Computational Methods in Applied Sciences

Manolis Papadrakakis
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# Computational Methods in Stochastic Dynamics

Volume 2

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# Computational Methods in Stochastic Dynamics

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*This book is dedicated to the memory of Prof. Gerhart I. Schuëller, a pioneer in the field of Computational Stochastic Dynamics*

# Preface

The considerable influence of inherent uncertainties on structural behavior has led the engineering community to recognize the importance of a stochastic approach to structural problems. Issues related to uncertainty quantification and its influence on the reliability of the computational models, are continuously gaining in significance. In particular, the problems of dynamic response analysis and reliability assessment of structures with uncertain system and excitation parameters have been the subject of continuous research over the last two decades as a result of the increasing availability of powerful computing resources and technology. This book is a follow up of a previous book with the same subject and focuses on advanced computational methods and software tools which can highly assist in tackling complex problems in stochastic dynamic/seismic analysis and design of structures. The selected chapters are authored by some of the most active scholars in their respective areas and represent some of the most recent developments in this field.

This edited book is primarily intended for researchers and post-graduate students who are familiar with the fundamentals and wish to study or to advance the state of the art on a particular topic in the field of computational stochastic structural dynamics. Nevertheless, practicing engineers could benefit as well from it as most code provisions tend to incorporate probabilistic concepts in the analysis and design of structures. The book consists of 21 chapters which are extended versions of papers presented at the recent COMPDYN 2011 Conference. The chapters can be grouped into several thematic topics including dynamic analysis of stochastic systems, reliability-based design, structural control and health monitoring, model updating, system identification, wave propagation in random media, seismic fragility analysis and damage assessment.

In Chap. 1, A. Batou and C. Soize examine the random dynamic response of a multibody system with uncertain rigid bodies. A stochastic model of an uncertain rigid body is constructed by modeling the mass, the center of mass and the tensor of inertia by random variables. The prior probability distributions of these random variables are computed using the maximum entropy principle under the constraints defined by the available information. Several uncertain rigid bodies are linked to each other in order to calculate the random response of a multibody dynamic system.

A numerical application consisting of five rigid bodies is proposed to illustrate the theoretical developments.

In Chap. 2, V. Papadopoulos and O. Kokkinos extend the concept of Variability Response Functions (VRFs) to linear stochastic systems under dynamic excitation. An integral form for the variance of the dynamic response of stochastic systems is considered, involving a Dynamic VRF (DVRF) and the spectral density function of the stochastic field modeling the uncertain system properties. The uncertain property considered is the flexibility of the system. The same integral expression can be used to calculate the mean response of a dynamic system using a Dynamic Mean Response Function (DMRF) which is a function similar to the DVRF. These integral forms are used to efficiently compute the mean and variance of the transient system response along with time dependent spectral-distribution-free upper bounds.

A. Kundu and S. Adhikari provide the theoretical development and simulation results of a novel Galerkin subspace projection scheme for damped linear dynamic systems with stochastic coefficients and homogeneous Dirichlet boundary conditions (Chap. 3). The fundamental idea is to solve the stochastic dynamic system in the frequency domain by projecting the solution into a reduced finite dimensional spatio-random vector basis spanning the stochastic Krylov subspace to approximate the response. Galerkin weighting coefficients are employed to minimize the error induced by the use of the reduced basis. The statistical moments of the solution are evaluated at all frequencies to illustrate and compare the stochastic system response with the deterministic case. The results are validated with direct Monte Carlo simulation for different correlation lengths and variability of randomness.

An efficient approach for modeling nonlinear systems subjected to general non-Gaussian excitations is developed by X.F. Xu and G. Stefanou in Chap. 4. This chapter describes the formulation of an $n$-th order convolved orthogonal expansion (COE) method. For linear vibration systems, the statistics of the output are directly obtained as the first-order COE about the underlying Gaussian process. The COE method is next verified by its application on a weakly nonlinear oscillator. In dealing with strongly nonlinear dynamics problems, a variational method is presented by formulating a convolution-type action and using the COE representation as trial functions.

In Chap. 5 by L. Pichler et al., various finite difference (FD) and finite element methods (FEM) are discussed for the numerical solution of the Fokker–Planck equation allowing the investigation of the evolution of the probability density function of linear and nonlinear systems. The results are compared using various numerical examples. Despite the greater numerical effort, the FEM is preferable over FD, because it yields more accurate results. However, at this moment the FEM is only suitable for dimension less or equal to 3. In the case of 3D and 4D problems, a stabilized multi-scale FEM provides a tool with a high order of accuracy, preserving numerical efficiency due to the fact that a coarser mesh size can be used.

There are various approaches to deal with uncertainty propagation in stochastic dynamics. In Chap. 6, M. Corradi et al. examine some classical structural problems in order to investigate which probabilistic approach better propagates the uncertainty from input to output, in terms of accuracy and computational cost. The examined methods are: Univariate Dimension Reduction methods, Polynomial Chaos Expansion, First-Order Second Moment method, and algorithms based on the Evidence Theory for epistemic uncertainty. The performances of these methods are compared in terms of moment estimations and probability density function construction corresponding to several scenarios of reliability-based design and robust design. The structural problems examined are: (i) the static, dynamic and buckling behavior of a composite plate, (ii) the reconstruction of the deformed shape of a beam from measured surface strains.

Chapter 7 by F. Steinigen et al. is devoted to enhanced computational algorithms to simulate the load-bearing behavior of reinforced concrete structures under dynamic loading. In order to take into account uncertain data of reinforced concrete, fuzzy and fuzzy stochastic analyses are presented. The capability of the fuzzy dynamic analysis is demonstrated by an example in which a steel bracing system and viscous damping connectors are designed to enhance the structural resistance of a reinforced concrete structure under seismic loading.

W. Verhaeghe et al. use the concept of interval fields to deal with uncertainties of spatial character arising in the context of groundwater transport models needed to predict the flow of contaminants (Chap. 8). The main focus of the chapter is on the application of interval fields to a geo-hydrological problem. The uncertainty taken into account is the material layers' hydraulic conductivity. The results presented are the uncertainties on the contaminant's concentration near a river. Another objective of the chapter is to define an input uncertainty elasticity of the output, i.e. to identify the locations in the model, whose uncertainties mostly influence the uncertainty on the output. Such a quantity indicates where to perform additional in situ point measurements to reduce the uncertainty on the output the most.

Although reliability analysis methods have matured in recent years, the problem of reliability-based structural design still poses a challenge in stochastic dynamics. In Chap. 9, A. Naess et al. extend their recently developed enhanced Monte Carlo approach to the problem of reliability-based design. The objective is to optimize a design parameter $\alpha$ so that the system, represented by a set of failure modes or limit states, achieves a target reliability. Monte Carlo sampling occurs at a range of values for $\alpha$ that result in failure probabilities larger than the target and thus the design problem essentially amounts to a statistical estimation of a high quantile. Several examples of the approach are provided in the chapter.

Chapter 10 by H. Jensen et al. presents a general framework for reliability-based design of base-isolated structural systems under uncertain conditions. The uncertainties about the structural parameters as well as the variability of future excitations are characterized in a probabilistic manner. Nonlinear elements composed by hysteretic devices are used for the isolation system. The optimal design problem is formulated as a constrained minimization problem which is solved by a sequential approximate optimization scheme. First excursion probabilities that account for the uncertainties in the system parameters as well as in the excitation are used to characterize the system reliability. The approach explicitly takes into account all nonlinear characteristics of the combined structural system (superstructure-isolation system) during the design process. Numerical results highlight the beneficial effects of isolation systems in reducing the superstructure response.

The influence of structural uncertainties on actively controlled smart beams is investigated in Chap. 11 by A. Moutsopoulou et al. The dynamical problem of a model smart composite beam is treated using a simplified modeling of the actuators and sensors, both being realized by means of piezoelectric layers. In particular, a practical robust controller design methodology is developed, which is based on recent theoretical results on $H_\infty$ control theory and $\mu$-analysis. Numerical examples demonstrate the vibration-suppression property of the proposed smart beams under stochastic loading.

The field of Structural Health Monitoring (SHM) has significantly evolved in the last years due to the technological advances and the evolution of advanced smart systems for damage detection and signal processing. In Chap. 12, G. Saad and R. Ghanem present a robust data assimilation approach based on a stochastic variation of the Kalman Filter where polynomial functions of random variables are used to represent the uncertainties inherent to the SHM process. The presented methodology is combined with a non-parametric modeling technique to tackle SHM of a four-story shear building subjected to a base motion consistent with the El-Centro earthquake and undergoing a preset damage in the first floor. The purpose of the problem is localizing the damage in both space and time, and tracking the state of the system throughout and subsequent to the damage time. The application of the introduced data assimilation technique to SHM enhances its applicability to a wide range of structural problems with strongly nonlinear dynamic behavior and with uncertain and complex governing laws.

The accurate prediction of the response of spacecraft systems during launch and ascent phase is a crucial aspect in design and verification stages which requires accurate numerical models. The enhancement of numerical models based on experimental data is denoted model updating and focuses on the improvement of the correlation between finite element (FE) model and test structure. In aerospace industry, the examination of the agreement between model and real structure involves the comparison of the modal properties of the structure. Chapter 13 by B. Goller et al. is devoted to the efficient model updating of a satellite in a Bayesian setting based on experimental modal data. A detailed FE model of the satellite is used for demonstrating the applicability of the employed updating procedure to large-scale complex aerospace structures.

In Chap. 14, B. Rosič and H. Matthies deal with the identification of properties of stochastic elastoplastic systems in a Bayesian setting. The inverse problem is formulated in a probabilistic framework where the unknown uncertain quantities are embedded in the form of their probability distributions. With the help of stochastic functional analysis, a new update procedure is introduced as a direct, purely algebraic way of computing the posterior, which is comparatively inexpensive to evaluate. Such description requires the solution of the convex minimization problem in a stochastic setting for which the extension of the classical optimization algorithm in predictor-corrector form is proposed as the solution procedure. The identification method is finally validated through a series of virtual experiments taking into account the influence of the measurement error and the order of the approximation on the posterior estimate.

Chapter 15 deals with the study of SH surface waves in a half space with random heterogeneities. C. Du and X. Su prove both theoretically and numerically that surface waves exist in a half space which has small, random density, but the mean value of the density is homogeneous. Historically, this type of half space is often treated as homogeneous using deterministic methods. In this investigation, a closed-form dispersion equation is derived stochastically and the frequency spectrum, dispersion equation and phase/group velocity are computed numerically to study how the random inhomogeneities will affect the dispersion properties of the half space with random density. The results of this research may find their application in various fields, such as in seismology and in non-destructive test/evaluation of structures with randomly distributed micro-cracks or heterogeneities.

The following six chapters are devoted to earthquake engineering applications. P. Jehel et al. (Chap. 16) investigate the seismic fragility of a moment-resisting reinforced concrete frame structure in the area of the Cascadia subduction zone situated in the South-West of Canada and the North-West of the USA. According to shaking table tests, the authors first validate the capability of an inelastic fiber beam/column element, using a recently developed concrete constitutive law, for representing the seismic behavior of the tested frame coupled to either a commonly used Rayleigh damping model or a proposed new model. Then, for each of the two damping models, they perform a structural fragility analysis and investigate the amount of uncertainty to be induced by damping models.

In Chap. 17 by Y. Vargas et al., a detailed study of the seismic response of a reinforced concrete building is conducted using a probabilistic approach in the framework of Monte Carlo simulation. The building is representative for office buildings in Spain but the procedures used and the results obtained can be extended to other types of buildings. The purpose of the work is twofold: (i) to analyze the differences when static and dynamic analysis techniques are used and (ii) to obtain a measure of the uncertainties involved in the assessment of structural vulnerability. The results show that static procedures are somehow conservative and that uncertainties increase with the severity of the seismic actions and with the damage. Low damage state fragility curves have little uncertainty while high damage state fragility curves show great scattering.

Seismic pounding can induce severe damage and losses in buildings. The corresponding risk is particularly relevant in densely inhabited metropolitan areas, due to the inadequate clearance between buildings. Chapter 18 by E. Tubaldi and M. Barbato proposes a reliability-based procedure for assessing the level of safety corresponding to a given value of the separation distance between adjacent buildings exhibiting linear elastic behavior. The seismic input is modeled as a non-stationary random process and the first-passage reliability problem corresponding to the pounding event is solved employing analytical techniques involving the determination of specific statistics of the response processes. The proposed procedure is applied to estimate the probability of pounding between linear single-degree-of-freedom systems and to evaluate the reliability of simplified design code formulae used to determine building separation distances. Furthermore, the capability of the proposed method to deal with complex systems is demonstrated by assessing the effectiveness of the use of viscous dampers in reducing the probability of pounding between adjacent buildings modeled as multi-degree-of-freedom systems.

In Chap. 19, A. Elenas provides a methodology to quantify the relationship between seismic intensity parameters and structural damage. First, a computer-supported elaboration of ground motion records provides several peak, spectral and energy seismic parameters. After that, nonlinear dynamic analyses are carried out to provide the structural response for a set of seismic excitations. Among the several response characteristics, the overall structure damage indices after Park/Ang and the maximum inter-storey drift ratio are selected to represent the structural response. Correlation coefficients are evaluated to express the grade of interrelation between seismic acceleration parameters and structural damage. The presented methodology is applied to a reinforced concrete frame building, designed according to the rules of the recent Eurocodes, and the numerical results show that the spectral and energy parameters provide strong correlation to the damage indices.

As demonstrated in the previous chapter, there is interdependence between seismic intensity parameters and structural damage. In Chap. 20, A. Elenas et al. proceed to the classification of seismic damage in buildings using an adaptive neuro-fuzzy inference system. The seismic excitations are simulated by artificial accelerograms and their intensity is described by seismic parameters. The proposed system is trained using a number of seismic events and tested on a reinforced concrete structure. The results show that the proposed fuzzy technique contributes to the development of an efficient blind prediction of seismic damage. The recognition scheme achieves correct classification rates over 90%.

The book closes with a study on damage identification of historical masonry structures under seismic excitation by G. De Matteis et al. (Chap. 21). The seismic behavior of a physical 1:5.5 scaled model of the church of the Fossanova Abbey (Italy) is examined by means of numerical and experimental analyses. As it mostly influences the seismic vulnerability of the Abbey, the central transversal three-bay complex of the church was investigated in detail by means of a shaking table test on a 1:5.5 scaled physical model in the Laboratory of the Institute for Earthquake Engineering and Engineering Seismology in Skopje. In this chapter, a brief review of the numerical activity related to the prediction of the shaking table test response of the model is first proposed. Then, the identification of frequency decay during collapse is performed through decomposition of the measured power spectral density matrix. Finally, the localization and evolution of damage in the structure is analyzed and the obtained numerical results show a very good agreement with the experimental data.

The book editors would like to express their deep appreciation to all contributors for their active participation in the COMPDYN 2011 Conference and for the time and effort devoted to the completion of their contributions to this volume. Special thanks are also due to the reviewers for their constructive comments and suggestions

which enhanced the quality of the book. Finally, the editors would like to thank the personnel of Springer for their most valuable support during the publication process.

Athens, Greece
April 2012

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George Stefanou
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# Contents

# Chapter 1
# Random Dynamical Response of a Multibody System with Uncertain Rigid Bodies

**Anas Batou and Christian Soize**

**Abstract** This work is devoted to the construction of the random dynamical response of a multibody system with uncertain rigid bodies. We construct a stochastic model of an uncertain rigid body by modeling the mass, the center of mass and the tensor of inertia by random variables. The prior probability distributions of these random variables are constructed using the maximum entropy principle under the constraints defined by the available information. A generator of independent realizations are then developed. Several uncertain rigid bodies can be linked each to the others in order to calculate the random response of a multibody dynamical system. An application is proposed to illustrate the theoretical development.

## 1 Introduction

This work is devoted to the construction of a probabilistic model of uncertainties for a rigid multibody dynamical system made up of uncertain rigid bodies. In some cases, the mass distribution inside a rigid body is not perfectly known and must be considered as random (for example, the distribution of passengers inside a vehicle) and therefore, this unknown mass distribution inside the rigid body induces uncertainties in the model of this rigid body. Here, we propose a new probabilistic modeling for uncertain rigid bodies in the context of the multibody dynamics. Concerning the modeling of uncertainties in multibody dynamical system, a very few previous researches have been carried out. These researches concerned parameters which describe the joints linking each rigid body to the others and the external sources (see [3, 8, 12, 13, 16]), but not rigid bodies themselves. In the field of uncertain rigid bodies, a first work has been proposed in [9, 10], in which the authors take into account uncertain rigid bodies for rotor dynamical systems using the nonparamet-

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ric probabilistic approach [19, 20] consisting in replacing the mass and gyroscopic matrices by random matrices.

In this paper, a general and complete stochastic model is constructed for an uncertain rigid body. The mass, the center of mass and the tensor of inertia which describe the rigid body are modeled by random variables. The prior probability distributions of the random variables are constructed using the maximum entropy principle [6, 7] from Information Theory [17, 18]. The generator of independent realizations corresponding to the prior probability distributions of these random quantities are developed and presented. Then, several uncertain rigid bodies can be linked each to the others in order to calculate the random response of an uncertain multibody dynamical system. The stochastic multibody dynamical equations are solved using the Monte Carlo simulation method.

Section 2 is devoted to the construction of the nominal model for the rigid multibody dynamical system by using the classical method. In Sect. 3, we propose a general probability model for an unconstrained uncertain rigid body and then, the uncertain rigid multibody dynamical system is obtained by joining this unconstrained uncertain rigid body to the other rigid bodies. The last section is devoted to an application which illustrates the proposed theory.

## 2 Nominal Model for the Rigid Multibody Dynamical System

In this paper, the usual model of a rigid multibody dynamical system for which all the mechanical properties are known will be called the nominal model. This section is devoted to the construction of the nominal model for a rigid multibody dynamical system. This nominal model is constructed as in [14, 15] and is summarized below.

### *2.1 Dynamical Equations for a Rigid Body of the Multibody System*

Let $\mathrm{RB}_i$ be the rigid body occupying a bounded domain $\Omega_i$ with a given geometry. Let $\xi$ be the generic point of the three dimensional space (see Fig. 1.1). Let $\mathbf{x} = (x_1, x_2, x_3)$ be the position vector of point $\xi$ defined in a fixed inertial frame $(O, x_{0,1}, x_{0,2}, x_{0,3})$, such that $\mathbf{x} = \overrightarrow{O\xi}$. A rigid body is classically defined by three quantities.

1. The first one is the mass $m_i$ of $\mathrm{RB}_i$ which is such that

$$m_i = \int_{\Omega_i} \rho(\mathbf{x})\, d\mathbf{x}, \tag{1.1}$$

   where $\rho(\mathbf{x})$ is the mass density.

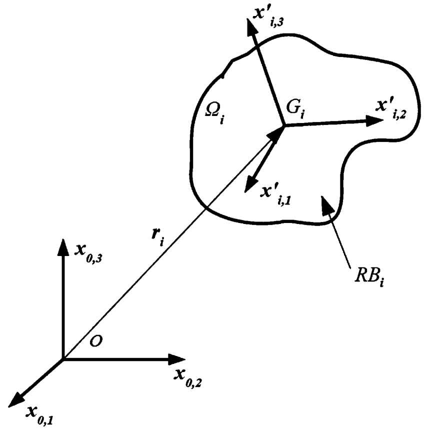


**Fig. 1.1** Rigid body $\mathrm{RB}_i$

2. The second quantity is the position vector $\mathbf{r}_i$ of the center of mass $G_i$, defined in the fixed inertial frame, by

$$\mathbf{r}_i = \frac{1}{m_i} \int_{\Omega_i} \mathbf{x} \rho(\mathbf{x})\, d\mathbf{x}. \tag{1.2}$$

3. Let $(G_i, x'_{i,1}, x'_{i,2}, x'_{i,3})$ be the local frame for which the origin is $G_i$ and which is deduced from the fixed frame $(O, x_{0,1}, x_{0,2}, x_{0,3})$ by the translation $\overrightarrow{OG_i}$ and a rotation defined by the three Euler angles $\alpha_i$, $\beta_i$ and $\gamma_i$. The third quantity is the positive-definite matrix $[J_i]$ of the tensor of inertia in the local frame such that

$$[J_i]\mathbf{u} = -\int_{\Omega_i} \mathbf{x}' \times \mathbf{x}' \times \mathbf{u} \rho\big(\mathbf{x}'\big)\, d\mathbf{x}', \quad \forall \mathbf{u} \in \mathbb{R}^3, \tag{1.3}$$

in which the vector $\mathbf{x}' = (x'_1, x'_2, x'_3)$ of the components of vector $\overrightarrow{G_i \xi}$ are given in $(G_i, x'_{i,1}, x'_{i,2}, x'_{i,3})$. In the above equation, $\mathbf{u} \times \mathbf{v}$ denotes the cross product between the vectors $\mathbf{u}$ and $\mathbf{v}$.

## *2.2 Matrix Model for the Rigid Multibody Dynamical System*

The rigid multibody dynamical system is made up of $n_b$ rigid bodies and ideal joints including rigid joints, joints with given motion (rheonomic constraints) and vanishing joints (free motion). The interactions between the rigid bodies are realized by these ideal joints but also by springs, dampers or actuators which produce forces between the bodies. In this paper, only $n_c$ holonomic constraints are considered. Let $\mathbf{u}$ be the vector in $\mathbb{R}^{6n_b}$ such that $\mathbf{u} = (\mathbf{r}_1, \ldots, \mathbf{r}_{n_b}, \mathbf{s}_1, \ldots, \mathbf{s}_{n_b})$ in which $\mathbf{s}_i = (\alpha_i, \beta_i, \gamma_i)$ is the rotation vector. The $n_c$ constraints are given by $n_c$ implicit equations which are globally written as $\boldsymbol{\varphi}(\mathbf{u}, t) = 0$. The $(6n_b \times 6n_b)$ mass matrix $[M]$ is defined by

$$[M] = \begin{bmatrix} [M^r] & 0 \\ 0 & [M^s] \end{bmatrix}, \tag{1.4}$$

where the $(3n_b \times 3n_b)$ matrices $[M^r]$ and $[M^s]$ are defined by

$$[M^r] = \begin{bmatrix} m_1[I_3] & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & m_{n_b}[I_3] \end{bmatrix}, \qquad [M^s] = \begin{bmatrix} [J_1] & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & [J_{n_b}] \end{bmatrix}, \tag{1.5}$$

in which $[I_3]$ is the $(3 \times 3)$ identity matrix. The function $\{\mathbf{u}(t) \in [0, T]\}$ is then the solution of the following differential equation (see [15])

$$\begin{bmatrix} [M] & [\boldsymbol{\varphi}_{\mathbf{u}}]^T \\ [\boldsymbol{\varphi}_{\mathbf{u}}] & [0] \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{q} - \mathbf{k} \\ -\frac{d}{dt}\boldsymbol{\varphi}_t - [\frac{d}{dt}\boldsymbol{\varphi}_{\mathbf{u}}]\dot{\mathbf{u}} \end{bmatrix}, \tag{1.6}$$

with the initial conditions

$$\mathbf{u}(0) = \mathbf{u}_0, \qquad \dot{\mathbf{u}}(0) = \mathbf{v}_0, \tag{1.7}$$

in which $\mathbf{k}(\dot{\mathbf{u}})$ is the vector of the Coriolis forces and where $[\boldsymbol{\varphi}_{\mathbf{u}}(\mathbf{u}(t), t)]_{ij} = \partial\varphi_i(\mathbf{u}(t), t)/\partial u_j(t)$ and $\boldsymbol{\varphi}_t = \partial\boldsymbol{\varphi}/\partial t$. The vector $\mathbf{q}(\mathbf{u}, \dot{\mathbf{u}}, t)$ is constituted of the applied forces and torques induced by springs, dampers and actuators. The vector $\boldsymbol{\lambda}(t)$ is the vector of the Lagrange multipliers. Equation (1.6) can be solved using an adapted integration algorithm (see for instance [2]).

## 3 Stochastic Model for a Multibody Dynamical System with Uncertain Rigid Bodies

Firstly, a stochastic model for an uncertain rigid body of the multibody dynamical system is proposed and secondly, the stochastic model for the multibody dynamical system with uncertain rigid bodies is constructed joining the stochastic model of the uncertain rigid bodies.

### *3.1 Stochastic Model for an Uncertain Rigid Body of the Multibody Dynamical System*

The properties of the nominal model of the rigid body $\mathrm{RB}_i$ are defined by its mass $\underline{m}_i$, the position vector $\underline{\mathbf{r}}_{0,i}$ of its center of mass $\underline{G}_i$ at initial time $t = 0$ and the matrix $[\underline{J}_i]$ of its tensor of inertia with respect to the local frame $(\underline{G}_i, \underline{x}'_{i,1}, \underline{x}'_{i,2}, \underline{x}'_{i,3})$. The probabilistic model of uncertainties for this rigid body is constructed by replacing these three parameters by the following three random variables: the random mass $M_i$, the random position vector $\mathbf{R}_{0,i}$ of its random center of mass $\mathbf{G}_i$ at initial time $t = 0$ and the random matrix $[\mathbf{J}_i]$ of its random tensor of inertia with respect to the random local frame $(\mathbf{G}_i, \underline{x}'_{i,1}, \underline{x}'_{i,2}, \underline{x}'_{i,3})$. The probability density functions

(PDF) of these three random variables are constructed using the maximum entropy principle (see [6, 7, 17]), that is to say, in maximizing the uncertainties in the model under the constraints defined by the available information.

### 3.1.1 Construction of the PDF for the Random Mass

(i) *Available information.*
Let $E\{\cdot\}$ be the mathematical expectation. The available information for the random mass $M_i$ is defined as follows. Firstly, the random variable $M_i$ must be positive almost surely. Secondly, the mean value of the random mass $M_i$ must be equal to the value $\underline{m}_i$ of the nominal model. Thirdly, as it is proven in [20], the random mass must verify the inequality $E\{M_i^{-2}\} < +\infty$ in order that a second-order solution exists for the stochastic dynamical system. In addition, it is also proven that this constraint can be replaced by $|E\{\log M_i\}| < +\infty$.

(ii) *Maximum entropy principle.*
The probability density function $\mu \mapsto p_{M_i}(\mu)$ of the random variable $M_i$ is constructed by maximizing the entropy under the constraints defined above. The solution of this optimization problem is the PDF of a gamma random variable defined on $]0, +\infty[$. This PDF depends on two parameters which are the nominal value $\underline{m}_i$ and the coefficient of variation $\delta_{M_i}$ of the random variable $M_i$ such that $\delta_{M_i} = \sigma_{M_i}/\underline{m}_i$ where $\sigma_{M_i}$ is the standard deviation of the random variable $M_i$. Therefore, the PDF of the random mass is completely defined by the mean value $\underline{m}_i$ and by the dispersion parameter $\delta_{M_i}$.

### 3.1.2 Construction of the PDF for the Random Position Vector $\mathbf{R}_{0,i}$

In this subsection, the PDF of the random initial position vector $\mathbf{R}_{0,i}$ of the center of mass of $\text{RB}_i$ at initial time $t = 0$ is constructed.

(i) *Available information.*
The position vector $\underline{\mathbf{r}}_{0,i}$ of the center of mass $\underline{G}_i$ at initial time $t = 0$ of the nominal model is given. However, the real position is not exactly known and $\underline{\mathbf{r}}_{0,i}$ only corresponds to a mean position. Consequently, there is an uncertainty about the real position and this is the reason why this position is modeled by the random vector $\mathbf{R}_{0,i}$. Some geometrical and mechanical considerations lead us to introduce an admissible domain $D_i$ of random vector $\mathbf{R}_{0,i}$. We introduce the vector $\mathbf{h}$ of the parameters describing the geometry of domain $D_i$. In addition, the mean value of the random vector $\mathbf{R}_{0,i}$ must be equal to the value $\underline{\mathbf{r}}_{0,i}$ of the nominal model. Therefore, the available information for random variable $\mathbf{R}_{0,i}$ can be written as

$$\mathbf{R}_{0,i} \in D_i(\mathbf{h}) \quad \text{a.s.,} \tag{1.8a}$$

$$E\{\mathbf{R}_{0,i}\} = \underline{\mathbf{r}}_{0,i} \in D_i(\mathbf{h}). \tag{1.8b}$$

(ii) *Maximum entropy principle.*
The probability density function $\mathbf{a} \mapsto p_{\mathbf{R}_{0,i}}(\mathbf{a})$ of random variable $\mathbf{R}_{0,i}$ is then constructed by maximizing the entropy with the constraints defined by the available information in Eqs. (1.8a) and (1.8b). The solution of this optimization problem depends on two parameters which are $\underline{\mathbf{r}}_{0,i}$ and vector-valued parameter $\mathbf{h}$, and is such that

$$p_{\mathbf{R}_{0,i}}(\mathbf{a};\mathbf{h}) = \mathbb{1}_{D_i(\mathbf{h})}(\mathbf{a}) C_0 e^{-\langle \boldsymbol{\lambda}, \mathbf{a} \rangle}. \tag{1.9}$$

The positive valued parameter $C_0$ and vector $\boldsymbol{\lambda}$ are the unique solution of the equations

$$C_0 \int_{D_i(\mathbf{h})} e^{-\langle \boldsymbol{\lambda}, \mathbf{a} \rangle}\, d\mathbf{a} = 1, \tag{1.10a}$$

$$C_0 \int_{D_i(\mathbf{h})} \mathbf{a} e^{-\langle \boldsymbol{\lambda}, \mathbf{a} \rangle}\, d\mathbf{a} = \underline{\mathbf{r}}_{0,i}. \tag{1.10b}$$

(iii) *Generator of independent realizations.*
The independent realizations of random variable $\mathbf{R}_{0,i}$ must be generated using the constructed PDF $p_{\mathbf{R}_{0,i}}$. Such a generator can be obtained using the Monte Carlo Markov Chain (MCMC) method (Metropolis–Hastings algorithm [5]).

### 3.1.3 Random Matrix $[\mathbf{J}_i]$ of the Random Tensor of Inertia

In this subsection, the random matrix $[\mathbf{J}_i]$ of the random tensor of inertia with respect to $(\mathbf{G}_i, \underline{x}'_{i,1}, \underline{x}'_{i,2}, \underline{x}'_{i,3})$ is defined and an algebraic representation of this random matrix is constructed. The mass distribution around the random center of mass $\mathbf{G}_i$ is uncertain and consequently, the tensor of inertia is also uncertain. This is the reason why the matrix $[\underline{J}_i]$ of the tensor of inertia of the nominal model with respect to $(\underline{G}_i, \underline{x}'_{i,1}, \underline{x}'_{i,2}, \underline{x}'_{i,3})$ is replaced by a random matrix $[\mathbf{J}_i]$ which is constructed by using the maximum entropy principle. We introduce the positive-definite matrix $[Z_i]$ independent of $m_i$ such that

$$[Z_i] = \frac{1}{m_i}\left\{ \frac{\mathrm{tr}([J_i])}{2}[I_3] - [J_i] \right\}. \tag{1.11}$$

Then $[J_i]$ can be calculated as a function of $[Z_i]$,

$$[J_i] = m_i \left\{ \mathrm{tr}\big([Z_i]\big)[I_3] - [Z_i] \right\}. \tag{1.12}$$

It can be proven that $[Z_i]$ is positive definite and that each positive definite matrix $[J_i]$ constructed using Eq. (1.12), where $[Z_i]$ is a given positive definite matrix, can be interpreted as the matrix of a tensor of inertia of a physical rigid body (see [1]). In the literature, the matrix $m_i[Z_i]$ is referred as to the Euler tensor. The probabilistic modeling $[\mathbf{J}_i]$ of $[J_i]$ consists in introducing the random matrix $[\mathbf{Z}_i]$ and in using Eq. (1.12) in which $m_i$ is replaced by the random variable $M_i$ and where $[Z_i]$ is replaced by $[\mathbf{Z}_i]$. We then obtain

$$[\mathbf{Z}_i] = \frac{1}{M_i}\left\{\frac{\mathrm{tr}([\mathbf{J}_i])}{2}[I_3] - [\mathbf{J}_i]\right\}, \tag{1.13}$$

$$[\mathbf{J}_i] = M_i\left\{\mathrm{tr}\big([\mathbf{Z}_i]\big)[I_3] - [\mathbf{Z}_i]\right\}. \tag{1.14}$$

(i) *Available information concerning random matrix* $[\mathbf{Z}_i]$.
Let us introduce (1) the nominal value $[\underline{Z}_i]$ of deterministic matrix $[Z_i]$ such that $[\underline{Z}_i] = (1/\underline{m}_i)\{\mathrm{tr}([\underline{J}_i])/2[I_3] - [\underline{J}_i]\}$ and (2) the upper bound $[Z_i^{\max}]$ of random matrix $[\mathbf{Z}_i]$. Then, the available information for $[\mathbf{Z}_i]$ can be summarized as follows,

$$\begin{gathered}
[\mathbf{Z}_i] \in \mathbb{M}_3^+(\mathbb{R}) \quad \text{a.s.,}\\
\big\{\big[Z_i^{\max}\big] - [\mathbf{Z}_i]\big\} \in \mathbb{M}_3^+(\mathbb{R}) \quad \text{a.s.,}\\
E\big\{[\mathbf{Z}_i]\big\} = [\underline{Z}_i],\\
E\big\{\log\big(\det[\mathbf{Z}_i]\big)\big\} = C_i^l, \quad \big|C_i^l\big| < +\infty,\\
E\big\{\log\big(\det\big(\big[Z_i^{\max}\big] - [\mathbf{Z}_i]\big)\big)\big\} = C_i^u, \quad \big|C_i^u\big| < +\infty.
\end{gathered} \tag{1.15}$$

For more convenience, random matrix $[\mathbf{Z}_i]$ is normalized as follow. Matrix $[\underline{Z}_i]$ being positive definite, its Cholesky decomposition yields $[\underline{Z}_i] = [\underline{L}_{Z_i}]^T[\underline{L}_{Z_i}]$ in which $[\underline{L}_{Z_i}]$ is an upper triangular matrix in the set $\mathbb{M}_3(\mathbb{R})$ of all the $(3 \times 3)$ real matrices. Then, random matrix $[\mathbf{Z}_i]$ can be rewritten as

$$[\mathbf{Z}_i] = [\underline{L}_{Z_i}]^T[\mathbf{G}_i][\underline{L}_{Z_i}], \tag{1.16}$$

in which the matrix $[\mathbf{G}_i]$ is a random matrix for which the available information is

$$\begin{gathered}
[\mathbf{G}_i] \in \mathbb{M}_3^+(\mathbb{R}) \quad \text{a.s.,}\\
\big\{\big[G_i^{\max}\big] - [\mathbf{G}_i]\big\} \in \mathbb{M}_3^+(\mathbb{R}) \quad \text{a.s.,}\\
E\big\{[\mathbf{G}_i]\big\} = [I_3],\\
E\big\{\log\big(\det[\mathbf{G}_i]\big)\big\} = C_i^{l'}, \quad \big|C_i^{l'}\big| < +\infty,\\
E\big\{\log\big(\det\big(\big[G_i^{\max}\big] - [\mathbf{G}_i]\big)\big)\big\} = C_i^{u'}, \quad \big|C_i^{u'}\big| < +\infty,
\end{gathered} \tag{1.17}$$

in which $C_i^{l'} = C_i^l - \log(\det[\underline{Z}_i])$, $C_i^{u'} = C_i^u - \log(\det[\underline{Z}_i])$ and where the matrix $[G_i^{\max}]$ is an upper bound for random matrix $[\mathbf{G}_i]$ and is defined by $[G_i^{\max}] = ([\underline{L}_{Z_i}]^T)^{-1}[Z_i^{\max}][\underline{L}_{Z_i}]^{-1}$.

(ii) *Maximum entropy principle.*
The probability distribution of random matrix $[\mathbf{G}_i]$ is constructed using the maximum entropy principle under the constraints defined by the available information given by Eq. (1.17). The probability density function $p_{[\mathbf{G}_i]}([G])$ with respect to the volume element $\widetilde{d}G$ of random matrix $[\mathbf{G}_i]$ is then written as

$$\begin{aligned}
p_{[\mathbf{G}_i]}\big([G]\big) = {}& \mathbb{1}_{\mathbb{M}_3^+(\mathbb{R})}\big([G]\big) \times \mathbb{1}_{\mathbb{M}_3^+(\mathbb{R})}\big(\big[G_i^{\max}\big] - [G]\big) \times C_{G_i} \times \big(\det[G]\big)^{-\lambda_l}\\
& \times \big(\det\big(\big[G_i^{\max}\big] - [G]\big)\big)^{-\lambda_u} \times e^{-\mathrm{tr}([\mu][G])},
\end{aligned} \tag{1.18}$$

in which the positive valued parameter $C_{G_i}$ is a normalization constant, the real parameters $\lambda_l < 1$ and $\lambda_u < 1$ are Lagrange multipliers relative to the two last constraints defined by Eq. (1.17) and the symmetric real matrix $[\mu]$ is a Lagrange multiplier relative to the third constraint defined by Eq. (1.17). This probability density function is a particular case of the Kummer-Beta matrix variate distribution (see [4, 11]) for which the lower bound is a zero matrix. Parameters $C_{G_i}$, $\lambda_l$, $\lambda_u$ and matrix $[\mu]$ are the unique solution of the equations

$$\begin{aligned} E\{\mathbb{1}_{\mathbb{M}_3^S(\mathbb{R})}([\mathbf{G}_i])\} &= 1, \\ E\{[\mathbf{G}_i]\} &= [I_3], \\ E\{\log(\det[\mathbf{G}_i])\} &= C_i^{l'}, \\ E\{\log(\det([G_i^{\max}] - [\mathbf{G}_i]))\} &= C_i^{u'}. \end{aligned} \tag{1.19}$$

For fixed values of $\lambda_l$ and $\lambda_u$, parameters $C_{G_i}$ and $[\mu]$ can be estimated using Eq. (1.19). In Eq. (1.19), since the parameters $C_i^{l'}$ and $C_i^{u'}$ have no real physical meaning, the parameters $\lambda_l$ and $\lambda_u$ are kept as parameters which then allows the "shape" of the PDF to be controlled. If experimental data are available for the responses of the dynamical system, then the two parameters $\lambda_l$ and $\lambda_u$ can be identified solving an inverse problem. If experimental data are not available, these two parameters allow a sensitivity analysis of the solution to be carried out with respect to the level of uncertainties.

(iii) *Properties for random matrix* $[\mathbf{J}_i]$.
It is proven in [1] that using Eq. (1.14) and the available information defined by Eq. (1.15), the following important properties for random matrix $[\mathbf{J}_i]$ can be deduced,

$$\left\{\frac{1}{2}\operatorname{tr}([\mathbf{J}_i])[I_3] - [\mathbf{J}_i]\right\} \in \mathbb{M}_3^+(\mathbb{R}) \quad \text{a.s.,} \tag{1.20a}$$

$$\{[\mathbf{J}_i^{\max}] - [\mathbf{J}_i]\} \in \mathbb{M}_3^+(\mathbb{R}) \quad \text{a.s.,} \tag{1.20b}$$

$$E\{[\mathbf{J}_i]\} = [\underline{J}_i], \tag{1.20c}$$

$$\{\lambda_l < -2\} \Rightarrow E\{\|[\mathbf{J}_i]^{-1}\|^2\} < +\infty, \tag{1.20d}$$

in which the random matrix $[\mathbf{J}_i^{\max}]$, which represents a random upper bound for random matrix $[\mathbf{J}_i]$, is defined by

$$[\mathbf{J}_i^{\max}] = M_i\{\operatorname{tr}([Z_i^{\max}])[I_3] - [Z_i^{\max}]\}. \tag{1.21}$$

It should be noted that Eq. (1.20a) implies that each realization of random matrix $[\mathbf{J}_i]$ corresponds to the matrix of a tensor of inertia of a physical rigid body. In addition, this equation implies that random matrix $[\mathbf{J}_i]$ is almost surely positive definite. Equation (1.20b) provides a random upper bound for random matrix $[\mathbf{J}_i]$. Equation (1.20c) corresponds to a construction for which the mean value of random matrix $[\mathbf{J}_i]$ is equal to the nominal value $[\underline{J}_i]$. Finally, Eq. (1.20d) is necessary for that the random solution of the nonlinear dynamical system be a second-order stochastic process.

(iv) *Generator of independent realizations for random matrix* $[\mathbf{J}_i]$.
The generator of independent realizations of random matrix $[\mathbf{G}_i]$ is based on the Monte Carlo Markov Chain (MCMC) (Metropolis–Hastings algorithm [5] with the PDF defined by Eq. (1.18). Then, independent realizations of random matrix $[\mathbf{Z}_i]$ are obtained using Eq. (1.16). Finally, independent realizations of random matrix $[\mathbf{J}_i]$ are obtained using Eq. (1.14) and independent realizations of random mass $M_i$.

## *3.2 Stochastic Matrix Model for a Multibody Dynamical System with Uncertain Rigid Bodies and Its Random Response*

In order to limit the developments, it is assumed that only one of the $n_b$ rigid bodies denoted by $\mathrm{RB}_i$ of the rigid multibody system is uncertain. The extension to several uncertain rigid bodies is straightforward. Let the $6n_b$ random coordinates be represented by the $\mathbb{R}^{6n_b}$-valued stochastic process $\mathbf{U} = (\mathbf{R}_1, \ldots, \mathbf{R}_{n_b}, \mathbf{S}_1, \ldots, \mathbf{S}_{n_b})$ indexed by $[0, T]$ and let the $n_c$ random Lagrange multipliers be represented by the $\mathbb{R}^{n_c}$-valued stochastic process $\boldsymbol{\Lambda}$ indexed by $[0, T]$. The deterministic Eq. (1.6) becomes the following stochastic equation

$$\begin{bmatrix} [\mathbf{M}] & [\boldsymbol{\varphi}_{\mathbf{u}}]^T \\ [\boldsymbol{\varphi}_{\mathbf{u}}] & [0] \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{U}} \\ \boldsymbol{\Lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{q} - \mathbf{K} \\ -\frac{d}{dt}\boldsymbol{\varphi}_t - [\frac{d}{dt}\boldsymbol{\varphi}_{\mathbf{u}}]\dot{\mathbf{U}} \end{bmatrix}, \tag{1.22}$$

$$\mathbf{U}(0) = \mathbf{U}_0, \qquad \dot{\mathbf{U}}(0) = \mathbf{v}_0, \quad \text{a.s.} \tag{1.23}$$

in which the vector $\mathbf{U}_0 = (\mathbf{r}_{0,1}, \ldots, \mathbf{R}_{0,i}, \ldots, \mathbf{r}_{0,n_b}, \mathbf{s}_{0,1}, \ldots, \mathbf{s}_{0,n_b})$ is random due to the random vector $\mathbf{R}_{0,i}$. For all given real vector $\dot{\mathbf{u}}$, the vector $\mathbf{K}(\dot{\mathbf{u}})$ of the Coriolis forces is random due to the random matrix $[\mathbf{J}_i]$. The random mass matrix $[\mathbf{M}]$ is defined by

$$[\mathbf{M}] = \begin{bmatrix} [\mathbf{M}^r] & 0 \\ 0 & [\mathbf{M}^s] \end{bmatrix}, \tag{1.24}$$

in which the $(3n_b \times 3n_b)$ random matrices $[\mathbf{M}^r]$ and $[\mathbf{M}]^s$ are defined by

$$[\mathbf{M}^r] = \begin{bmatrix} m_1[I_3] & & \cdots & & 0 \\ & \ddots & & & \\ \vdots & & M_i[I_3] & & \vdots \\ & & & \ddots & \\ 0 & & \cdots & & m_{n_b}[I_3] \end{bmatrix}, \tag{1.25}$$

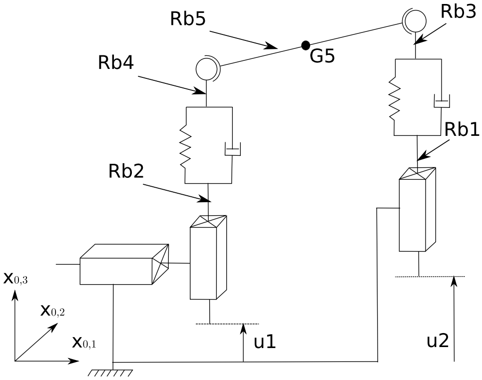


**Fig. 1.2** Rigid multibody system

$$\left[\mathbf{M}^s\right] = \begin{bmatrix} [J_1] & & \cdots & & 0 \\ & \ddots & & & \\ \vdots & & [\mathbf{J}_i] & & \vdots \\ & & & \ddots & \\ 0 & & \cdots & & [J_{n_b}] \end{bmatrix}. \tag{1.26}$$

Random Eqs. (1.22) and (1.23) are solved using the Monte Carlo simulation method.

# 4 Application

In this section, we present a numerical application which validates the methodology presented in this paper.

## 4.1 Description of the Nominal Model

The rigid multibody model is made up of five rigid bodies and six joints which are described in the fixed frame $(O, x_{0,1}, x_{0,2}, x_{0,3})$ (see Fig. 1.2). The plan defined by $(O, x_{0,1}, x_{0,2})$ is identified below as the "ground". The gravity forces in the $x_{0,3}$-direction are taken into account.

(i) *Rigid bodies.*
In the initial configuration, the rigid bodies $Rb1$, $Rb2$, $Rb3$ and $Rb4$ are cylinders for which the axes follow the $x_{0,3}$-direction. In the initial configuration, the rigid body $Rb5$ is supposed to be symmetric with respect to the planes $(G5, x_{0,1}, x_{0,2})$ and $(G5, x_{0,1}, x_{0,3})$ in which $G5$ is the center of mass of $Rb5$.

(ii) *Joints.*

– The joint *Ground*-$Rb1$ is made up of a prismatic joint following $x_{0,3}$-direction. The displacement following $x_{0,3}$-direction (see Fig. 1.2), denoted

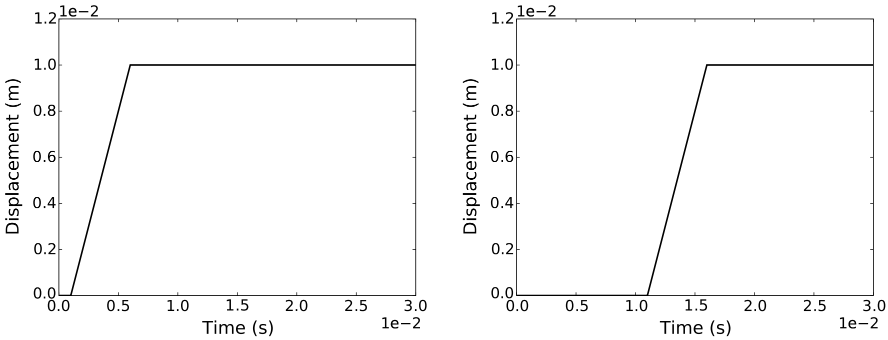


**Fig. 1.3** Imposed displacement $u1(t)$ (*left figure*) and $u2(t)$ (*right figure*)

by $u1(t)$, is imposed. The joint *Ground*-*Rb*2 is a prismatic joint following $x_{0,3}$-direction. The displacement following $x_{0,3}$-direction (see Fig. 1.2), denoted by $u2(t)$, is imposed. The displacement following $x_{0,1}$-direction is unconstrained. Imposed displacements $u1(t)$ and $u2(t)$ are plotted in Fig. 1.3 for $t$ in [0, 0.03] s.
- The joints *Rb*1-*Rb*3 and *Rb*2-*Rb*4 are constituted of 6D spring-dampers.
- Finally, the joints *Rb*3-*Rb*5 and *Rb*4-*Rb*5 are $x_{0,2}$-direction revolute joints.

## *4.2 Random Response of the Stochastic Model*

Rigid body *Rb*5*a* is considered as uncertain and is therefore modeled by a random rigid body. As explained in Sect. 3, the elements of inertia of the uncertain rigid Body *Rb*5 are replaced by random quantities. The fluctuation of the response is controlled by four parameters $\delta_{M_5}$, $\mathbf{h}$, $\lambda_l$ and $\lambda_u$. A sensitivity analysis is carried out with respect to these four parameters. Statistics on the transient response are estimated using the Monte Carlo simulation method with 500 independent realizations. The initial velocities and angular velocities are zero. The observation point $P_{obs}$ belongs to *Rb*5. Four different cases are analyzed:

1. *Case 1*: $M_5$ is random, $\mathbf{R}_{0,5}$ is deterministic and $[\mathbf{Z}_5]$ is deterministic.
   We choose $\delta_{M_5} = 0.5$. The confidence region, with a probability level $P_c = 0.90$, of the random acceleration of point $P_{obs}$ is plotted in Fig. 1.4. It can be noted that the acceleration is sensitive to the mass uncertainties.
2. *Case 2*: $M_i$ is deterministic, $\mathbf{R}_{0,5}$ is random and $[\mathbf{Z}_5]$ is deterministic.
   The domain of $\mathbf{R}_{0,5}$ is supposed to be a parallelepiped which is centered at point $(0, 0, 0.55)$ for which its edges are parallel to the directions $x_{0,1}$, $x_{0,2}$ and $x_{0,3}$ and for which the lengths following these three directions are respectively 0.5, 0.2 and 0.02. The confidence region, with a probability level $P_c = 0.90$, of the random acceleration of point $P_{obs}$ is plotted in Fig. 1.5. We can remark that the angular acceleration is sensitive to uncertainties on initial center of mass of *Rb*5.

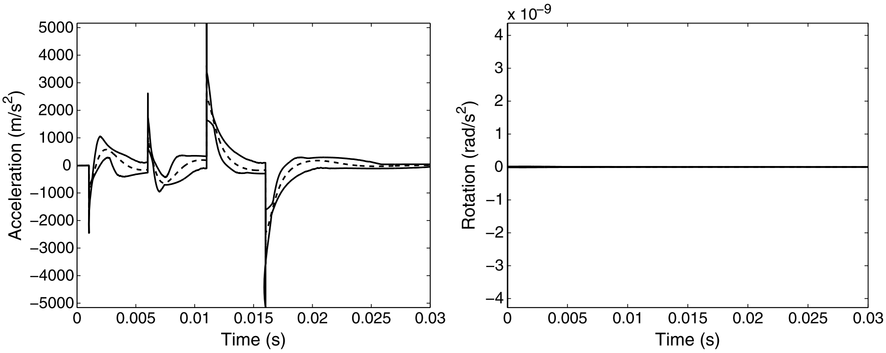


**Fig. 1.4** Random transient acceleration of point $P_{obs}$, Case 1: confidence region (*upper* and *lower solid lines*) and mean response (*dashed line*); $x_{0,3}$-acceleration (*left figure*) and $x_{0,1}$-angular acceleration (*right figure*)

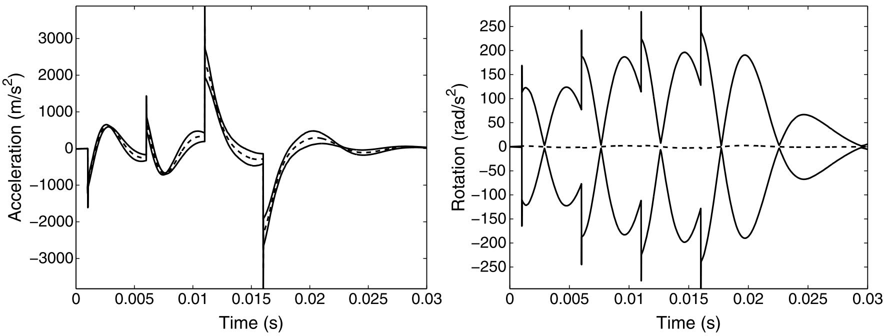


**Fig. 1.5** Random transient acceleration of point $P_{obs}$, Case 2: confidence region (*upper* and *lower solid lines*) and mean response (*dashed line*); $x_{0,3}$-acceleration (*left figure*) and $x_{0,1}$-angular acceleration (*right figure*)

3. *Case 3*: $M_5$ is deterministic, $\mathbf{R}_{0,5}$ is deterministic and $[\mathbf{Z}_5]$ is random.
   We choose $\lambda_l = -5$ and $\lambda_u = -5$ for random matrix $[\mathbf{Z}_5]$. The confidence region, with a probability level $P_c = 0.90$, of the random acceleration of point $P_{obs}$ is plotted in Fig. 1.6. We can remark, as it was expected, that the angular acceleration is very sensitive to uncertainties on the tensor of inertia. We can also remark a high sensitivity of the acceleration.
4. *Case 4*: $M_5$, $\mathbf{R}_{0,5}$ and $[\mathbf{Z}_5]$ are random.
   The values of the parameters of the PDF are those fixed in the three previous cases. The confidence region, with a probability level $P_c = 0.90$, of the random acceleration of point $P_{obs}$ is plotted in Fig. 1.7. It can be viewed that (1) the randomness on the acceleration is mainly due to the randomness of mass $M_5$, (2) the randomness on the angular acceleration is mainly due to the randomness of the initial position $\mathbf{R}_{0,5}$ of the center of mass and the random tensor $[\mathbf{Z}_5]$.

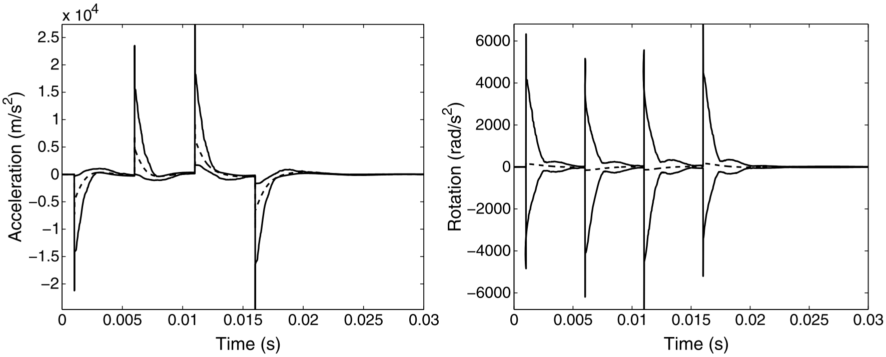


**Fig. 1.6** Random transient acceleration of point $P_{obs}$, Case 3: confidence region (*upper* and *lower solid lines*) and mean response (*dashed line*); $x_{0,3}$-acceleration (*left figure*) and $x_{0,1}$-angular acceleration (*right figure*)

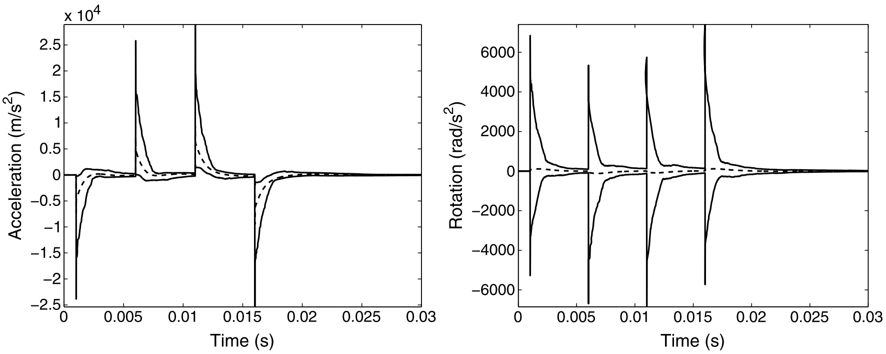


**Fig. 1.7** Random transient acceleration of point $P_{obs}$, Case 4: confidence region (*upper* and *lower solid lines*) and mean response (*dashed line*); $x_{0,3}$-acceleration (*left figure*) and $x_{0,1}$-angular acceleration (*right figure*)

# 5 Conclusion

We have presented a complete and general probabilistic modeling of uncertain rigid bodies taking into account all the known mechanical and mathematical properties defining a rigid body. This probabilistic model of uncertainties is used to construct the stochastic equations of uncertain multibody dynamical systems. The random dynamical responses can then be calculated. In the proposed probabilistic model, the mass, the center of mass and the tensor of inertia are modeled by random variables for which the prior probability density functions are constructed using the maximum entropy principle under the constraints defined by all the available mathematical, mechanical and design properties. Several uncertain rigid bodies can be linked each to the others in order to obtain the stochastic dynamical model of the uncertain multibody dynamical system. The theory proposed has been illustrated analyzing a simple example. The results obtained clearly show the role played by uncertainties and the sensitivity of the responses due to uncertainties on (1) the mass (2) the center of mass and (3) the tensor of inertia. Such a prior stochastic model allows the robustness of the responses to be analyzed with respect to uncertainties. If experimental data were available on the responses, then the parameters which control the level of uncertainties could be estimated by solving an inverse stochastic problem.

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# Chapter 2
# Dynamic Variability Response for Stochastic Systems

**Vissarion Papadopoulos and Odysseas Kokkinos**

**Abstract** In this study we implement the concept of Variability Response Functions (VRFs) in dynamic systems. The variance of the system response can be readily estimated by an integral involving the Dynamic VRF (DVRF) and the uncertain system parameter power spectrum. With the proposed methodology spectral and probability distribution-free upper bounds can be easily derived. Also an insight is provided with respect to the mechanisms controlling the system's response. The necessarily asserted conjecture of independence of the DVRF to the spectral density and the marginal probability density is validated numerically through brute-force Monte Carlo simulations.

**Keywords** Dynamic Variability Response Functions · Stochastic finite element analysis · Upper bounds · Stochastic dynamic systems

## 1 Introduction

Over the past two decades a lot of research has been dedicated to the stochastic analysis of structural systems involving uncertain parameters in terms of material or geometry with the implementation of stochastic finite element methodologies. Although these methods have proven to be highly accurate and computationally efficient for a variety of problems, there is still a wide range of problems in stochastic mechanics involving combinations of strong non-linearities and/or large variations of system properties as well as non-Gaussian system properties that can be solved with reasonable accuracy only through a computationally expensive Monte Carlo simulation approach [3–5, 12].

In all aforementioned cases, the spectral/correlation characteristics and the marginal probability distribution function (pdf) of the stochastic fields describing

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