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Tata Lectures on Theta II

David Mumford

With the collaboration of
C. Musili, M. Nori, E. Previato,
M. Stillman, and H. Umemura

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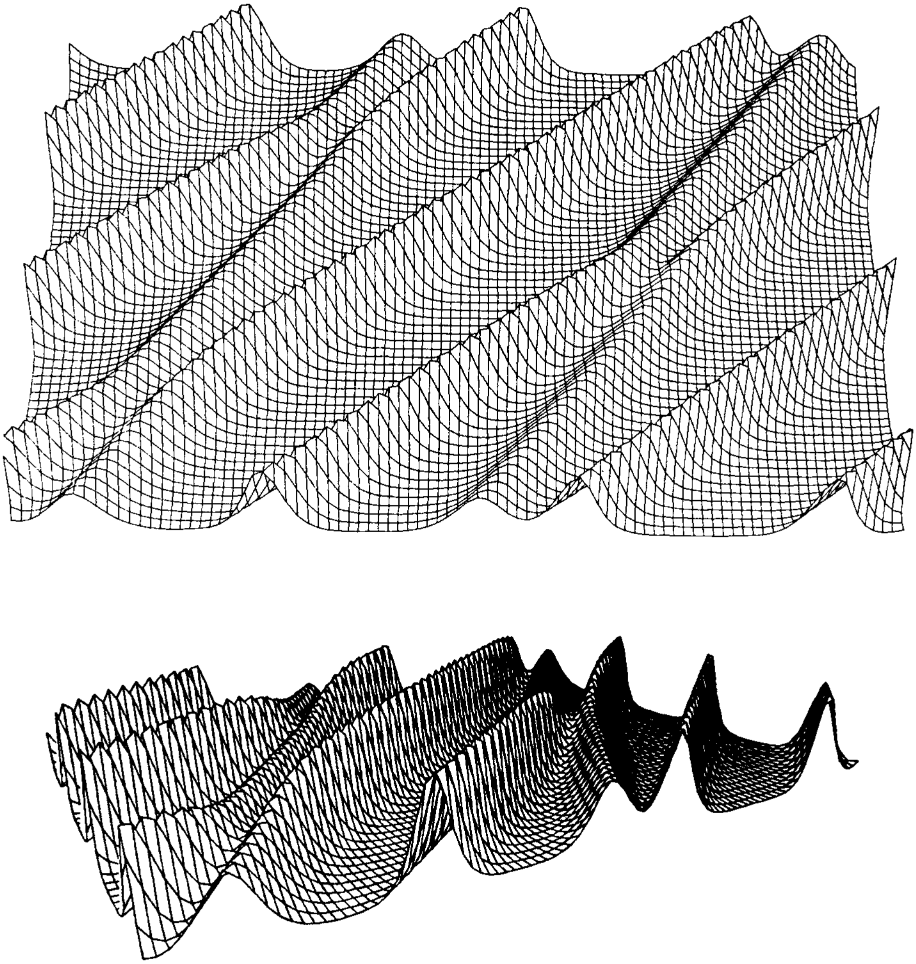

Almost periodic solution of K-dV given by the genus 2 $\mathfrak{p}$-function $D^2\log\text{-}\vartheta(z,\Omega)$ with $\Omega = \begin{pmatrix} 10 & 2 \\ 2 & 10 \end{pmatrix}$.

An infinite train of fast solitons crosses an infinite train of slower solitons (see Ch. IIIa, §10,IIIb, §4).

Two slow waves appear in the pictures: Note that each is shifted backward at every collision with a fast wave.

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Jacobian theta functions and differential equations

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CHAPTER III

Jacobian theta functions and Differential Equations

Introduction to Chapter III

In the first chapter of this book, we analyzed the classical analytic function

$$\vartheta(z,\tau) = \sum e^{\pi i n^2 \tau + 2\pi i n z}$$

of 2 variables, explained its functional equations and their geometric significance and gave some idea of its arithmetic applications. In the second chapter, we indicated how ϑ generalizes when the scalar z is replaced by a vector variable $\vec{z} \in \mathbb{C}^g$ and the scalar τ by a $g \times g$ symmetric period matrix Ω. The geometry was more elaborate, and it led us to the concept of abelian varieties: complex tori embeddable in complex projective space. We also saw how these functions arise naturally if we start from a compact Riemann surface X of genus g and attempt to construct meromorphic functions on X by the same methods used when $g = 1$.

However, a very fundamental fact is that as soon as $g \geq 4$, the set of $g \times g$ symmetric matrices Ω which arise as period matrices of Riemann surfaces C depends on fewer parameters than $g(g+1)/2$, the number of variables in Ω. Therefore, one expects that the Ω's coming from Riemann surfaces C, and the corresponding tori X_Ω, also known as the Jacobian variety Jac(C) of C, will have <u>special properties</u>. Surprisingly, these special properties are rather subtle. I have given elsewhere (Curves and their Jacobians, Univ. of Mich. Press, 1975), a survey of some of these special properties. What I want to explain in this chapter are some of the special function-theoretic properties that ϑ possesses when Ω comes from a Riemann surface. One of the most striking properties is that from these special ϑ's one can produce solutions of many important non-linear partial differential equations that have arisen in applied mathematics. For an arbitrary Ω, general considerations of functional dependence say that $\vartheta(\vec{z},\Omega)$ must always satisfy many non-linear PDE's: but if $g \geq 4$, these equations are not known explicitly. Describing them is a very interesting problem. But in contrast when Ω comes from a Riemann surface, and especially when the Riemann surface is hyperelliptic, ϑ satisfies quite simple non-linear PDE's of fairly low degree. The best known examples are the Korteweg-de Vries (or KdV) equation and the Sine-Gordan equation in the hyperelliptic case, and somewhat more complicated Kadomstev-Petriashvili (or KP) equation for general Riemann surfaces. We wish to explain these facts in this chapter.

The structure of the chapter was dictated by a second goal, however. As background, let me recall that for all $g \geq 2$, the natural projective embeddings of the general tori X_Ω lie in very high-dimensional projective space, e.g., $\mathbb{P}_{(3^g-1)}$ or $\mathbb{P}_{(4^g-1)}$ and their image in these projective spaces is given by an even larger set of polynomials equations derived from Riemann's theta relation. The complexity of this set of equations has long been a major obstacle in the theory of abelian varieties. It forced mathematicians, notably A. Weil, to develop the theory of these varieties purely abstractly without the possibility of motivating or illustrating results with explicit projective examples of dimension greater than 1. I was really delighted, therefore, when I found that J. Moser's use of hyperelliptic theta functions to solve certain non-linear ordinary differential equations leads directly to a very simple projective model of the corresponding tori X_Ω. It turned out that the ideas behind this model in fact go back to early work of Jacobi himself (Crelle, 32, 1846). It therefore seemed that these elementary models, and their applications to ODE's and PDE's are a very good introduction to the general algebro-geometric theory of abelian varieties, and this Chapter attempts to provide such an introduction.

In the same spirit, one can also use hyperelliptic theta functions to solve explicitly algebraic equations of arbitrary degree. It was shown by Hermite and Kronecker that algebraic equations of degree 5 can be solved by elliptic modular functions and elliptic integrals. H. Umemura, developing ideas of Jordan, has shown how a simple expression involving hyperelliptic theta functions and hyperelliptic integrals can be used to write down the roots of any algebraic equation. He has kindly written up his theory as a continuation of the exposition below.

The outline of the book is as follows. The first part deals entirely with hyperelliptic theta functions and hyperelliptic jacobians:

§0 reviews the basic definitions of algebraic geometry, making the book self-contained for analysts without geometric background.

§§1-4 present the basic projective model of hyperelliptic jacobians and Moser's use of this model to solve the Neumann system of ODE's.

§5 links the present theory with that of Ch. 2, §§2-3.

§§6-9 shows how this theory can be used to solve the problem of characterizing hyperelliptic period matrices Ω among all matrices Ω. This result is new, but it is such a natural application of the theory that we include it here rather than in a paper.

§§10-11 discuss the theory of McKean-vanMoerbeke, which describes "all" the differential identities satisfied by hyperelliptic theta functions, and especially the Matveev-Iits formula giving a solution of Kd V. We present the Adler-Gel'fand-Manin-et-al description of Kd V as a completely-integrable dynamical system in the space of pseudo-differential operators.

The second part of the chapter takes up general jacobian theta functions (i.e., $\vartheta(\vec{z},\Omega)$ for Ω the period matrix of an arbitrary Riemann surface). The fundamental special property that all such ϑ's have is expressed by the "trisecant" identity, due to John Fay (Theta functions on Riemann Surface, Springer Lecture Notes 352), and the Chapter is organized around this identity:

§1 is a preliminary discussion of the "Prime form" $E(x,y)$ — a gadget defined on a compact Riemann surface X which vanishes iff $x = y$.

§2 presents the identity.

§§3-4 specialize the identity and derive the formulae for solutions of the KP equation (in general) and KdV, Sine-Gordan (in the hyperelliptic case).

§5 is only loosely related, but I felt it was a mistake not to include a discussion of how algebraic geometry describes and explains the soliton solutions to KdV as limits of the theta-function solutions when g of the 2g cycles on X are "pinched".

The third part of the chapter by Hiroshi Umemura derives the formula mentioned above for the roots of an arbitrary algebraic equation in terms of hyperelliptic theta functions and hyperelliptic integrals.

There are two striking unsolved problems in this area: the first, already mentioned, is to find the differential identities in $\vec{z}$ satisfied by $\vartheta(\vec{z},\Omega)$ for general Ω. The second is called the "Schottky problem": to characterize the jacobians X_Ω among all abelian varieties, or to characterize the period matrices Ω of Riemann surfaces among all Ω. The problem can be understood in many ways: (a) one can seek <u>geometric</u> properties of X_Ω and especially of the divisor Θ of zeroes of $\vartheta(\vec{z},\Omega)$ to characterize jacobians or (b) one can seek a set of modular forms in Ω whose vanishing implies comes from a Riemann surface. One can also simplify the problem by (a) seeking only a generic characterization: conditions that define the jacobians plus possibly some other irritating components, or (b) seeking identities involving auxiliary variables: the characterization then says that X_Ω is a jacobian iff $\exists$ choices of the auxiliary variables such that the identities hold. In any case, as this book goes to press substantial progress is being made on this exciting problem. I refer the reader to forthcoming papers:

E. Arbarello, C. De Concini, On a set of equations characterizing Riemann matrices,

T. Shiota, Soliton equations and the Schottky problem,

B. van Geemen, Siegel modular forms vanishing on the moduli space of curves,

G. Welters, On flexes of the Kummer varieties.

The material for this book dates from lectures at the Tata Institute of Fundamental Research (Spring 1979), Harvard University (fall 1979) and University of Montreal (Summer 1980). Unfortunately, my purgatory as Chairman at Harvard has delayed their final preparation for 3 years. I want to thank many people for help and permissions, especially Emma Previato for taking notes that are the basis of Ch. IIIa, Mike Stillman for taking notes that are the basis of Ch. IIIb, Gert Sabidusi for giving permission to include the Montreal section here rather than in their publications, and S. Ramanathan for giving permission to include the T.I.F.R. section here. Finally, I would like to thank Birkhauser-Boston for their continuing encouragement and meticulous care.

§0. Review of background in algebraic geometry.

We shall work over the complex field $\mathbb{C}$.

Definition 0.1. An affine variety is a subset $X \subset \mathbb{C}^n$, defined as the set of zeroes of a prime ideal $\mathfrak{p} \subset \mathbb{C}[X_1,\ldots,X_n]$; $X = \{x \in \mathbb{C}^n \mid f(x) = 0 \text{ for all } f \in \mathfrak{p}\}$[1]. X will sometimes be denoted by $V(\mathfrak{p})$ or by $V(f_1,\cdots,f_k)$ if $f_1,\cdots,f_k$ generate $\mathfrak{p}$.

A morphism between two affine varieties X,Y is a polynomial map $f: X \longrightarrow Y$, i.e., if $(X_1,\cdots,X_n) \in X$, then the point $f(X_1,\ldots,X_n)$ has coordinates $Y_i = f_i(X_1,\ldots,X_n)$, where $f_i \in \mathbb{C}[X_1,\ldots,X_n]$; following this definition, we will identify isomorphic varieties, possibly lying in different (dimensional) $\mathbb{C}^n$'s.

A variety is endowed with several structures:

a) 2 topologies; the "complex topology", induced as a subspace of $\mathbb{C}^n$, with a basis for the open sets given by $\{(x_1,\ldots,x_n) \mid |x_i-a_i| < \varepsilon, \text{ all } i\}$, and the "Zariski topology" with basis $\{(x_1,\ldots,x_n) \mid f(x) \neq 0\}$, $f \in \mathbb{C}[X_1,\ldots,X_n]$.

b) the affine ring $R(X) = \mathbb{C}[X_1,\ldots,X_n]/\mathfrak{p}$, which can be viewed as a subring of the ring of $\mathbb{C}$-valued functions on X since $\mathfrak{p}$ is the kernel of the restriction homomorphism defined on $\mathbb{C}$-valued polynomial functions on $\mathbb{C}^n$, by the Nullstellensatz.

c) the function field $\mathbb{C}(X)$, which is the field of fractions of $R(X)$; the local rings $\mathcal{O}_x$ and $\mathcal{O}_{Y,X}$, where x is a point, Y a subvariety of X, defined by $\mathcal{O}_x = \{f/g \mid f,g \in R(X) \text{ and } g(x) \neq 0\}$, with maximal ideal $m_x = \{f/g \in \mathcal{O}_x \mid f(x) = 0\}$, $\mathcal{O}_{Y,X} = \{f/g \mid f,g \in R(X), g \not\equiv 0 \text{ on } Y\} = R(X)_{\mathfrak{q}}$[2] if $Y = Y(\mathfrak{q})$;

1) If a polynomial $f \in \mathbb{C}[X_1,\cdots,X_n]$ is zero at every point of V then $f \in \mathfrak{p}$; this is Hilbert's Nullstellensatz.

2) We denote by $A_{\mathfrak{q}}$ the localization of a domain A with respect to its prime ideal $\mathfrak{q}$, $A_{\mathfrak{q}} = \{a/b \mid a,b \in A,\ b \notin \mathfrak{q}\}$.

(notice: if $x \in Y$, $R(X) \subset \mathcal{O}_x \subset \mathcal{O}_{Y,X} \subset \mathbb{C}(X)$); the structure sheaf $\mathcal{O}_X$, subsheaf of the constant sheaf $U \longmapsto \mathbb{C}(X)$, which assigns to any Zariski-open subset U of X the ring $\bigcap_{x \in U} \mathcal{O}_x = \Gamma(U, \mathcal{O}_X)$; and a dimension given by $\dim X = \mathrm{tr.d}_{\mathbb{C}}\mathbb{C}(X)$. dim X is related to the Krull dimension of $\mathcal{O}_{Y,X}$ (maximum length of a chain of prime ideals), by:

<u>Proposition 0.2.</u> $\dim X - \dim Y = \text{Krull dim. } \mathcal{O}_{Y,X}$.

d) the Zariski tangent-space at $x \in X$, which can be defined in a number of equivalent ways:

$T_{X,x}$ = vector space of derivations $d: R(X) \longrightarrow \mathbb{C}$ centered at x (i.e., satisfying the product rule $d(fg) = f(x)dg+g(x)df$); or

$T_{X,x} = (m_x/m_x^2)^\vee$, the space of linear functions on m/m^2; or

$T_{X,x}$ = the space of n-tuples $(\dot{x}_1,\cdots,\dot{x}_n)$ such that for all $f \in \mathfrak{p}$, $f(x_1+\varepsilon\dot{x}_1,\cdots,x_n+\varepsilon\dot{x}_n) \equiv 0 \bmod \varepsilon^2$,

where from a derivation d a linear function $\ell(X_i-x_i) = dX_i$ and an n-tuple $(\dot{x}_1,\cdots,\dot{x}_n)$ with $dX_i = \dot{x}_i$ are obtained; this sets up the bijection. This vector is also written customarily as $\sum_{i=1}^{n} (\dot{x}_i)\partial/\partial x_i$;

<u>Proposition 0.3.</u> $\exists$ <u>a non-empty Zariski open subset</u> $U \subset X$ <u>such that</u> $\mathrm{tr.d.}_{\mathbb{C}}\, \mathbb{C}(X) = \dim T_{X,x}$ <u>for all</u> $x \in U$; <u>if</u> $x \notin U$, <u>then</u> $\dim T_{X,x} > \dim X$.

U is called the set of "smooth" points of X, X-U the "singular locus". It can be shown from this proposition that U (with the complex topology) is locally homeomorphic to $\mathbb{C}^d$, where d is $\mathrm{tr.deg.}_{\mathbb{C}}\mathbb{C}(X)$.

Lemma 0.4. For any $x \in X$, $\exists$ a fundamental system of Zariski neighborhoods U of x such that U is isomorphic to an affine variety.

In fact, for any $f \in R(X)$ such that $f(x) \neq 0$, $U_f = \{y \in X \mid f(y) \neq 0\}$ is a neighborhood of x and if $R_X = \mathbb{C}[X_1,\ldots,X_n]/\mathfrak{P}$, then U_f is isomorphic to the subvariety of $\mathbb{C}^{n+1}$ is defined by the ideal $(\mathfrak{P}, X_{n+1}f(X_1,\ldots,X_n)-1$; the isomorphism is realized by

$$(x_1,\cdots,x_n) \longmapsto (x_1,\cdots,x_n, f(x_1,\cdots,x_n)^{-1}).$$

But we need a more subtle definition of morphism from an open set to an affine variety.

Definition 0.5. $f\colon \underset{\substack{\cap \text{open} \\ X}}{U} \longrightarrow Y$ is a morphism if (equivalently):

(1) for any $g \in R(Y)$, thought of as a complex-valued function on Y, $g \circ f \in \Gamma(U, \mathcal{O}_X)$

(2) $\exists$ $g_{ik}, h_k \in \mathbb{C}[X_1,\cdots,X_m]$ such that for any $(x_1,\cdots,x_n) \in U$ there is a suitable k such that $h_k(x) \neq 0$, and the i-th coordinate of $f(x_1,\cdots,x_n)$ is given by $\dfrac{g_{ik}(x_1,\cdots,x_n)}{h_k(x_1,\cdots,x_n)}$ whenever $h_k(x) \neq 0$.

(n.b. there may not exist a single expression $f(X_1,\cdot\cdot,X_m)_i = \dfrac{g_i(X_1,\cdot\cdot,X_n)}{h(X_1,\cdot\cdot,X_n)}$, with $h^{-1} \in \Gamma(U,\mathcal{O}_X)$.

Theorem 0.6 (Weak Zariski's Main Theorem). If $f\colon X \longrightarrow Y$ is an injective morphism between affine varieties of the same dimension and Y is smooth, then f is an isomorphism of X with an open subset of Y.

The *product* of affine varieties is categorical, i.e., given $X \subset \mathbb{C}^n$ and $Y \subset \mathbb{C}^m$ affine varieties, i) $X \times Y$ is an affine variety (in $\mathbb{C}^{n+m}$), ii) the projections are morphisms, iii) if Z is an affine variety and morphisms $Z \longrightarrow X$, $Z \longrightarrow Y$ are given, then there is a unique morphism $Z \longrightarrow X \times Y$ making a commutative diagram

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow & \downarrow & \searrow & \\ X & \leftarrow & X\times Y & \rightarrow & Y \end{array}$$

Definition 0.7. *A variety in general is obtained by an atlas of affine varieties*: $X = \bigcup_{\alpha \in S} X_\alpha$, S *a finite set*, $X_\alpha \subset \mathbb{C}^{n_\alpha}$, *glued by isomorphisms*

$$\begin{array}{ccc} U_{\alpha,\beta} & \subset & X_\alpha \\ \phi_{\alpha\beta}\big\downarrow \;\big\uparrow \phi^{-1}_{\alpha\beta} & & \\ U_{\beta,\alpha} & \subset & X_\beta \end{array}$$

(*where* $U_{\alpha,\beta}$ *is a nonempty Zariski-open subset of* X_α), *such that one of the equivalent (separation) conditions holds*:

(1) X is Hausdorff in the "complex topology" (a subset of X being open in the complex topology if and only if its intersections with X_α are open for all α's)

(2) the graph of $\phi_{\alpha\beta}, \Gamma_{\alpha\beta} \subset X_\alpha \times X_\beta$ is Zariski-closed.

(3) for any valuation ring $R \subset \mathbb{C}(X) = \mathbb{C}(X_\alpha)$ (any α, for the function field of $U_f \subset U_{\alpha\beta}$ coincides with that of X_α, hence $\phi_{\alpha\beta}$ identifies $\mathbb{C}(X_\alpha)$ and $\mathbb{C}(X_\beta)$) there is at most one point $x \in X$

such that $R \succ \mathcal{O}_x$ (R "dominates" $\mathcal{O}_x$, or R is "centered" at x, i.e., $R \supset \mathcal{O}_x$ and $\mathfrak{m}_R \supset \mathfrak{m}_x$).

(4) for all affine varieties Y and morphisms $f,g: Y \longrightarrow X$, the set $\{y \in Y \mid f(y) = g(y)\}$ is (Zariski) closed in Y.

Such an X carries:

(a') 2 topologies (the complex and the Zariski; as with the complex topology, a subset of X is Zariski-open if and only if its intersection with all the X_α's is Zariski-open)

(c') the function field $\mathbb{C}(X)$; the local rings[3)] $\mathcal{O}_x = \{f/g \mid f,g \in R(X_\alpha),\ g(x) \neq 0\}$ if $x \in X_\alpha$; the structure sheaf $U \longmapsto \bigcap_{x \in U} \mathcal{O}_x$

(d') the Zariski tangent space $T_{x,X} = T_{x,X_\alpha}$ if $x \in X_\alpha$.

$f: X \longrightarrow Y$ is a morphism between two varieties if the restriction res $f: U_\alpha \cap f^{-1}(V_\beta) \longrightarrow V_\beta$ is a morphism for all α,β's or, equivalently, if for any open set $U \subset Y$ and $g \in \Gamma(U,\mathcal{O}_Y)$, $g \circ f \in \Gamma(f^{-1}U,\mathcal{O}_X)$ is satisfied.

Key example. Projective varieties, defined by homogeneous ideals $\mathfrak{p} \subset \mathbb{C}[X_0,\ldots,X_n]$, as

$$V(\mathfrak{p}) = \{(x_0,\cdots,x_n) \in \mathbb{P}^n \mid f(x) = 0 \text{ for all } f \in \mathfrak{p}\};$$

an atlas is given by $V(\mathfrak{p})_i = \{p \in V(\mathfrak{p}) \mid X_i(p) \neq 0\}$.

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3) A variety X can even be defined as a set of local rings $\{\mathcal{O}_x\}$ with the same fraction field $\mathbb{C}(X)$. Then the topology on X is defined as follows - for each $f \in \mathbb{C}(X)$, let U_f be the set of the local rings containing f.

The product of varieties is again a variety; we take $(U\times V)_{\Sigma f_i g_i}$ to be a basis for the open sets in $X\times Y$, where U,V are open subsets of X,Y isomorphic to affine varieties, $f_i \in \Gamma(U,\mathcal{O}_X)$, $g_i \in \Gamma(V,\mathcal{O}_Y)$ and $(U\times V)_{\Sigma f_i g_i}$ is the set of points $(x,y) \in U\times V$ such that $\Sigma f_i(x) g_i(y) \neq 0$.

The product of projective varieties is again a projective variety, for instance the map $(x_i, y_j) \longmapsto (x_i y_j)$ embeds $\mathbb{P}^n \times \mathbb{P}^m$ into $\mathbb{P}^{(n+1)(m+1)-1}$ and the image is given on the affine pieces $(\mathbb{P}^{n+m+nm})_{X_{hk}}$ by the equations $s_{ij} = s_{ih}s_{kj}$ for all $i \neq h$ and $j \neq k$, where $s_{ij} = X_{ij}/X_{hk}$.

<u>Definition</u> 0.8. <u>A variety</u> X <u>is complete (or proper) if one of the following equivalent condition holds</u>:

(1) X <u>is compact in the complex topology</u>

(2) $\exists$ <u>a surjective birational morphism</u> $f: X' \longrightarrow X$, X' <u>projective</u>

(3) <u>for all valuation rings</u> $R \subset \mathbb{C}(X)$, $\exists\, x \in X$ <u>such that</u> $R \succ \mathcal{O}_x$

(4) <u>for all varieties</u> Y, $Z \subset X\times Y$ <u>closed</u>, $pr_2 Z$ <u>is closed in</u> Y.

A subvariety of a variety X is an irreducible locally closed subset Y of X; the variety structure is given by the sheaf $\mathcal{O}_Y$ which assigns to any open subset V of Y the ring

$$\Gamma(V,\mathcal{O}_Y) = \left\{ \mathbb{C}\text{-valued functions } f \text{ on } V \;\middle|\; \begin{array}{l} \forall x \in V,\ \exists \text{ a neighborhood } U \\ \text{of } x \text{ in } X \text{ and a function} \\ f_1 \in \Gamma(U,\mathcal{O}_X) \text{ such that} \\ f = \text{restriction to } U\cap V \text{ of } f_1 \end{array} \right\}$$

So, any open subset of X is a subvariety; but a subvariety which is a complete variety must be closed.

Divisors and linear systems.

The theory of divisors is based on a fundamental result of Krull.

0.9. If R is a noetherian integrally closed integral domain, then

a) for all $\mathfrak{p} \subset R$, $\mathfrak{p}$ minimal prime ideal, $R_{\mathfrak{p}}$ is a discrete valuation ring.

b) $R = \bigcap_{\substack{\mathfrak{p} \text{ min.} \\ \text{prime}}} R_{\mathfrak{p}}$.

Thus if $\mathrm{ord}_{\mathfrak{p}}$ = valuation attached to $R_{\mathfrak{p}}$, and K is the fraction field of R, we get an exact sequence:

$$1 \longrightarrow R^* \longrightarrow K^* \longrightarrow \left[\begin{array}{l}\text{free abel. group} \\ \text{on min. prime ideals}\end{array}\right]$$

$$f \longmapsto \sum_{\mathfrak{p}} \mathrm{ord}_{\mathfrak{p}} f \cdot [\mathfrak{p}] = (f) \quad .$$

Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_n$ be the primes occurring positively in (f),

$\mathfrak{p}'_2, \cdots, \mathfrak{p}'_m$ " " " negatively in (f), then

Corollary 0.10. For all prime ideals $\mathfrak{p}$ in R,

$$f \in R_{\mathfrak{p}} \iff \mathfrak{p} \not\supset \mathfrak{p}'_i \quad \text{any } i$$

$$f^{-1} \in R_{\mathfrak{p}} \iff \mathfrak{p} \not\supset \mathfrak{p}_i \quad \text{any } i, \text{ hence}$$

$$\left(\begin{array}{l}\text{neither } f \text{ or } f^{-1} \text{ are in } R_{\mathfrak{p}} \\ \text{"}f \text{ is indeterminate at } \mathfrak{p}\text{"}\end{array}\right) \iff \mathfrak{p} \supset \mathfrak{p}_i + \mathfrak{p}'_j \quad \text{for some } i,j$$

(in particular, if f is indeterminate at $\mathfrak{p}$, then $\mathfrak{p}$ is not a minimal prime ideal).

We will apply Krull's result to the following geometrical situation:

<u>Theorem</u> 0.11: <u>If</u> $X = \bigcup X_\alpha$ <u>is a smooth variety, then</u> R_{X_α} <u>is integrally closed, the minimal primes</u> $\mathfrak{p}$ <u>in</u> R_{X_α} <u>are the codimension one (closed) subvarieties</u> Y <u>of</u> X <u>which meet</u> X_α , <u>and</u> $(R_{X_\alpha})_{\mathfrak{p}} = \mathcal{O}_{Y,X}$.

Idea of the proof: for all points $P \in X_\alpha$, the hypothesis of being smooth means $\dim \mathfrak{m}_P/\mathfrak{m}_P^2 = \dim X = \text{Krull-dim. } \mathcal{O}_P$, i.e., $\mathcal{O}_P$ is "regular" (this can be taken as a definition). One proves that a regular local ring is integrally closed, hence $\mathcal{O}_P$ is integrally closed. Since, for any affine variety, $R_{X_\alpha} = \bigcap_{P \in X_\alpha} \mathcal{O}_P$ [5), R_{X_α} is integrally closed. The rest of the statement follows from the:

<u>Lemma</u> 0.12. <u>A (closed) subvariety</u> Y <u>of</u> Z <u>is maximal</u> $\Longleftrightarrow$ $\dim Y = \dim Z-1$.

(This follows from (0.2), or else can be used to prove (0.2).)

Thus the map $f \longmapsto (f)$ defines a homomorphism

$$\mathbb{C}(X)^* \longrightarrow \text{Div } X = \begin{bmatrix} \text{free abel. group} \\ \text{on codim. 1 subvar.} \end{bmatrix}$$

Elements of Div X are called divisors on X and 2 divisors D_1, D_2 are called linearly equivalent (written $D_1 \equiv D_2$) if $D_1 - D_2 = (f)$, some $f \in \mathbb{C}(X)^*$.

The corollary 0.10 has the following geometrical meaning: for any $f \in \mathbb{C}(X)^*$, set $(f) = (f)_0 - (f)_\infty$ with $(f)_0$ (zero-divisor) and $(f)_\infty$ (pole-divisor) both positive divisors, and let, for any divisor $D = \sum n_i Y_i$, $\text{supp } D = \cup Y_i$; then

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5) If $x/y \in \bigcap_{P \in X_\alpha} \mathcal{O}_P$, consider the ideal $A = \{z \in R_{X_\alpha} \mid z \cdot \frac{x}{y} \in R_{X_\alpha}\}$; since $x/y \in \mathcal{O}_P$, x/y can be written w/z, with $w \in R_{X_\alpha}$, $z \in R_{X_\alpha} - M_P$, so $A \not\subset M_P$. Therefore A is not contained in any maximal ideal, so $A = R_{X_\alpha}$. This means that $1 \in A$, i.e., $\frac{x}{y} \in R_{X_\alpha}$.

$$f \in \mathcal{O}_P \iff P \notin \text{supp }(f)_\infty$$

$$f^{-1} \in \mathcal{O}_P \iff P \notin \text{supp }(f)_0$$

$$f \text{ is indeterminate at } P \iff P \in \text{supp }(f)_0 \cap \text{Supp }(f)_\infty.$$

Moreover, if X is a smooth affine variety of dimension 1 with affine ring R, then R is a Dedekind domain, so all its ideals are products of prime ideals. If $f \in R$, let:

$(f) = \Sigma n_i Y_i$ where Y_i corresponds to the prime ideal $\mathfrak{p}_i$ in R.

Then:

Corollary 0.14.

$$f \cdot R = \prod_i \mathfrak{p}_i^{n_i}.$$

We define $\text{Div}^+(X)$ to be the semi-group in Div(X) of divisors with only positive coefficients.

We define Pic(X) as the cokernel:

$$\mathbb{C}(X)^* \longrightarrow \text{Div } X \xrightarrow{\pi} \text{Pic}(X) \longrightarrow 0 ,$$

i.e., as the obstruction to finding rational functions with given zeroes and poles. Elements of Pic(X) are called divisor classes.

Example. $\text{Pic }(\mathbb{P}^n) = \mathbb{Z}$. In fact, any hypersurface is given by the zeroes of a homogeneous polynomial. The degree of a divisor $D = \Sigma n_i Y_i$ is defined by $\deg D = \Sigma n_i \deg Y_i$ where $\deg Y_i$ is the degree of the irreducible homogeneous polynomial defining it. Then any divisor of degree zero comes from a rational function, and degree gives an isomorphism $\text{Pic}(\mathbb{P}^n) \xrightarrow{\sim} \mathbb{Z}$.

Suppose D is a positive divisor; we define the vector space

$$\mathcal{L}(D) = \{f \in \mathbb{C}(X)^* \mid (f)+D \geq 0\} \cup \{0\},$$

Note: The condition $(f)+D \geq 0$ is equivalent to $(f)_\infty \leq D$ (the poles of f are bounded by D). Note that $\mathcal{L}(D)$ is a sub-vector space of $\mathbb{C}(X)$.

Lemma 0.15. If X is proper, $\dim \mathcal{L}(D) < \infty$ and for all $f \in \mathbb{C}(X)^*$ $(f) = 0$ if and only if $f \in \mathbb{C}^*$.

In this case, we form the associated projective space $\mathbb{P}(\mathcal{L}(D))$ of one-dimensional subspaces of $\mathcal{L}(D)$ and note:

$$\mathbb{P}(\mathcal{L}(D)) \cong \left[\pi^{-1}(\pi D) \cap \mathrm{Div}^+(X)\right] = \left[\begin{array}{l}\text{fibre through D of} \\ \mathrm{Div}^+(X) \xrightarrow{\pi} \mathrm{Pic}(X)\end{array}\right]$$

$$\begin{array}{ccc} \mathbb{P}(\mathcal{L}(D)) & & \pi^{-1}(\pi D) \cap \mathrm{Div}^+(X) \\ \Psi & & \Psi \\ \text{line}\{\alpha.f \mid \alpha \in k\} & \longleftrightarrow & \text{divisor } (f)+D \end{array}$$

These projective spaces and their linear subspaces are the so-called "linear systems" of divisors. $\mathbb{P}(\mathcal{L}(D))$ is denoted $|D|$.

If $L \subset |D|$ is a linear subspace of dimension k, set

$$B(L) = \bigcap_{E \in L} \mathrm{Supp}\, E, \text{ the "base locus" of } L.$$

The fundamental construction associated to linear systems is the map

$$\varphi_L : (X-B(L)) \longrightarrow L^\vee,$$

where $L^\vee$ is the projective space of hyperplanes in L, given by

$$x \longmapsto [\text{hyperplane in L consisting of the } E \in L \text{ s.t. } x \in \mathrm{Supp}\, E]$$

φ_L is a morphism. To prove this and to describe φ_L explicitly, let's choose a projective basis of L, i.e., k+1 points which are not contained in a hyperplane:

$$E,\ E+(f_1), E+(f_2), \cdots, E+(f_k).$$

Set $f_o = 1$; the map

$$x \longmapsto (f_o(x), \cdots, f_k(x))$$

is defined on the open set X-Supp E since the poles of f_i are all contained in Supp E; it coincides with φ_L on X-Supp E, as we see if we let coordinates on L be $c_o, \cdots, c_k$ and note: for $x \notin$ Supp E,

$$x \in \text{Supp}\left(E + \left(\sum_{i=0}^{k} c_i f_i\right)\right) \iff \sum_{i=0}^{k} c_i\ f_i(x) = 0.$$

hence $\varphi_L(x)$ = hyperplane in L with coefficients $f_0(x), --, f_k(x)$

= pt. of $L^\vee$ with homogeneous coordinates $f_0(x), ---, f_k(x)$.

§1. Divisors on hyperelliptic curves.

Given a finite number of distinct elements $a_i \in \mathbb{C}$, $i \in S$, let $f(t) = \prod_{i \in S}(t-a_i)$. We form the plane curve C_1 defined by the equation

$$s^2 = f(t).$$

The polynomial $s^2-f(t)$ is irreducible, so $(s^2-f(t))$ is a prime ideal, and C_1 is a 1-dimensional affine variety in $\mathbb{C}^2$. In fact, C_1 is smooth. To prove this, we will calculate the dimension of the Zariski-tangent space at each point, i.e., the space of solutions $(\dot{s},\dot{t}) \in \mathbb{C}^2$ to the equation

$$(s+\varepsilon\dot{s})^2 = \prod(t+\varepsilon\dot{t}-a_i) \bmod \varepsilon^2 \qquad \text{for } (s,t) \in C_1.$$

That is equivalent to the equation

$$2s\dot{s} = \dot{t}\cdot\sum_{j \in S}\prod_{i \neq j}(t-a_i);$$

if $s \neq 0$, the solutions are all linearly dependent since $\dot{s} = \frac{\dot{t}}{2s}\left(\sum_{j \in S}\prod_{i \neq j}(t-a_i)\right)$; if $s = 0$, we get from the equation of the curve $\prod_{i \in S}(t-a_i) = 0$, hence $t = a_i$ for some i; thus $0 = \dot{t}\cdot\prod_{j \neq i}(a_i-a_j)$, so $\dot{t} = 0$. Thus at all points, the Zariski tangent space is one-dimensional.

We add points at infinity by introducing a second chart:

$$C_2: \qquad s'^2 = \prod_{i \in S}(1-a_it') \qquad \text{if } \#S = 2k$$

$$s'^2 = t'\cdot\prod_{i \in S}(1-a_it') \qquad \text{if } \#S = 2k-1,$$

glued by the isomorphism $t' = \frac{1}{t}$

$$s' = \frac{s}{t^k}$$

between the open sets $t \neq 0$ of C_1 and $t' \neq 0$ of C_2.
The points at ∞ of C_1 are:

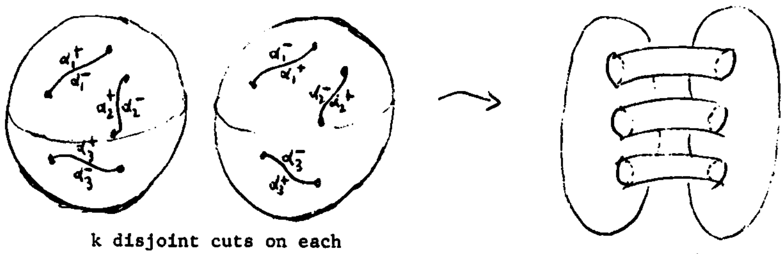
∞_1, ∞_2 given by $t' = 0$, $s' = \pm 1$ if #S even

∞ " " $t' = 0 = s'$ if #S odd.

On the resulting variety C we can define a morphism $\pi: C \longrightarrow \mathbb{P}^1$. Let t and $t' = 1/t$ be affine coordinates in $\mathbb{P}^1$; then define π by

$$(s,t) \longmapsto t, \text{ on the chart } C_1,$$
$$(s',t') \longmapsto t', \text{ on the chart } C_2.$$

π is 2:1 except over the set B of the "branch points" consisting in the a_i's, and ∞ in the case #S odd. The number of branch points is therefore an even number 2k in both cases. Topologically C is a surface with k-1 handles, so we say that it is of genus $g = k-1$; this is called the genus of the curve. This is usually visualized by defining 2 continuous functions $+\sqrt{f(t)}$, $-\sqrt{f(t)}$ for $t \in \mathbb{P}^1-$(k"cuts") and reconstructing C by glueing the 2 open pieces of C defined by $s = +\sqrt{f(t)}$ and $s = -\sqrt{f(t)}$:



k disjoint cuts on each copy of $\mathbb{P}^1$

glueing α_i^+'s, α_i^-'s

Since C is smooth, the affine rings of C_1 and C_2 are Dedekind domains[1)], and their local rings $\mathcal{O}_x$ are discrete valuation rings.

$$\iota: \quad (s,t) \longmapsto (-s,t)$$

is an automorphism of C, that flips the sheets of the covering, hence is an involution, with the set of orbits $C/\{\pm 1\} \cong \mathbb{P}^1$. $\pi^{-1}(B)$ is the set of fixed points of ι .

We want to prove that C is actually a projective variety.

Let

$$D = \begin{cases} k(\infty_1 + \infty_2) & \text{if } \#S \text{ is even} \\ 2k\infty & \text{if } \#S \text{ is odd.} \end{cases}$$

Lemma 1.1. $1,t,t^2,\cdots,t^k,s$ is a basis for the vector space $\mathcal{L}(D)$.

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1) We already know that the tangent space to the curve at each point has the right dimension, in each of the two affine pieces; but it's also easy to see directly that $\mathbb{C}[t,s]/(s^2-\Pi(t-a_i))=R$ is integrally closed, the reason being that $\Pi(t-a_i)$ is a square-free discriminant over the U.F.D. $\mathbb{C}[t]$. If we let σ be the automorphism which sends (s,t) to $(-s,t)$, then the general element of the quotient field of R is $a+bs$, with $a,b\in\mathbb{C}(t)$, and for all $a+bs$ integral over R, $(a+bs)+\sigma(a+bs)=2a$ and $(a+bs)\cdot\sigma(a+bs) = a^2-b^2d$ are in $\mathbb{C}(t)$ and are integral over $\mathbb{C}[t]$, which is integrally closed. Thus $2a\in\mathbb{C}[t]$, $a^2-db^2\in\mathbb{C}[t]$, so $db^2\in\mathbb{C}[t]$; since d is square-free and $\mathbb{C}[t]$ is U.F.D. we conclude $b\in\mathbb{C}[t]$, hence $a+bs\in R$.

Proof: The function field of X, $\mathbb{C}(t)[\sqrt{\Pi(t-a_i)}]$, has an involution over $\mathbb{C}(t)$, that interchanges ∞_1, ∞_2, or fixes the point ∞, hence sends $\mathcal{L}(D)$ into itself. Thus $\mathcal{L}(D)$ splits into the sum of the +1 and -1 eigenspaces of ι,

$$\mathcal{L}(D) = [\mathcal{L}(D) \cap \mathbb{C}(t)] \oplus [\mathcal{L}(D) \cap s\mathbb{C}(t)].$$

If $h(t) \in \mathcal{L}(D) \cap \mathbb{C}(t)$, since it has no poles for finite values of t, then it must be a polynomial in t, $h \in \mathbb{C}[t]$. On the other hand, in the case #S even, the maximum ideal of $\mathcal{O}_{\infty_1} = R(C_2)_{(t',s'-1)}$ is generated by t' since the equation of the curve gives $s'-1 = ((s'+1)^{-1}\cdot(\Pi(1-t'a_i)-1) \in (t')R(C_2)_{(t',s'-1)}$; in the case #S odd the max. ideal of $\mathcal{O}_\infty = R(C_2)_{(t',s')}$ is generated by s', since $t' = s'^2\ (\Pi(1-t'a_i))^{-1}$; thus

$$v_{\infty_1}(t') = -v_{\infty_1}(t) = 1 \text{ (similarly } v_{\infty_2}(t') = 1), \text{ or } v_\infty(t') = -v_\infty(t) = 2,$$

$$\text{i.e., } (t)_\infty = \begin{cases} \infty_1 + \infty_2, & \#S \text{ even} \\ 2\infty, & \#S \text{ odd} \end{cases}$$

So in order for $(h)_\infty$ to be $\leq D$ we must have $\deg h \leq k$.

Now consider $h(t) \in \mathcal{L}(D) \cap s\mathbb{C}(t)$, $h = sg(t)$ with $g(t) \in \mathbb{C}(t)$; g may only have poles in C_1 where s has zeroes, i.e., in the set $\{P_i = (0,a_i)\}$. The order of vanishing of s at P_i is 1 and that of $(t-a_i)$ is 2, since the max. ideal in $\mathcal{O}_{P_i}$ is generated by s and $(t-a_i)$ and $(t-a_i) = s^2\cdot(\prod_{j\neq i}(t-a_j))^{-1}$. That prevents $g(t) \in \mathbb{C}(t)$

from having a pole at P_i, because the product sg(t) would still have a pole at P_i. Thus the only poles of g(t) must be at ∞, i.e., g(t) must be a polynomial in t; but now

$$(s)_\infty = (s't^k)_\infty = \begin{cases} k(\infty_1+\infty_2) & \#S \text{ even} \\ (2k-1)\infty & \#S \text{ odd} \end{cases}$$

hence g(t) must be constant in order to have $D+(sg(t)) \geq 0$. This proves the lemma.

Now a projective base for $|D|$ is $D, D+(t), \cdots, D+(t^k)$, $D+(s)$; since $(t)_\infty = \begin{cases} \infty_1+\infty_2 \\ 2\infty \end{cases}$ we have $D+(t^k) = k\cdot(t)_0$ = either $k(O_1+O_2)$ or $2kO$

where O is a branch point in the 2nd case, and where O_1, O_2 are the two points in the fiber over the point $t = 0$ of $\mathbb{P}^1$ in the 1st case. Hence $|D|$ contains D and $k\cdot(t)_0$ whose supports are disjoint, hence $B|D|$ is empty.

Thus explicitly, corresponding to $|D|$, we get $\varphi_{|D|}: C \longrightarrow \mathbb{P}^{k+1}$

by $(s,t) \longmapsto (1,t,t^2,\cdots,t^k,s)$ on X - Supp $D = C_1$,

and $(s',t') \longmapsto (t'^k,\ldots\ldots,t',1,s')$ on C_2.

Note that these 2 maps do agree on the overlap: $(1,t,t^2,\cdots t^k,s) \sim \frac{1}{t^k}(1,t,\ldots t^k,s) = (t'^k,\cdots,t',1,s')$.

This map, which is an isomorphism of C with its image, makes C into a projective curve.