of friction in technical handbooks and textbooks on experimental physics. A high degree of reliability, however, can hardly be ascribed to them, since, according to what was said in the previous section, such numerical results depend strongly on the secondary circumstances of the experiments.[206]

The behavior of the coefficient of friction for higher velocities remains, in any case, an open question that first became relevant in the development of railroads. Between the brake pad and the collar of the wheel we have true sliding friction without lubrication, while the friction between the wheel and the rail is generally, at least by intention, rolling friction, and becomes sliding friction only in the exceptional case that the wheels skid on the rails. Concerning the friction between the brake pad and the wheel, the experiments of *Douglas Galton*,*) which were carried out in full scale on various English rail lines with the support of the Westinghouse firm, stand out above all others. Galton's experimental van contained an entire series of tachometers and dynamometers. The tachometers provided the rotational velocity of the wheels and the progressional velocity of the van. The sliding velocity between the wheel and the brake pad is equal to the first of these velocities, and the possible sliding velocity between the wheel and the rail is equal to the difference of the two. The dynamometers measured 1) the braking force with which the brake pad was pressed to the wheel, and therefore the force that acts as the normal force N for the friction between the brake and the wheel; 2) the frictional resistance W between the brake and

*) Institution of Mechanical Engineers Proceedings 1878, 1879; see, in particular, 1879, p. 172 or Engineering 1879, p. 371 or Reports of the British Association (Dublin) 1878.

the wheel; 3) the resistance against the forward motion of the car per axle, which comprises air resistance, rolling friction on the rail, etc., which, however, if sliding on the rails occurs, represents in essence the frictional resistance W' of this sliding. All these tachometers and dynamometers registered automatically, and thus provided the temporal changes of the relevant velocities and forces. The very interesting diagrams that were obtained for these quantities can only be indicated here. We must restrict ourselves to the conclusions to be extracted from them regarding the variability of the friction coefficients. The coefficient of friction between the brake pad and the wheel results from division of the measured forces W and N, while the coefficient of friction between the wheel and the rail follows from the likewise measured friction W' and the portion of the van weight G that falls on the individual wheel.[207]

The corresponding friction coefficients for different experiments with the same velocities naturally show no complete agreement among themselves; their magnitudes depend on the weather and the related dampness condition of the sliding surfaces, as well as on the cleanliness of the surfaces and the braking duration, which obviously causes an increase in temperature and thus a change in the surface condition. It occurred, for example, that after the braking force had acted for 20 seconds, the friction coefficient was reduced to half of its original value. Nevertheless, the mean of the friction coefficients obtained from many (up to 100) experiments show a clear regularity; namely, a *continuous decrease of the friction coefficient with increasing velocity.* Fig. 72 gives the coefficient of friction between the brake pad and the wheel (brake pad of cast iron, wheel rim of steel). The solid line is the mean value of the observations, and the dotted lines are the largest and smallest values for each velocity. The entire strip between these boundary lines is to be imagined as filled with observation points that thicken along the mean line. One first recognizes the difference between the static coefficient of friction and that of the considered motion. But the figure further shows that the

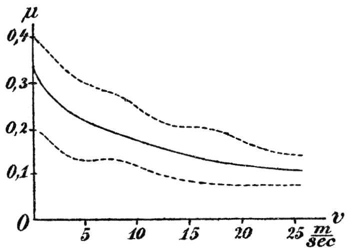

Fig. 72.

coefficient of friction for the velocity 60 km/hr (= 16,7 m/sec = mean express train velocity) amounts to less than half of the static coefficient of friction (0,33), and that the coefficient of friction for 90 km/hr (= 25 m/sec = highest current permissible speed in Germany) amounts to no more than a third of the static coefficient of friction. This naturally leaves the dependence between μ and v uncertain, and thus expressible through one formula or another. We thus disregard the statement of such a formula as well as the numerical values, since the figure depicts all that is to be concluded from the degree of precision of the observations, and since any formula used to represent the observations would be highly arbitrary. — Concerning the coefficient of sliding friction between the rail and the wheel, Galton's experiments give a less certain conclusion; as much as can be discerned from them without doubt is that this coefficient also decreases continually with increasing velocity from its greatest value for small velocity.

Older French and more recent experiments conducted in Germany*) yielded essentially the same results.

But it is to be considered that Galton's experiments refer to a very extended velocity range. For a moderately changing velocity, the variation of the friction coefficient is also only small; according to the curve one has, for example, the Galton mean value of approximately $\mu = 0{,}27$ to 0,23 for $v = 2$ to 6 m/sec. These differences are, with respect to the general uncertainty of friction values, undoubtedly to be neglected. Coulomb's assumption of a velocity-independent coefficient of friction is thus affirmed, in the first approximation, for a *moderate* velocity range. For our application to the top, in which velocities of the order of express train velocities certainly do not come into consideration, we may thus set, with Coulomb, the coefficient of friction equal to a constant.

The lack of dependence of the coefficient of friction on the size of the contact area that is stated in the Coulomb law may also be tested experimentally. According to the Coulomb law, a prism of base area 10 cm^2 and height 4 cm that moves on a fixed plane must be subjected to the same frictional force as a prism of base area 20 cm^2 and height 2 cm, since, assuming the same material, the total normal force against the support, namely the weight of the prism, is the same in each case,

*) Cf. Organ der Fortschr. des Eisenbahnwesens 1889, p. 114. The brake pad consisted here of cast steel. The experiments were conducted not on the track, but rather in a workshop.[208]

while the normal forces per unit area in the two cases have the ratio $2:1$. It appears that this result is well confirmed in experiments,*) in so far as a perceptible deformation of the support is not caused by a very large specific pressure. —

It is also well known that the process of rolling two surfaces upon one another, which (apparently) occurs without sliding, is also associated in a small degree with energy dissipation. The law of *rolling friction* that is usually adopted for the calculation of this energy loss was likewise constructed by *Coulomb*. Here it is useful to speak not of a *frictional single-force*, as for sliding, but rather of a frictional moment that is to be overcome at each instant by the turning-moment that is employed for the rolling. If the total normal force at the contact between the support and the "roller" is again denoted by N and the frictional moment is denoted by M, then one sets

$$M = \nu N.$$

The quantity ν is called the *coefficient of rolling friction*; as the equation shows, it is not a pure number like the coefficient of sliding friction, but rather has the dimension of length. This coefficient is also to be regarded as a material or surface constant. If one wishes to replace M by a frictional single-force W on the outer surface, then one must set the latter equal to M/r, where r is the radius of the roller, or, for a non-circular perimeter of the roller, is equal to the radius of curvature at the considered place on the circumference; this single-force W is therefore directly proportional to the normal force and inversely proportional to the radius.

A beautiful work of O. Reynolds**) gives some information on the mechanism of rolling friction. Reynolds shows that rolling is always accompanied by a certain sliding due to elastic deformation.

If one assumes, for simplicity, that the support is essentially softer than the roller (support of rubber, roller of iron), then one can disregard the deformation of the roller, and consider only that of the support. The latter will consist of a trough-shaped depression, so that the support is stretched in the middle of the depression, and is raised and

*) Perry l. c. p. 67.

**) On rolling Friction, London Philos. Transactions Vol. 166, Part I (1876) and Ges. Werke Bd. I, p. 110.

compressed on the sides. The contact no longer occurs at a geometric point, but along the surface of the trough, whose midpoint we denote as the mean contact point P. In the adjacent figure, a number of points are marked on the roller and the support. The points on the roller are equidistant. The points on the support were equidistant before the deformation; the changed distances thus show the sense of the distortion that has occurred. The points marked by the same numbers are the mean contact points that successively coincide with one another in the rolling process, which is made possible by the elongation of the support at the mean contact position due to the rolling process. In an instantaneous state, however, these points do not coincide. Point 4 of the instantaneous state must, for example, slide along the small section of the rolling circumference between the two points 4 in the figure before it has become the mean contact point. The same holds for each point of the contact surface. In this process, therefore, a certain sliding friction in fact occurs.

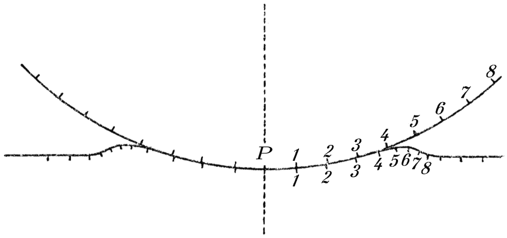

Fig. 73.

The total energy loss of the rolling friction will be made up in part by the energy loss associated with the sliding friction, and in part by the work required in the elastic deformation, in so far, namely, as the latter occurs irreversibly.

Reynolds was able to demonstrate experimentally that the representation in Fig. 73 is correct. Under the conditions assumed above, the circumference of the roller develops not on the natural upper surface of the support, but rather on the extended surface at P. If the path that the roller has covered after one rotation is measured on the support that has returned to its natural length, then this path must be somewhat shorter than the circumference of the wheel. Reynolds has, in fact, demonstrated this experimentally for the case in which the roller is harder than the support. The opposite must occur, and according to Reynolds does occur, if the support is considerably harder than the roller. If both are of the same material, then the path measured on the support is again somewhat smaller than the circumference of the roller, as would be shown by a more detailed entrance into the preceding deliberation.

The Reynolds investigation is not carried out to a precise measurement of the magnitude of the frictional resistance and a verification of the Coulomb assumption. It is not very probable that this simple assumption corresponds precisely to reality, considering the complicated nature of the process. —

In addition to sliding and rolling friction, one speaks in the third place of *boring* friction, particularly when one body rotates on another about the normal to the contact point. Since the contact is assumed in this case to be pointlike, and since the contact point is assumed to be a point on the rotation axis, sliding of the two bodies on one another does not, theoretically, take place. This circumstance has given occasion for the introduction of the special term "boring friction." However, the process is reduced immediately to sliding friction if only one assumes a somewhat extended contact between the bodies. One can then speak of a mean radius a of the contact surface, and may reduce the forces distributed on rings about the rotation axis to sliding friction at this mean distance a. These forces may obviously be composed into a turning-moment about the normal to the contact surface, whose magnitude is calculated from the law of sliding friction as

$$M = \mu' N, \quad \mu' = \mu a.$$

The factor of proportionality μ' can be denoted as the *coefficient of boring friction*; it has the dimension of length and depends, in addition to the material and the nature of the surface, on the extent of the contact surface. It should naturally not be claimed from the equation $\mu' = \mu a$ that the coefficient of boring friction would be predetermined from that of sliding friction if one could measure the size of the contact surface. Rather, this equation provides only an indication of the meaning of the coefficient μ', and an approximate magnitude of the ratio of boring to sliding friction that will be of use to us in the next section.

We have explicitly restricted ourselves in this report to *dry* friction, even though friction under the application of a *lubricant* is of predominant interest in practice. The current understanding is that lubrication friction is subject to a completely different law; it depends, namely, on