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Foreword by Prof. Dr. Ming-Jun Lai

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Foreword

Multivariate splines are multivariate piecewise polynomial functions defined over a triangulation (in $\mathbb{R}^2$) or tetrahedral partition (in $\mathbb{R}^3$) or a simplicial partition (in $\mathbb{R}^d, d \geq 3$) with a certain smoothness. These functions are very flexible and can be used to approximate any known or unknown complicated functions. Usually one can divide any domain of interest into triangle/tetrahedron/simplex pieces according to a given function to be approximated and then using piecewise polynomials to approximate the given function. They are also computer compatible in the sense of computation, as only additions and multiplications are needed to evaluate these functions on a computer. Thus, they are a most efficient and effective tool for numerically approximating known or unknown functions. In addition, multivariate spline functions are very important tools in the area of applied mathematics as they are widely used in approximation theory, computer aided geometric design, scattered data fitting/interpolation, numerical solution of partial differential equations such as flow simulation and image processing including image denoising.

These functions have been studied in the last fifty years. In particular, univariate splines have been studied thoroughly. One understands univariate splines very well in theory and computation. One typical monograph on univariate spline functions is the well-known book written by Carl de Boor. It is called ”A Practical Guide to Splines” published first in 1978 by Springer Verlag. Another popular book is Larry Schumaker’s monograph ”Spline Functions: Basics and Applications” which is now in second edition published by Cambridge University Press in 2008. Theory and computation of bivariate splines are also understood very well. For approximation properties of bivariate splines, there is a monograph "Spline Functions on Triangulation" authored by Ming-Jun Lai and Larry Schumaker, published by Cambridge University Press in 2007. For using bivariate splines for scattered data fitting/interpolation and numerical solution of partial differential equations, one can find the paper by G. Awanou, Ming-Jun Lai and P. Wenston in 2006 useful. In their paper, they explained how to compute bivariate splines without constructing explicit basis functions and use them for data fitting and numerical solutions of

PDEs including 2D Navier-Stokes equations and many flow simulations and image denoising simulations. These numerical results can be found at www.math.uga.edu/˜mjlai.

Approximation properties of spline functions defined on spherical triangulations can also be found in the monograph by M.-J. Lai and L. L. Schumaker mentioned above. A numerical implementation of spherical spline functions to reconstruct geopotential around the Earth can be found in the paper by V. Baramidze, M.-J. Lai, C. K. Shum, and P. Wenston in 2009. In addition, some basic properties of trivariate spline functions can also be found in the monograph by Lai and Schumaker. For example, Lai and Schumaker describe in their monograph how to construct C^1 and C^2 macro-element functions over the Alfeld split of tetrahedral partitions and the Worsey-Farin split of tetrahedral partitions. However, how to construct locally supported basis functions such as macro-elements for smoothness $r > 2$ was not discussed in the monograph. In addition, how to find interpolating spline functions using trivariate spline functions of lower degree without using any variational formulation is not known.

In his thesis, these two questions are carefully examined and fully answered. That is, Michael Matt considers trivariate macro-elements and trivariate local Lagrange interpolation methods. The macro-elements of arbitrary smoothness, based on the Alfeld and the Worsey-Farin split of tetrahedra, are described very carefully and are illustrated with many examples. Michael Matt has spent a great deal of time and patience to write down in detail how to determine for C^r macro-elements for $r = 3,4,5,6$ and etc.. One has to point out that macro-elements are locally supported functions and they form a superspline subspace which has much smaller dimension than the whole spline space defined on the same tetrahedral partition. As this superspline subspace possesses the same full approximation power as the whole spline space, they are extremely important as these are enough to use for approximate known and unknown functions with a much smaller dimension. Thus, they are most efficient. The results in this thesis are an important extension of the literature on trivariate macro-elements known so far. Michael Matt also describes two methods for local Lagrange interpolation with trivariate splines. Both methods, for once continuously differentiable cubic splines based on a type-4 partition and for splines of degree nine on arbitrary tetrahedral partitions, are examined very detailed. Especially the method based on arbitrary tetrahedral partitions is a crucial contribution to this area of researchsince it is the first method for trivariate Lagrange interpolation for two times continuously differentiable splines.

To summarize, the results in this dissertation extend the construction theory of trivariate splines of C^r macro-elements for any $r \geq 1$. The construction schemes for finding interpolatory splines over arbitrary tetrahedral partition presented in this thesis are highly recommended for any application practitioner.

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Michael Andreas Matt

Abstract

In this work, we construct two trivariate local Lagrange interpolation methods which yield optimal approximation order and C^r macro-elements based on the Alfeld and the Worsey-Farin split of a tetrahedral partition. The first interpolation method is based on cubic C^1 splines over type-4 cube partitions, for which numerical tests are given. The other one is the first trivariate Lagrange interpolation method using C^2 splines. It is based on arbitrary tetrahedral partitions using splines of degree nine. In order to obtain this method, several new results on C^2 splines over partial Worsey-Farin splits are required. We construct trivariate macro-elements based on the Alfeld, where each tetrahedron is divided into four subtetrahedra, and the Worsey-Farin split, where each tetrahedron is divided into twelve subtetrahedra, of a tetrahedral partition. In order to obtain the macro-elements based on the Worsey-Farin split we construct minimal determining sets for C^r macro-elements over the Clough-Tocher split of a triangle, which are more variable than those in the literature.

Zusammenfassung

In dieser Arbeit konstruieren wir zwei Methoden zur lokalen Lagrange-Interpolation mit trivariaten Splines, welche optimale Approximationsordnung besitzen, sowie C^r Makro-Elemente, welche auf der Alfeld- und der Worsey-Farin-Unterteilung einer Tetraederpartition basieren. Eine Interpolationsmethode basiert auf kubischen C^1 Splines auf Typ-4 Würfelpartitionen. Für diese werden auch numerische Tests angegeben. Die nächste Methode ist die erste zur lokalen Lagrange-Interpolation mit trivariaten C^2 Splines. Sie basiert auf beliebigen Tetraederpartitionen und verwendet Splines vom Grad neun. Um diese Methode zu erhalten, werden einige neue Resultate zu C^2 Splines über partiellen Worsey-Farin-Unterteilungen benötigt. Wir konstruieren trivariate Makro-Elemente beliebiger Glattheit, die auf der Alfeld-Unterteilung, bei der jeder Tetraeder in vier Subtetraeder unterteilt ist, und der Worsey-Farin-Unterteilung, bei welcher jeder Tetraeder in zwölf Subtetraeder unterteilt ist, einer Tetraederpartition beruhen. Um die Makro-Elemente, die auf der Worsey-Farin-Unterteilung basieren, zu erzeugen, konstruieren wir minimal bestimmende Mengen für C^r Makro-Elemente über der Clough-Tocher-Unterteilung eines Dreiecks, welche, im Vergleich zu den bereits in der Literatur bekannten, variabler sind.

Contents

1 Introduction

Multivariate splines play an important role in several areas of applied mathematics. Due to their efficient computability and their approximation properties, they are widely used for the construction and reconstruction of surfaces and volumes, the interpolation and approximation of scattered data, and many other fields in computer aided geometric design and numerical analysis, such as the solution of partial differential equations.

In this thesis we consider the space of multivariate splines of degree d defined on a tessellation Δ of a polyhedral domain $\Omega \subset \mathbb{R}^n$ into n-simplices, which is given by

$$\mathcal{S}_d^r(\Delta) := \{s \in C^r(\Omega) : s|_T \in \mathcal{P}_d^n \text{ for all } T \in \Delta\},$$

where $C^r(\Omega)$ is the space of functions of C^r smoothness and $\mathcal{P}_d^n$ is the space of multivariate polynomials of degree d. We are mainly interested in the case $n = 3$, and to some extend $n = 2$, and there we mostly consider certain subspaces, the spaces of supersplines, which fulfill supersmoothness conditions at vertices and edges of the tetrahedra or triangles, and in some cases also additional individual smoothness conditions.

A common approach for the construction and reconstruction of volumes is interpolation. For a tessellation Δ of a domain $\Omega \subset \mathbb{R}^n$, a set $\mathcal{L} := \{\kappa_1, \ldots, \kappa_m\}$ is called a Lagrange interpolation set for an m-dimensional spline space $\mathcal{S}_d^r(\Delta)$, provided that for each function $f \in C(\Omega)$ there exists a unique spline $s_f \in \mathcal{S}_d^r(\Delta)$, such that

$$s_f(\kappa_i) = f(\kappa_i), \quad i = 1, \ldots, m,$$

holds. If also derivatives of a sufficiently smooth function f are interpolated, the corresponding set $\mathcal{H}$ is called a Hermite interpolation set.

An important property of interpolation sets is locality. An interpolation set is called local, provided that the value of an interpolant s_f at a point $\xi \in \Omega$ only depends on data values in a finite environment of ξ. Thus, the interpolant can be determined by solving several smaller linear system of equations at a time, which is very useful for the implementation

M. A. Matt, *Trivariate Local Lagrange Interpolation and Macro Elements of Arbitrary Smoothness*, DOI 10.1007/978-3-8348-2384-7_1,

of interpolation methods. A further desirable property is the stability of an interpolation method. Roughly speaking, a method is called stable if a small modification of a data value only leads to a small change of the corresponding interpolant. These two properties are important in order to achieve optimal approximation order. The approximation order of a spline space $\mathcal{S}_d^r(\Delta)$ is the biggest natural number k, such that

$$dist(f, \mathcal{S}_d^r(\Delta)) := \inf\{\|f - s\| : s \in \mathcal{S}_d^r(\Delta)\} \leq K|\Delta|^k$$

holds for a constant $K > 0$ that only depends on f, d, and the smallest angles of Δ, where $|\Delta|$ is the mesh size of Δ. Then, for an interpolation method, the approximation error $|f - s_f|$ is considered, where s_f is the interpolant of f constructed by the method. Thus, the approximation order gives the rate of convergence of the error for refinements of the tessellation Δ. It was shown by Ciarlet and Raviart [23] that $d + 1$ is an upper bound for the approximation order, which implies that $k = d + 1$ is the optimal approximation order. Following de Boor and Jia [36], the optimal approximation order cannot be obtained by every interpolation method.

The terminology ”spline function” was first used in Schoenberg [91, 92], though earlier papers were concerned with splines without actually using this name (see [86, 88]). Univariate splines were extensively studied and many results on approximation and interpolation, as well as on the dimension of univariate spline spaces, are known (see [32, 69, 93], and references therein).

In recent years a lot of research has been done on bivariate splines (see [2–8, 8, 14, 15, 20, 22, 25–29, 31, 34, 38, 39, 44–53, 56–61, 70, 71, 73–80, 82, 85, 87, 89, 90, 94, 100, 101]).

In contrast, much less is known about trivariate splines, especially for spline spaces with a low degree of polynomials compared to the degree of smoothness (see [12, 16, 17, 21, 41, 43, 62, 105]). Due to the complex structure of these spline spaces many problems, such as the construction of local interpolation sets, are still open. One approach are macro-element methods, which are mostly based on the refinement of the tetrahedra of a given partition into smaller subtetrahedra (see [1, 9–11, 13, 18, 55, 63, 64, 96–99, 103, 104]). Another approach is based on regular tetrahedral partitions (see [40, 42, 72, 81, 83, 95]).

In this thesis, we consider local Lagrange interpolation methods for C^1 and C^2 splines and C^r macro-elements based on the Alfeld and the Worsey-Farin split of tetrahedra.

Local Lagrange interpolation methods can be used to interpolate scattered data and to construct and reconstruct volumes. We consider local Lagrange interpolation with C^1 cubic splines on type-4 cube partitions. There are already a few articles on trivariate local Lagrange interpolation on various tetrahedral partitions (see [41–43, 81, 95]). Though the method by Matt and Nürnberger [67] presented here is the first one that was actually implemented. Hence, it is the first time that it can also be shown numerically that the spline space corresponding to the trivariate Lagrange interpolation method yields optimal approximation order. We also construct local and stable Lagrange interpolation sets for C^2 splines on arbitrary tetrahedral partitions. It can be seen from above that there exists some literature on bivariate Lagrange interpolation methods based on C^2 and also some literature on local Lagrange interpolation with trivariate C^1 splines. Here, we construct the first trivariate Lagrange interpolation method for C^2 splines. We also show that the interpolation set is local and stable and that the corresponding spline space yields optimal approximation order.

We also examine C^r macro-element methods based on the Alfeld and the Worsey-Farin refinement of a tetrahedron. Macro-elements can be used for example to numerically solve partial differential equations, to simulate properties of materials or economic models. In the bivariate case there exists a vast literature on macro-elements, also for those of type C^r (see [6, 7, 59, 61, 106]). In the trivariate setting there exists some literature on C^1 and C^2 macro-elements (see [1, 9–11, 13, 18, 55, 64, 96–99, 103, 104]), though only polynomial C^r macro-elements are known so far (see [63]). Their degree of polynomials is $d = 8r + 1$, which is quite high. By splitting the tetrahedra with the Alfeld and the Worsey-Farin split, we obtain C^r macro-elements with a significantly lower degree of polynomials. We also show that for the corresponding Hermite interpolation methods the considered superspline spaces yield optimal approximation order.

This thesis is divided into the following chapters: In chapter 2, we consider the fundamentals of multivariate spline theory. First, we investigate tessellations of a polyhedral domain $\Omega \subset \mathbb{R}^n$ into n-simplices, especially tetrahedral partitions, as well as some special tessellations for $n = 2$ and $n = 3$. Subsequently, we introduce multivariate splines and supersplines, and their basis, the multivariate polynomials. Then, we examine the Bernstein-Bézier techniques for multivariate splines. Using barycentric coordinates, the multivariate Bernstein polynomials, and thus the B-form of polynomials, can be defined. Following, we consider the de Casteljau algorithm for efficient evaluation of polynomials in the B-form, which can also be used for subdivision. Next, we describe smoothness conditions for polynomials in the B-form on adjacent n-simplices, especially for $n = 2$ and $n = 3$, which were introduced by Farin [39] and de Boor [33]. Subsequently, we consider the concept of minimal determining sets, which can be used to characterize spline spaces. Minimal determining sets also play an important role, since their cardinality is equal to the dimension of the corresponding spline space. Next, we define the Lagrange and Hermite interpolation and some properties of interpolation sets, such as locality and stability. Finally, we introduce nodal minimal determining sets and define the approximation order of spline spaces.

In chapter 3, we consider minimal determining sets for C^1 and C^2 splines on partial Worsey-Farin splits of a tetrahedron, which were defined in chapter 2. First, we review the results of Hecklin, Nürnberger, Schumaker, and Zeilfelder [42] for C^1 splines. Subsequently, we construct several minimal determining sets for different C^2 superspline spaces. We also state some lemmata that show how a spline defined on a tetrahedral partition $\Delta \setminus T$ can be extended to a spline defined on Δ, where the tetrahedron T is refined with a partial Worsey-Farin split. To this end, additional smoothness conditions are used. The minimal determining sets and the lemmata considered in this chapter are then applied in chapters 4 and 5.

In chapter 4, we present the work on local Lagrange interpolation with trivariate C^1 splines on type-4 cube partitions by Matt and Nürnberger [67]. At first the type-4 cube partition is defined. Therefore, the cubes of a cube partition $\diamond$ are divided into five classes. Afterwards, each cube is split into five tetrahedra, according to the classification of the cubes. Then, following this classification and the location of the tetrahedra in the cubes, we chose the interpolation points and refine some of the tetrahedra with a partial Worsey-Farin split. Subsequently, it is shown that the set of interpolation points chosen is a Lagrange interpolation set for the space of C^1 splines of degree three on the final tetrahedral partition. The final partition is obtained by refining some of the tetrahedra even further, though by the time a spline is determined on these tetrahedra this is not needed and thus omitted at the moment, in order to keep the computation of the spline less complex. We also give a nodal minimal determining set for the interpolation method. Following that, we show that the interpolation method yields optimal approximation order. Finally, numerical tests and visualizations of the Marschner-Lobb test function (see [65]) are given.

In chapter 5, we construct a local Lagrange interpolation method for C^2 splines of degree nine on arbitrary tetrahedral partitions. The construction is based on a decompositions of a tetrahedral partition Δ into classes of tetrahedra. This decomposition induces an order of the tetrahedra in Δ, according to their number of common vertices, edges, and faces. In contrast to the decomposition used in [43] to create a local Lagrange interpolation method for C^1 splines, the number of common vertices, edges, and faces has to be considered simultaneously in order to construct a local Lagrange interpolation set for C^2 splines. Next, as with the Lagrange interpolation method considered in chapter 4, some of the tetrahedra of the partition Δ have to be refined with partial Worsey-Farin splits according to the number of common edges with the previous tetrahedra in the order imposed by the decomposition. Then, we construct a superspline space based on the refined partition, which is endowed with several additional smoothness conditions corresponding to the lemmata in chapter 3. Subsequently, we construct a local and stable Lagrange interpolation set. It is notable that the interpolation set is 11-local which is very low, compared to the number of 24 classes needed to create the method and especially when compared to the C^1 method for local Lagrange interpolation on arbitrary tetrahedral partitions by Hecklin, Nürnberger, Schumaker, and Zeilfelder [43] which is 10-local. We also give a nodal minimal determining set for the superspline space considered in this chapter. Finally, we examine the approximation order of the spline space considered in this chapter and show that it is optimal.

In chapter 6, we consider the minimal determining sets for bivariate C^r macro-elements based on the Clough-Tocher split of a triangle constructed by Matt [66]. First, we consider conditions on the minimal degree of polynomials and the minimal degrees of supersmoothnesses in order to construct C^r macro-elements based on non-split triangles and triangles refined with a Clough-Tocher or a Powell-Sabin split, respectively. These are used throughout this chapter and in chapters 7 and 8, in order to derive the minimal conditions for macro-elements based on the Alfeld and the Worsey-Farin split of a tetrahedron. Then, minimal determining sets for C^r splines with various degrees of polynomials and supersmoothnesses based on the Clough-Tocher split of a triangle are examined. In case the minimal degrees are applied, the macro-elements reduce to those constructed in [59]. These minimal determining sets are needed for the construction of trivariate C^r macro-elements over the Worsey-Farin split of a tetrahedron in chapter 8. Subsequently, we illustrate the minimal determining sets of the C^r

macro-elements with several examples for $r = 0, \ldots, 9$. These are also used in the examples for minimal determining sets for macro-elements based on the Worsey-Farin split of a tetrahedron in chapter 8.

In chapter 7, we consider the trivariate C^r macro-elements based on the Alfeld split of a tetrahedron by Lai and Matt [54]. It is firstly shown which restrictions for the degree of polynomials, as well as the supersmoothness conditions, have to be fulfilled in order to construct macro-elements over the Alfeld split of a tetrahedron. These can be derived from the restrictions for bivariate macro-elements considered in chapter 6. Thus, it can be seen that the degree of polynomials and supersmoothnesses is the lowest possible to construct such macro-elements. Subsequently, a corresponding superspline space and minimal determining sets for the macro-elements are examined, first on one tetrahedron divided with the Alfeld split and then on a refined tetrahedral partition. Since these minimal determining sets are quite complex, we illustrate them for C^r macro-elements on a single tetrahedron for $r = 1, \ldots, 6$. It can be seen that for $r = 1, 2$ the macro-elements reduce to those in [62] in section 18.3 and 18.7. In the following, we consider nodal minimal determining sets for the macro-elements. First, nodal minimal determining sets for the macro-element defined on one Alfeld split tetrahedron are analyzed, and then for macro-elements over a refined tetrahedral partition. Finally, we examine a Hermite interpolation set for C^r splines over the Alfeld split of tetrahedra and show that it yields optimal approximation order.

In chapter 8, we examine the C^r macro-elements based on the Worsey-Farin split of tetrahedra by Matt [66]. We first consider the conditions for the minimal degree of polynomials and the minimal degrees of supersmoothnesses needed to construct C^r macro-elements based on the Worsey-Farin split. Again, these can be derived from the corresponding conditions for bivariate macro-elements considered in chapter 6. Following, we present a superspline space which can be used to define C^r macro-elements based on the Worsey-Farin split of a tetrahedron and a corresponding minimal determining set. We also give a minimal determining set for a superspline space defined on a tetrahedral partition, where each tetrahedron is refined with a Worsey-Farin split. Subsequently, we illustrate minimal determining sets for C^r macro-elements based on the Worsey-Farin split of one tetrahedron for $r = 1, \ldots, 6$. For $r = 1$ and $r = 2$ the macro-elements reduce to those considered by Lai and Schumaker [62] in sections 18.4 and 18.8, respectively. Next, we examine nodal minimal determining sets for the C^r macro-elements, first for macro-elements based on a single Worsey-

Farin split tetrahedron and then on a whole tetrahedral partitions that has been refined with the Worsey-Farin split. We conclude this chapter with a Hermite interpolation set for C^r splines over the Worsey-Farin split of tetrahedra and consider the approximation order, which is optimal.

The next chapter contains the references. It is followed by appendix A, where we state several bivariate lemmata concerned with minimal determining sets for splines based on triangles refined with the Clough-Tocher split. These lemmata are needed in chapter 3 in order to proof the theorems on minimal determining sets for splines on tetrahedra refined with partial Worsey-Farin splits. For a better understanding we illustrate the minimal determining sets constructed here.

2 Preliminaries

In this chapter, we consider some results and techniques of multivariate spline theory. Since we are mainly interested in trivariate splines in this work, and as an aid to some extent also bivariate splines, we state most of the results and definitions shown in this chapter combined in the multivariate setting. However, to ease the understanding, we also show some of theses results for the trivariate and bivariate case. In section 2.1, we define tessellations of a domain in $\mathbb{R}^n$. Furthermore, we show some refinement schemes of triangles and tetrahedral partitions and introduce some notation and the Euler relations for tetrahedra. In section 2.2, we examine multivariate polynomials, which form a basis for the subsequently considered multivariate splines and supersplines. In the next section, we describe the Bernstein-Bézier techniques for multivariate splines. These are based on the barycentric coordinates, which are needed to define Bernstein polynomials and the resulting B-form of polynomials that is used throughout this dissertation. Subsequently the de Casteljau algorithm, which can be used to efficiently evaluate polynomials in the B-form, as well as smoothness conditions between two polynomials are considered. Finally, the concept of minimal determining sets is introduced. In the last section, we define the problem of interpolation and give another characterization of spline spaces, nodal minimal determining sets. In the end we define the approximation order of spline spaces.

2.1 Tessellations of $\mathbb{R}^n$

In this section we define tessellations of a polyhedral subset of $\mathbb{R}^n$ into n-simplices. Especially the cases $n = 2,3$ are of interest here. Then, we define certain special refinements of triangles and tetrahedral partitions. Moreover, some notation concerning the relations of tetrahedra are established, as well as the trivariate Euler relations.

M. A. Matt, *Trivariate Local Lagrange Interpolation and Macro Elements of Arbitrary Smoothness*, DOI 10.1007/978-3-8348-2384-7_2,

2.1.1 Tessellations of $\mathbb{R}^n$ into n-simplices

First, since the methods considered in this dissertation are mostly in the trivariate setting, tetrahedral partitions are denominated. Thereafter, tessellations of $\mathbb{R}^n$, $n \in \mathbb{N}$, into $n-$simplices are defined, which are also needed later on in this chapter.

Definition 2.1:
Let Ω be a polyhedral subset of $\mathbb{R}^3$. If Ω is divided into tetrahedra T_i, $i = 1,\ldots,N$, such that the intersection of two different tetrahedra is either empty, a common vertex, a common edge, or a common face, then $\Delta := \{T_1,\ldots,T_N\}$ is called a **tetrahedral partition** of Ω.

A tetrahedron $T := \langle v_1, v_2, v_3, v_4 \rangle$ is called non-degenerated provided that it has nonzero volume.

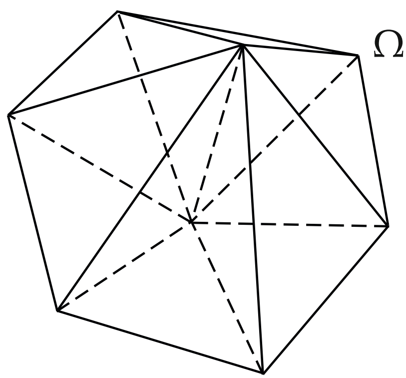

Figure 2.1: Tetrahedral partition of a polyhedral domain Ω.

Definition 2.2:
Let Ω be a polyhedral subset of $\mathbb{R}^n$, $n \in \mathbb{N}$. If Ω is divided into n-simplices T_i, $i = 1,\ldots,N$, such that the intersection of two different n-simplices is either empty or a common $k-$simplex, for $k = 0,\ldots,n-1$, then $\Delta := \{T_1,\ldots,T_N\}$ is called a **tessellation** of Ω into **n-simplices**.

An n-simplex $T := \langle v_1,\ldots,v_{n+1} \rangle$ is called non-degenerated if the points $v_i \in \mathbb{R}^n$, $i = 1,\ldots,n+1$, are linear independent.

In the bivariate case a tessellation of Ω into 2-simplices is called a **triangulation**.

2.1.2 Special n-simplices

In this subsection, special refinements of n-simplices are considered. These are, the Clough-Tocher and the Powell-Sabin split of a triangle, as well the Alfeld and Worsey-Farin split of a tetrahedron and a tetrahedral partition. Moreover, partial Worsey-Farin splits of a tetrahedron are considered.

2.1.2.1 Refinements of triangles

Definition 2.3 (Clough-Tocher split (cf. [24])):
Let $F := \langle v_1, v_2, v_3 \rangle$ be a triangle in $\mathbb{R}^2$, and let v_F be a point strictly inside F. Then the Clough-Tocher split of F is obtained by connecting v_F to the three vertices of F. The resulting refinement is denoted by F_{CT}, which consists of the three subtriangles $F_i := \langle v_i, v_{i+1}, v_F \rangle$, $i = 1,2,3$, where $v_4 := v_1$.

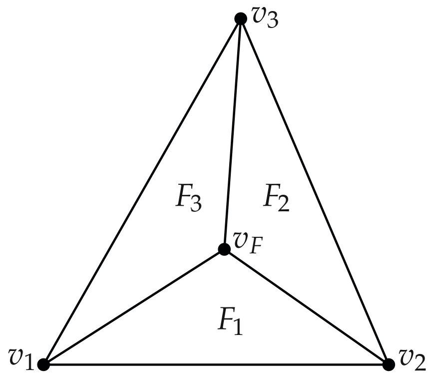

Figure 2.2: Clough-Tocher split of a triangle $F := \langle v_1, v_2, v_3 \rangle$ at the split point v_F in the interior of F.

Definition 2.4 (Powell-Sabin split (cf. [85]):
Let $F := \langle v_1, v_2, v_3 \rangle$ be a triangle in $\mathbb{R}^2$, v_F be a point strictly inside F, and let $v_{e,i}$, $i = 1,2,3$, be points strictly in the interior of the edges $e_i := \langle v_i, v_{i+1} \rangle$, $i = 1,2,3$, respectively, where $v_4 := v_1$. Then the Powell-Sabin split of F is obtained by connecting v_F to the three vertices of F and to the three points $v_{e,i}$, $i = 1,2,3$. The resulting refinement is denoted

by F_{PS} and consists of the six subtriangles $F_i := \langle v_i, v_{e,i}, v_F \rangle$, $i = 1,2,3$, and $\widetilde{F}_i := \langle v_{e,i}, v_{i+1}, v_F \rangle$, $i = 1,2,3$, where $v_4 := v_1$.

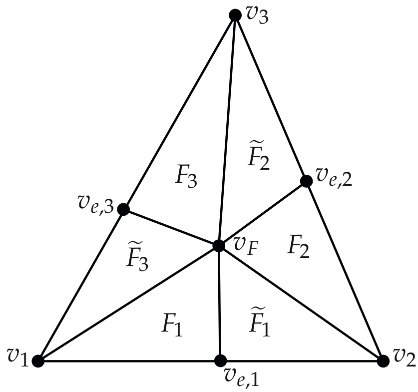

Figure 2.3: Powell-Sabin split of a triangle $F := \langle v_1, v_2, v_3 \rangle$ at the split point v_F in the interior of F, and the points $v_{e,i}$, $i = 1,2,3$, in the interior of the edges of F.

Remark 2.5:
In order to construct the Clough-Tocher split of a triangle $F := \langle v_1, v_2, v_3 \rangle$, usually the barycenter $v_F := \frac{v_1+v_2+v_3}{3}$ of F is chosen as split point.

To construct the Powell-Sabin split of a triangle F, the incenter of F is chosen as the split point v_F in the interior of F. As split points in the interior of the edges of F, the centers of the edges are chosen. In case F shares an edge with another triangle $\widetilde{F}$, then the split point in the common edge is chosen as the intersection of the line connecting the incenters v_F of F and $v_{\widetilde{F}}$ of $\widetilde{F}$ with this edge. To chose the incenters of the triangles as split points ensures, that the line connecting two interior split points of neighboring triangles intersects the common edge. This property is needed in order to obtain C^r, $r > 0$ smoothness, which is explained later on in this chapter, across this edge.

2.1.2.2 Refinements of tetrahedral partitions

Definition 2.6 (Alfeld split (cf. [1])):
Let $T := \langle v_1, v_2, v_3, v_4 \rangle$ be a tetrahedron in $\mathbb{R}^3$, and let v_T be the barycenter of T. Then the Alfeld split of T is constructed by connecting v_T to the four vertices of T. The obtained refinement is denoted by T_A, which consists of the four subtetrahedra $T_i := \langle v_i, v_{i+1}, v_{i+2}, v_T \rangle$, $i = 1, \ldots, 4$, where $v_5 := v_1$ and $v_6 := v_2$. For a tetrahedral partition Δ, the partition obtained by applying the Alfeld split to each tetrahedron in Δ is denoted by Δ_A.

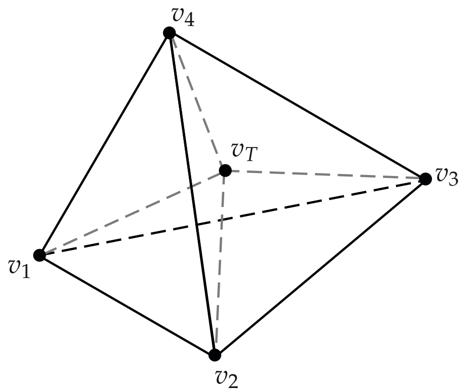

Figure 2.4: Alfeld split of a tetrahedron $T := \langle v_1, v_2, v_3, v_4 \rangle$ at the split point v_T in the interior of T.

Definition 2.7 (Worsey-Farin split (cf. [103])):
Let Δ be a tetrahedral partition and for each tetrahedron T in Δ let v_T be the incenter of T. For each interior face F of Δ, let v_F be the intersection of F and the straight line connecting the two incenters of the tetrahedra sharing F. In case F is on the boundary of Δ, the point v_F is chosen as the barycenter of F. Then the Worsey-Farin split of a tetrahedron $T := \langle v_1, v_2, v_3, v_4 \rangle$ is constructed by connecting v_T to the four vertices of T and the four points $v_{F,i}$, $i = 1, \ldots, 4$, in the interior of the faces $F_i := \langle v_i, v_{i+1}, v_{i+2} \rangle$, where $v_5 := v_1$ and $v_6 := v_2$, and by connecting the points $v_{F,i}$, $i = 1, \ldots, 4$, to the three vertices of the corresponding face F_i, respectively. The obtained refinement is denoted by T_{WF}, which consists of the

twelve subtetrahedra $T_{i,j} := \langle u_{i,j}, u_{i,j+1}, v_{F,i}, v_T \rangle$, $i = 1,\ldots,4$, $j = 1,2,3$, where $u_{i,j} := v_{i+j-1}$, with $u_{i,4} := u_{i,1}$ and $u_{i,j} := v_{i+j-5}$ for $i+j > 5$ and $j \neq 4$. We define Δ_{WF} to be the refined tetrahedral partition obtained by dividing each tetrahedron of Δ with the Worsey-Farin split.

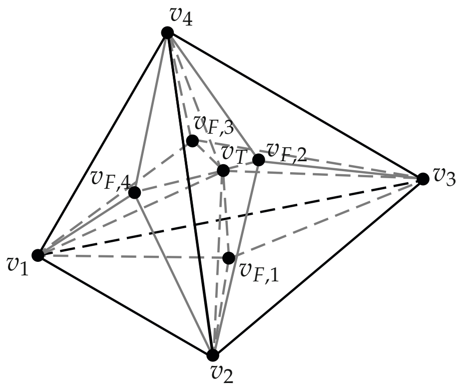

Figure 2.5: Worsey-Farin split of a tetrahedron $T := \langle v_1, v_2, v_3, v_4 \rangle$ at the split point v_T in the interior of T and the split points $v_{F,i}$, $i = 1,\ldots,4$, in the interior of the faces of T.

Note that by choosing the incenters of the tetrahedra as interior split points, it is ensured that the line connecting two of these split points from neighboring tetrahedra intersects the common face F (cf. Lemma 16.24 in [62]).

Definition 2.8 (Partial Worsey-Farin splits (cf. [42])):
Let T be a tetrahedron in $\mathbb{R}^3$, and let v_T be the incenter of T. Given an integer $0 \leq m \leq 4$, let $F_1,\ldots,F_m$ be distinct faces of T, and for each $i = 1,\ldots,m$, let $v_{F,i}$ be a point in the interior of F_i. Then the m-th order partial Worsey-Farin split T^m_{WF} of T is defined as the refinement obtained by the following steps:

1. connect v_T to each of the four vertices of T;

2. connect v_T to the points $v_{F,i}$ for $i = 1,\ldots,m$;

3. connect $v_{F,i}$ to the three vertices of F_i for $i = 1,\ldots,m$.

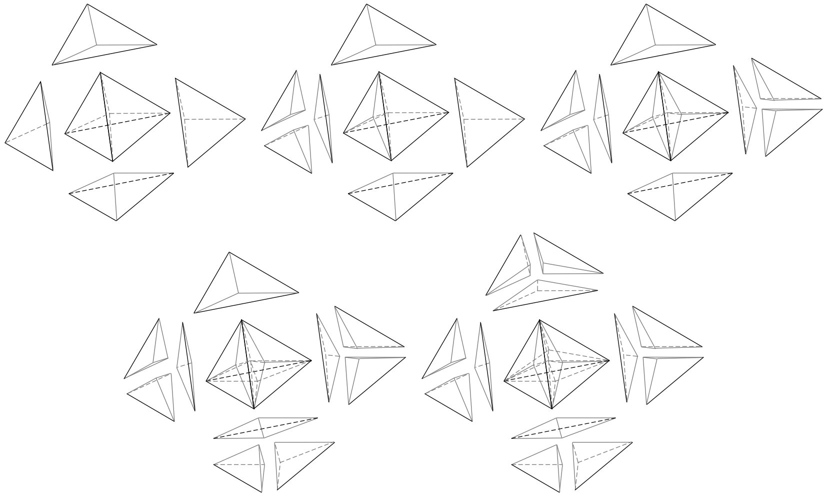

Figure 2.6: Partial Worsey-Farin splits of order m for $m = 0, \ldots, 4$, of a tetrahedron.

Note that in case T shares a face F with another tetrahedron, that is refined with a partial Worsey-Farin split, the point v_F is chosen as the intersection of the straight line connecting the incenters of the two tetrahedra and the face F. Else, the point v_F is chosen as the barycenter of F. Thus, the split points are chosen in the same way as for the Worsey-Farin split of a tetrahedron (cf. Definition 2.7).

It can easily be seen that the m-th order partial Worsey-Farin split of a tetrahedron results in $4 + 2m$ subtetrahedra. For $m = 0$ the partial Worsey-Farin split reduces to the Alfeld split (cf. Definition 2.6), although here the incenter is chosen as split point and not the barycenter. For the case $m = 4$ the partial Worsey-Farin split results in the Worsey-Farin split (cf. Definition 2.7).

2.1.3 Notation and Euler relations

In this subsection some notation are established and the Euler relations are stated for tetrahedral partitions Δ.

Notation 2.9:
For a tetrahedral partition Δ we define the following sets:

$\mathcal{V}_I, \mathcal{V}_B, \mathcal{V}$	:	Set of inner, boundary, and all vertices of Δ
$\mathcal{E}_I, \mathcal{E}_B, \mathcal{E}$	:	Set of inner, boundary, and all edges of Δ
$\mathcal{F}_I, \mathcal{F}_B, \mathcal{F}$	:	Set of inner, boundary, and all faces of Δ
N	:	Cardinality of the set of tetrahedra in Δ

Following Leonhard Euler, the following properties hold:

$$\begin{aligned} \#\mathcal{V}_B &= 2N - \#\mathcal{F}_I + 2 \\ N &= \#\mathcal{V}_I - \#\mathcal{E}_I + \#\mathcal{F}_I + 1 \end{aligned}$$

In order to characterize the tetrahedra of a partition Δ, the following terms and definitions are used.

Two tetrahedra $T, \widetilde{T} \in \Delta$ **touch** each other, if they share a common vertex, or a common edge. They are called **neighbors**, if they share a common face.

Two triangular faces $F, \widetilde{F} \in \mathcal{F}$ are called **neighbors**, if there is a tetrahedron $T \in \Delta$ where $F, \widetilde{F} \in T$. Two faces F and $\widetilde{F}$ of two tetrahedra T and $\widetilde{T}$ are called **degenerated** if they lie in a common plane.

The next definition deals with certain subsets of a tetrahedral partition Δ. It is equivalent for tessellations Δ of a domain $\Omega \subseteq \mathbb{R}^n$.

Definition 2.10:
Let $T \in \Delta$ and

$$\text{star}^0(T) := T.$$

Then, for $n \geq 1$, $\text{star}^n(T)$ is defined inductively as

$$\text{star}^n(T) := \bigcup \{ \widetilde{T} \in \Delta : \ \widetilde{T} \cap \text{star}^{n-1}(T) \neq \emptyset \}.$$

2.2 Splines

In this section first multivariate polynomials are defined. These form a basis for the space of multivariate splines, which are introduced afterwards. Furthermore, special subspaces of the spline space, the spaces of multivariate supersplines, are defined.

2.2.1 Multivariate polynomials

Definition 2.11:
Let $d \in \mathbb{N}_0$ and $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Then

$$\mathcal{P}_d^n = \text{span}\{x_1^{i_1} \cdot \ldots \cdot x_n^{i_n} : i_1, \ldots, i_n \geq 0, i_1 + \ldots + i_n \leq d\}$$

is the space of polynomials in n dimensions of total degree $\leq d$, where $\dim(\mathcal{P}_d^n) = \binom{d+n}{n}$.

Thus, each multivariate polynomial $p \in \mathcal{P}_d^n$ can be written in the form

$$p(x_1, \ldots, x_n) = \sum_{\substack{i_1 + \ldots + i_n \leq d \\ i_1, \ldots, i_n \geq 0}} a_{i_1 \ldots i_n} x_1^{i_1} \cdot \ldots \cdot x_n^{i_n},$$

where the $a_{i_1 \ldots i_n} \in \mathbb{R}$ are linear factors. This representation of multivariate polynomials is also called the monomial form.

2.2.2 Multivariate splines

In this subsection multivariate splines are defined. Therefore, first the space of smooth functions has to be denominated.

Definition 2.12:
Let Ω be a domain in $\mathbb{R}^n$, $n \in \mathbb{N}$, and let $r \in \mathbb{N}_0$. Then

$$C^r(\Omega) := \{f : \Omega \longrightarrow \mathbb{R} : f \text{ is } r\text{-times continuous differentiable}\}$$

is the space of functions of C^r smoothness, or also C^r continuity. For $C^0(\Omega)$, we also write $C(\Omega)$.

Now, the space of multivariate splines can be defined.

Definition 2.13:
Let $r, d \in \mathbb{N}_0$, with $0 \leq r < d$, and let $\Omega \subset \mathbb{R}^n, n \in \mathbb{N}$, be a polyhedral domain and Δ a tessellation of Ω into $n-$simplices. Then

$$\mathcal{S}_d^r(\Delta) = \{s \in C^r(\Omega) : s_{|T} \in \mathcal{P}_d^n \ \forall \ T \in \Delta\}$$

is called the **space of multivariate splines** of C^r smoothness of degree d.

Functions contained in $\mathcal{S}_d^r(\Delta)$ are piecewise polynomials of degree d which have C^r continuity at each intersection of two simplices in Δ.

2.2.3 Multivariate supersplines

For certain problems, as the partial Worsey-Farin splits in chapter 3, therewith also the Lagrange interpolation methods in chapter 4 and 5, or the macro-elements presented in chapter 7 and 8, additional smoothness conditions are needed, i.e. at the vertices and edges of a tetrahedral partition. The subspace of splines that fulfill these additional conditions is called the space of multivariate superspline. Thus, for a given tessellation Δ of a domain Ω into $n-$simplices, $n \in \mathbb{N}$, a superspline subspace of $\mathcal{S}_d^r(\Delta)$ is to be of $C^{\rho_{k,i}}$ smoothness at the $k-$simplices $v_{k,i}$ in Δ, for $k = 0, \dots, n-1$, with $r \leq \rho_{n-1,i} \leq \dots \leq \rho_{0,i} < d$.

Definition 2.14:
Let r, d be in $\mathbb{N}_0$, with $0 \leq r < d$, Ω a polyhedral domain in $\mathbb{R}^n$, Δ a tessellation of Ω into $n-$simplices, $n \in \mathbb{N}$, N_k the cardinality of the set of $k-$simplices in Δ, for $k = 0, \dots, n-1$, and let $\{v_{k,1}, \dots, v_{k,N_k}\}$ be the set of $k-$simplices in Δ. Let $\rho_k := (\rho_{k,1}, \dots, \rho_{k,N_k})$, $\rho_{k,i} \in \mathbb{N}$, for $i = 1, \dots, N_k$ and $k = 0, \dots, n-1$, where the $\rho_{k,i}$ are to fulfill $r \leq \min_{i=1,\dots,N_{n-1}} \rho_{n-1,i} \leq \max_{i=1,\dots,N_{n-1}} \rho_{n-1,i} \leq \dots \leq \min_{i=1,\dots,N_0} \rho_{0,i} \leq \max_{i=1,\dots,N_0} \rho_{0,i} < d$. Then the space

$$\mathcal{S}_d^{r,\rho_0,\dots\rho_{n-1}}(\Delta) = \{s \in \mathcal{S}_d^r(\Delta) : s \in C^{\rho_{k,i}}(v_{k,i}),\ i = 1, \dots, N_k,\ k = 0, \dots, n-2\}$$

is called a **superspline subspace** of $\mathcal{S}_d^r(\Delta)$.

Following Definition 2.14, for $\Omega \subset \mathbb{R}^2$ and a corresponding triangulation Δ a spline in $\mathcal{S}_d^{r,\rho}$ consists of bivariate polynomials of degree d that join with C^r smoothness at the edges of Δ and C^ρ smoothness at the vertices of Δ. Accordingly, for $\Omega \subset \mathbb{R}^3$ and a tetrahedral partition Δ, splines in $\mathcal{S}_d^{r,\rho_1,\rho_2}$ are piecewise polynomials of degree d that join with C^r smoothness at the triangular faces of Δ, C^{ρ_2} smoothness at the edges of Δ, and C^{ρ_1} smoothness at the vertices of Δ.

2.3 Bernstein-Bézier techniques

In this section, we consider the well known Bernstein-Bézier techniques, which are used to examine multivariate splines. The basis for these techniques can be found in [14, 33, 39, 62], among others. We first introduce barycentric coordinates, that form a basis for the Bernstein-Bézier techniques. Subsequently, we introduce the Bernstein polynomials and the

B-form of a polynomial, which will be used throughout this work. Then, the de Casteljau algorithm for the evaluation of polynomials in the B-form, and also the subdivision of polynomials, is considered (cf. [37]). In the next subsection, smoothness conditions between adjacent n-simplices are investigated. Finally, the concept of minimal determining sets is introduced, which can be used to characterize spline spaces and to consider their dimension.

2.3.1 Barycentric coordinates

In this subsection, we consider barycentric coordinates. These are very efficient for the representation of points in $\mathbb{R}^n$ relative to an n-simplex T, since they are affine invariant, in contrast to the usual Cartesian coordinates. Note that here and in the course of this work, we will assume that T is a non-degenerated n-simplex.

Definition 2.15:
Let $T := \langle v_1, \ldots, v_{n+1} \rangle$ be an n-simplex in $\mathbb{R}^n$ with vertices $v_1, \ldots, v_{n+1}$. Then there exist unique **barycentric coordinates** $\phi_1, \ldots, \phi_{n+1} \in \mathcal{P}_1^n$, that satisfy

$$\sum_{i=1}^{n+1} \phi_i = 1,$$

as well as the interpolation property

$$\phi_i(v_j) = \delta_{i,j} = \begin{cases} 1, & \text{for } i = j, \\ 0, & \text{for } i \neq j, \end{cases} \quad i, j = 1, \ldots, n+1.$$

They can be explicitly computed as

$$\phi_i(v) = \frac{\left| \begin{pmatrix} 1 & \ldots & 1 & \ldots & 1 \\ v_1 & \ldots & v & \ldots & v_{n+1} \end{pmatrix} \right|}{\left| \begin{pmatrix} 1 & \ldots & 1 & \ldots & 1 \\ v_1 & \ldots & v_i & \ldots & v_{n+1} \end{pmatrix} \right|}, \quad \text{for } i = 1, \ldots, n+1, \text{ and for all } v \in \mathbb{R}^n.$$

Moreover, for each point $v \in \mathbb{R}^n$

$$v = \phi_1(v)v_1 + \ldots + \phi_{n+1}(v)v_{n+1}$$

holds.