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Structure of Solutions of Variational Problems

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Preface

This book is devoted to the presentation of progress made during the last 10 years in the studies of the structure of approximate solutions of variational problems considered on subintervals of a real line. We present the results on properties of approximate solutions which are independent of the length of the interval, for all sufficiently large intervals. The results in this book deal with the so-called turnpike property of the variational problems. To have this property means, roughly speaking, that the approximate solutions of the problems are determined mainly by the integrand (objective function) and are essentially independent of the choice of interval and endpoint conditions, except in regions close to the endpoints. Turnpike properties are well known in mathematical economics. The term was first coined by P. Samuelson in 1948 when he showed that an efficient expanding economy would spend most of the time in the vicinity of a balanced equilibrium path (also called a von Neumann path). Now it is well known that the turnpike property is a general phenomenon which holds for large classes of variational problems. For these classes of problems using the Baire category approach, it was shown that the turnpike property holds for a generic (typical) problem. In this book we are interested in individual (non-generic) turnpike results and in sufficient and necessary conditions for the turnpike phenomenon.

Haifa, Israel — Alexander J. Zaslavski

Contents

1

Introduction

In this chapter we introduce and discuss turnpike properties and describe the structure of the book.

The study of optimal control problems and variational problems defined on infinite intervals and on sufficiently large intervals has been a rapidly growing area of research [3, 4, 7, 9, 10, 11, 12, 13, 16, 18, 19, 20, 23, 25, 30, 31, 32, 33, 39, 40, 51]. These problems arise in engineering [1, 21, 53], in models of economic growth [2, 15, 17, 24, 29, 34, 35, 37, 51], in infinite discrete models of solid-state physics related to dislocations in one-dimensional crystals [6, 38], and in the theory of thermodynamical equilibrium for materials [14, 22, 26, 27, 28].

Our book consists of four chapters. Here, in Chap. 1, we describe its structure. In Chap. 2 we study the structure of approximate solutions of nonautonomous variational problems with continuous integrands $f : [0,\infty) \times R^n \times R^n \to R^1$, where R^n is the n-dimensional Euclidean space. In Chap. 3, we study turnpike properties for autonomous variational problems with integrands $f : R^n \times R^n \to R^1$. Finally, in Chap. 4 we study the structure of approximate solutions of variational problems with integrands $f : R^n \times R^n \to R^1$ which are convex functions on $R^n \times R^n$.

Now we describe the structure of the book. We begin in Chap. 2 with the study of the structure of solutions of the variational problems:

$$\int_{T_1}^{T_2} f(t, z(t), z'(t))dt \to \min,\ z(T_1) = x,\ z(T_2) = y, \qquad (P)$$

$$z : [T_1, T_2] \to R^n \text{ is an absolutely continuous function,}$$

where $T_1 \geq 0$, $T_2 > T_1$, $x, y \in R^n$, and $f : [0,\infty) \times R^n \times R^n \to R^1$ belongs to a complete metric space of integrands $\mathcal{M}$ which is introduced in Sect. 2.1. Note that these integrands do not satisfy any convexity assumption which is usually used in the calculus of variations.

It is well known that the solutions of the problems (P) exist for integrands f which satisfy two fundamental hypotheses concerning the behavior of the

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integrand as a function of the last argument (derivative): one that the integrand should grow superlinearly at infinity and the other that it should be convex [8, 36]. Moreover, certain convexity assumptions are also necessary for properties of lower semicontinuity of integral functionals which are crucial in most of the existence proofs. For integrands f which do not satisfy the convexity assumption the existence of solutions of the problems (P) is not guaranteed, and in this situation we consider δ-*approximate solutions*.

Let $T_1 \geq 0$, $T_2 > T_1$, $x, y \in R^n$, and $f : [0,\infty) \times R^n \times R^n \to R^1$ be an integrand, and let δ be a positive number. We say that an absolutely continuous (a.c.) function $u : [T_1, T_2] \to R^n$ satisfying $u(T_1) = x$, $u(T_2) = y$ is a δ-approximate solution of the problem (P) if

$$\int_{T_1}^{T_2} f(t, u(t), u'(t))dt \leq \int_{T_1}^{T_2} f(t, z(t), z'(t))dt + \delta$$

for each a.c. function $z : [T_1, T_2] \to R^n$ satisfying $z(T_1) = x,\ z(T_2) = y$.

In Chap. 1 we deal with the so-called turnpike property of the variational problems (P) associated with an integrand f. To have this property means that there exists a bounded continuous function $X_f : [0,\infty) \to R^n$ depending only on f such that for each pair of positive numbers $K, \epsilon > 0$, there exist positive constants $L = L(K,\epsilon)$ and $\delta = \delta(K,\epsilon)$ depending on ϵ, K such that if $u : [T_1, T_2] \to R^n$ is an δ-approximate solution of the problem (P) with

$$T_2 - T_1 \geq L,\ |u(T_i)| \leq K,\ i = 1, 2,$$

then

$$|u(t) - X_f(t)| \leq \epsilon \text{ for all } t \in [T_1 + \tau_1, T_2 - \tau_2],$$

where $\tau_1, \tau_2 \in [0, L]$.

(The precise description of the turnpike property is given in Sect. 2.1.)

If the integrand f possesses the turnpike property, then the solutions of variational problems with f are essentially independent of the choice of time interval and values at the endpoints except in regions close to the endpoints of the time interval. If a point t does not belong to these regions, then the value of a solution at t is closed to a trajectory X_f ("turnpike") which is defined on the infinite time interval and depends only on f. This phenomenon has the following interpretation. If one wish to reach a point A from a point B by a car in an optimal way, then one should turn to a turnpike, spend most of time on it, and then leave the turnpike to reach the required point.

Turnpike properties are well known in mathematical economics. The term was first coined by Samuelson in 1948 (see [35]) where he showed that an efficient expanding economy would spend most of the time in the vicinity of a balanced equilibrium path (also called a von Neumann path). This property was further investigated for optimal trajectories of models of economic dynamics (see, e.g., [2, 15, 17, 24, 29, 34, 37, 51]). Many turnpike results are collected in [51].

In the classical turnpike theory the function f does not depend on the variable t, is strictly convex on the space $R^n \times R^n$, and satisfies a growth condition common in the literature. In this case, the turnpike property can be established, the turnpike X_f is a constant function, and its value is a unique solution of the maximization problem $f(x, 0) \to \min$, $x \in R^n$.

It was shown in our research, which was summarized in [51], that the turnpike property is a general phenomenon which holds for large classes of variational and optimal control problems without convexity assumptions. For these classes of problems a turnpike is not necessarily a constant function (singleton) but may instead be an nonstationary trajectory (in the discrete time nonautonomous case) [51] or an absolutely continuous function on the interval $[0, \infty)$ as it was described above (in the continuous time nonautonomous case) [44, 45, 51] or a compact subset of the space X (in the autonomous case) [39, 40, 41, 43, 50, 51].

More precisely, in Chap. 2 of [51] we study the turnpike properties for variational problems with integrands which belong to a subspace $\mathcal{M}_{co}$ of $\mathcal{M}$. The subspace $\mathcal{M}_{co} \subset \mathcal{M}$ consists of integrands $f \in \mathcal{M}$ such that the function $f(t, x, \cdot) : R^n \to R^1$ is convex for any $(t, x) \in [0, \infty) \times R^n$. In Chap. 2 of [51] we showed that the turnpike property holds for a generic integrand $f \in \mathcal{M}_{co}$. Namely, we established the existence of a set $\mathcal{F}_{co} \subset \mathcal{M}_{co}$ which is a countable intersection of open everywhere dense sets in $\mathcal{M}_{co}$ such that each $f \in \mathcal{F}_{co}$ has the turnpike property.

In [45] we extend this turnpike result of [51] established for the space $\mathcal{M}_{co}$ to the space of integrands $\mathcal{M}$. We show the existence of a set $\mathcal{F} \subset \mathcal{M}$ which is a countable intersection of open everywhere dense sets in $\mathcal{M}$ such that each $f \in \mathcal{F}$ has the turnpike property. In [45] we show that an integrand $f \in \mathcal{F}$ has a turnpike X_f which is a bounded continuous function.

Therefore according to the results of [45, 51], the turnpike property is a general phenomenon which holds for large classes of variational problems. For these classes of problems, using the Baire category approach, it was shown that the turnpike property holds for a generic (typical) problem. In this book we are interested in individual (nongeneric) turnpike results and in sufficient and necessary conditions for the turnpike phenomenon.

In Chap. 2 we consider the following question. Assume that $f \in \mathcal{M}$ and $X : [0, \infty) \to R^n$ is a bounded continuous function. How to verify if the integrand f possesses the turnpike property and X is its turnpike? In Sect. 2.1 we introduce two properties (P1) and (P2) and show that f has the turnpike property if and only if f possesses the properties (P1) and (P2). The property (P2) means that all approximate solutions of the corresponding infinite horizon variational problem have the same asymptotic behavior while the property (P1) means that if an a.c. function $v : [0, T] \to R^n$ is an approximate solution and T is large enough, then there is $\tau \in [0, T]$ such that $v(\tau)$ is close to $X(\tau)$. This result, which was obtained in [47], is stated in Sect. 2.1 while it is proved in Sects. 2.2–2.6.

In Chap. 2 we also consider the strong turnpike property, which was introduced and studied in [49], for an integrand $f \in \mathcal{M}$ such that the function $f(t, x, \cdot) : R^n \to R^1$ is convex for any $(t, x) \in [0, \infty) \times R^n$. We say that the integrand f possesses the strong turnpike property if the turnpike property holds for f with the function X_f being the turnpike, and moreover, in the definition of the turnpike property, $\tau_1 = 0$ if

$$|v(T_1) - X_f(T_1)| \leq \delta$$

and $\tau_2 = 0$ if

$$|v(T_2) - X_f(T_2)| \leq \delta.$$

This additional condition means that if at the point T_1 (T_2, respectively) the value of the approximate solution v is closed to a trajectory X_f, then the value of the solution v at t is closed to a trajectory X_f for all t except in a region close to the endpoint T_2 (T_1, respectively).

In Sect. 2.1 we state Theorem 2.4 (the second main result of Chap. 2) which was obtained in [49]. According to this result, the integrand f possesses the strong turnpike property with the function X_f being the turnpike if and only if the properties (P1) and (P2) hold and the function X_f is a unique solution of the corresponding infinite horizon variational problem associated with the integrand f.

In Chap. 3 we study turnpike properties for autonomous variational problems with continuous integrands $f : R^n \times R^n \to R^1$ which belong to the subspace of functions $\mathcal{A} \subset \mathcal{M}$ introduced in Sect. 3.1. It should be mentioned that $\mathcal{A}$ is a large space of autonomous integrands which was considered in Chaps. 3–5 of [51]. In [41] and in Chap. 3 of [51] we study the structure of approximate solutions of variational problems (P) with integrands $f \in \mathcal{A}$ and show the existence of a subset $\mathcal{F} \subset \mathcal{A}$ which is a countable intersection of open everywhere dense subsets of $\mathcal{A}$ such that each integrand $f \in \mathcal{F}$ has a turnpike property, where the turnpike is a nonempty compact subset of the n-dimensional Euclidean space R^n. More precisely, we show there (using the Baire category approach) that for a generic (typical) integrand $f \in \mathcal{A}$, there exists a nonempty compact set $H(f) \subset R^n$ such that the following property holds:

For each $\epsilon > 0$ there exists a constant $L > 0$ such that if v is a solution of problem (P), then for most of $t \in [T_1, T_2]$ the set $v([t, t + L])$ is equal to $H(f)$ up to ϵ with respect to the Hausdorff metric.

Note that for a generic integrand $f \in \mathcal{A}$ the turnpike $H(f)$ is a nonempty compact subset of R^n which is not necessarily a singleton. It should be mentioned that there exists $f \in \mathcal{A}$ which does not have this turnpike property. In Chap. 3 we prove individual (nongeneric) turnpike results for autonomous variational problems with integrands $f \in \mathcal{A}$.

Let $f \in \mathcal{A}$. A locally absolutely continuous (a.c.) function $v : R^1 \to R^n$ is called (f)-minimal if

$$\sup\{|v(t)| : t \in R^1\} < \infty$$