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Preface

This book is an attempt to give, as much as possible, a self-contained presentation of some of the main ideas involved in the mathematical analysis of the Sherrington–Kirkpatrick model and closely related mixed p-spin models of spin glasses. Certain topics, such as the high-temperature region and phase transition, are not covered and can be found in the comprehensive manuscript of Michel Talagrand [66].

In 1975 David Sherrington and Scott Kirkpatrick introduced in [58] a model of a spin glass—a disordered magnetic alloy that exhibits unusual magnetic behavior. This model is also often interpreted as a question about a typical behavior of the optimization problem $\max_{\sigma\in\Sigma_N} H_N(\sigma)$ for a certain function $H_N(\sigma)$ on the space of *spin configurations* $\Sigma_N = \{-1,+1\}^N$. This means that the parameters of $H_N(\sigma)$ are modeled as random variables and one would like to understand the asymptotic behavior of the average $\mathbb{E}\max_{\sigma\in\Sigma_N} H_N(\sigma)$ in the *thermodynamic (infinite-volume) limit*, as the size of the system N goes to infinity. We will see in Chap. 1 that, in order to solve this problem, it is enough to compute the limit of the *free energy*,

$$\lim_{N\to\infty}\frac{1}{N}\mathbb{E}\log\sum_{\sigma\in\Sigma_N}\exp\beta H_N(\sigma),$$

for each *inverse temperature* parameter $\beta = 1/T > 0$, and the formula for this limit was proposed by Sherrington and Kirkpatrick in [58] based on the so-called replica formalism. At the same time, they observed that their *replica symmetric solution* exhibits "unphysical behavior" at low temperature, which means that it can only be correct at high temperature.

Several years later Giorgio Parisi proposed [51, 52] another *replica symmetry breaking solution* within replica theory, now called the *Parisi ansatz*, which was consistent at any temperature $T\geq 0$ and, moreover, was in excellent agreement with computer simulations. The key feature of the celebrated Parisi ansatz was the choice of an *ultrametric parametrization* of the replica matrix in the computation of the free energy based on the replica approach. A rich theory emerged in the physics literature during the subsequent interpretation of the Parisi solution in terms of some physical properties of the *Gibbs measure* of the model

$$G_N(\sigma) = \frac{\exp \beta H_N(\sigma)}{\sum_{\rho \in \Sigma_N} \exp \beta H_N(\rho)}.$$

In particular, in the work of Parisi [53], the *order parameter* in the ultrametric parametrization of the replica matrix was related to the distribution of the *overlap* $R_{1,2} = N^{-1} \sum_{i=1}^{N} \sigma_i^1 \sigma_i^2$ of two spin configurations $\sigma^1, \sigma^2 \in \Sigma_N$ sampled from the Gibbs measure. The Parisi ansatz was further interpreted in terms of the geometric structure of the Gibbs measure in the work of Mézard et al. [37, 38], where it was understood, for example, that the ultrametricity of the replica matrix corresponds to the ultrametricity of the support of the Gibbs measure in the infinite-volume limit. Such reinterpretation of the Parisi solution formed a beautiful general physical theory of the model, which was described in the famous book of Mézard, Parisi, and Virasoro, "Spin Glass Theory and Beyond," [40]. In some sense, this also opened a path to a rigorous mathematical theory of the model.

Around the same time, motivated by the developments in the SK model, Bernard Derrida proposed two simplified models of spin glasses—the random energy model, REM, in [16, 17], and the generalized random energy model, GREM, in [18, 19]. The REM can be viewed as a formal limit of the family of the so-called pure p-spin models, in which the SK model corresponds to $p = 2$, and its Hamiltonian $H_N(\sigma)$ is given by an i.i.d. sequence of Gaussian random variables with variance N indexed by $\sigma \in \Sigma_N$, which is a rather classical object. The GREM combines several random energy models in a hierarchical way with the ultrametric structure built into the model from the beginning. Even though these simplified models do not shed light on the Parisi ansatz in the SK model directly, the structure of the Gibbs measures in these models was predicted to be, in some sense, identical to that of the SK model in the infinite-volume limit. For example, Derrida and Toulouse showed in [20] that the Gibbs weights in the REM have the same distribution in the thermodynamic limit as the Gibbs weights of the pure states (clusters of spin configurations) in the SK model; this latter distribution was computed earlier in [37] using the replica method. Independently, Mézard et al. [39] illustrated the connection between the REM and the SK model from a different point of view and, finally, de Dominicis and Hilhorst [15] demonstrated a similar connection between the Gibbs measure of the GREM and the global structure of the Gibbs measure in the SK model predicted by the Parisi ansatz.

The realization that the structure of the Gibbs measure in the SK model predicted by the Parisi replica theory coincides with the structure of the Gibbs measure in the GREM, which is much simpler than the SK model, turned out to be a very important step toward a deeper understanding of the Parisi ansatz. In particular, motivated by this connection with the SK model, in his seminal paper [56], David Ruelle gave an alternative explicit description of the Gibbs measure in the GREM in the thermodynamic limit in terms of a certain family of Poisson processes. As a result, one could now study the properties of these measures, nowadays called the *Ruelle probability cascades*, using the entire arsenal of the theory of Poisson processes. Some of these properties were already described in the original paper of Ruelle [56], while other important properties, which express certain invariance features of these measures, were discovered later by Erwin Bolthausen and Alain-Sol Sznitman in [10]. We will study the Ruelle probability cascades, including their invariance properties, in Chap. 2.

Another breakthrough in the mathematical analysis of the SK model came at the end of the nineties with the discovery of the two so-called *stability properties* of the Gibbs measure in the SK model in the work of Stefano Ghirlanda and Francesco Guerra [25] and Michael Aizenman and Pierluigi Contucci [1]. It was clear that these stability properties, known as the Ghirlanda–Guerra identities and the Aizenman–Contucci stochastic stability, impose very strong constraints on the structure of the Gibbs measure, but the question was whether they lead all the way to the Ruelle probability cascades. The Aizenman–Contucci stochastic stability is identical to one part of the Bolthausen–Sznitman invariance property for the Ruelle probability cascades. The fact that the Ghirlanda–Guerra identities also hold for the Ruelle probability cascades was first proved by Michel Talagrand in [62] in the case corresponding to the REM and, soon after, by Anton Bovier and Irina Kurkova [11] in the general case corresponding to the GREM. This means that both the Aizenman–Contucci stochastic stability and the Ghirlanda–Guerra identities, which were discovered in the setting of the SK model, also appear in the setting of the Ruelle probability cascades, suggesting some connection between the two.

The first partial answer to the above question was given in an influential work of Louis-Pierre Arguin and Michael Aizenman [5] who proved that, under a technical assumption that the overlap takes finitely many values in the thermodynamic limit, the Aizenman–Contucci stochastic stability implies the ultrametricity predicted by the Parisi ansatz. Soon after, it was shown in [43] that, under the same technical assumption, the Ghirlanda–Guerra identities also imply ultrametricity (an elementary proof can be found in [47]). Another approach was proposed by Talagrand in [65]. However, since at low temperature the overlap does not necessarily take finitely many values in the thermodynamic limit, all these results were not directly applicable to the SK model. Nevertheless, they strongly suggested that the stability properties can explain the Parisi ansatz and, indeed, the fact that the Ghirlanda–Guerra identities imply ultrametricity in general, without any technical assumptions, was proved in [50]. This means that the Ghirlanda–Guerra identities characterize the Ruelle probability cascades, which confirms the prediction of the physicists that the Gibbs measure in the SK model coincides with (or can be approximated by) the Ruelle probability cascades.

Even though the proof of this result, which will be given at the end of Chap. 2, is based only on the Ghirlanda–Guerra identities, it is important to mention that the Aizenman–Contucci stochastic stability played an important role in the discovery. It started with an observation made by Talagrand in 2007 (private communication, see also [66]) who noticed that in the setting of the Ruelle probability cascades the Ghirlanda–Guerra identities are contained in the Bolthausen–Sznitman invariance.

Talagrand's observation was reversed in [49] where the Ghirlanda–Guerra identities were combined with the Aizenman–Contucci stochastic stability and expressed as one unified stability property for the Gibbs measure in the SK model, which is the exact analogue of the Bolthausen–Sznitman invariance property in the setting of the Ruelle probability cascades. From this unified stability property one can derive a new invariance property that will appear in Sect. 2.5, where it will be used to prove the ultrametricity of the Gibbs measure in the SK model predicted by the Parisi ansatz. However, this new invariance property can be obtained much more easily as a direct consequence of the Ghirlanda–Guerra identities, which means that the Ghirlanda–Guerra identities alone explain the Parisi ansatz in the SK model and, for this reason, the Aizenman–Contucci stochastic stability will not be discussed in the book, even though behind the scenes it played a very important role. In some sense, this is good news because the Aizenman–Contucci stability is a more subtle property to work with than the Ghirlanda–Guerra identities, especially in the infinite-volume limit.

Once the structure of the Gibbs measure is understood, we will be in a position to prove the celebrated Parisi formula for the free energy. This will be the main focus of Chap. 3. The proof is based on two key results in the mathematical theory of the SK model—the replica symmetry breaking interpolation bound of Guerra [27] and the cavity computation scheme of Aizenman et al. [2]. The main idea of Guerra [27] can be viewed as a very clever interpolation between the SK model and the Ruelle probability cascades, which implies, due to monotonicity, that the Parisi formula is, in fact, an upper bound on the free energy of the SK model. Following this breakthrough discovery of Guerra, Talagrand proved in his famous tour-de-force paper [64] that the Parisi formula, indeed, gives the free energy in the SK model in the thermodynamic limit. Talagrand's ingenious proof finds a way around the Parisi ansatz for the Gibbs measure, but it is quite complicated. In Chap. 3 we will describe a much more direct approach to the matching lower bound based on the Aizenman–Sims–Starr cavity computation and the fact that the Gibbs measure can be approximated by the Ruelle probability cascades. Another advantage of this approach is that it yields the Parisi formula for all mixed p-spin models, while Talagrand's proof worked only for mixed p-spin models for even $p \geq 2$. For simplicity of notation, we only consider models without the external field, but all the results hold with obvious modifications in the presence of the external field.

In Chap. 4, we will study the Gibbs measure in the mixed p-spin models in more detail and describe the joint distribution of all spins in terms of the Ruelle probability cascades. This chapter is motivated by a different family of mean-field spin glass models that includes the random K-sat and diluted p-spin models, for which the main predictions of the physicists remain open and, since we can prove these predictions (in a certain sense) in the setting of the mixed p-spin models, we use it as an illustration of what is expected in these other models.

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Contents

Chapter 1
The Free Energy and Gibbs Measure

In Sect. 1.1, we will introduce the Sherrington–Kirkpatrick model and a family of closely related mixed p-spin models and give some motivation for the problem of computing the free energy in these models. A solution of this problem in Chap. 3 will be based on a description of the structure of the Gibbs measure in the thermodynamic limit and in this chapter we will outline several connections between the free energy and Gibbs measure. At the same time, we will introduce various ideas and techniques, such as the Gaussian integration by parts, Gaussian interpolation, and Gaussian concentration, that will play essential roles in the key results of this chapter and throughout the book. In the last section, we will prove the Dovbysh–Sudakov representation for Gram-de Finetti arrays, which will allow us to define a certain analogue of the Gibbs measure in the thermodynamic limit. As a first step, we will prove the Aldous–Hoover representation for exchangeable and weakly exchangeable arrays. In Sect. 1.4, we will give a classic probabilistic proof of this result for weakly exchangeable arrays and, for a change, in the Appendix we will prove the representation for exchangeable arrays using a different approach, based on more recent ideas of Lovász and Szegedy in the framework of limits of dense graph sequences. We will describe another application of the Aldous–Hoover representations for exchangeable arrays in Chap. 4.

1.1 The Sherrington–Kirkpatrick Model

The Sherrington–Kirkpatrick model originated in physics and was introduced in 1975 as a model for a spin glass—a disordered magnetic alloy that exhibits unusual magnetic behavior. However, even in the physics literature, it is often motivated as a pure optimization problem. We will continue this tradition and consider the following scenario, called the Dean's problem. Suppose we have a group of N people indexed by the elements of $\{1,\ldots,N\}$ and a collection of parameters g_{ij} for $1 \le i < j \le N$, called the interaction parameters, which describe how much people i

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and j like or dislike each other. Naturally, a positive parameter means that they like each other and a negative parameter means that they dislike each other. However unrealistic it may seem, we will assume that the feeling is mutual. We will consider different ways to divide a group into two subgroups and it will be convenient to describe them using vectors of ± 1 labels with the agreement that people with the same label belong to the same group. Therefore, vectors

$$\sigma = (\sigma_1, \ldots, \sigma_N) \in \Sigma_N = \{-1, +1\}^N$$

describe 2^N possible such partitions. For a given configuration σ, let us write $i \sim j$ whenever $\sigma_i \sigma_j = 1$ or, in other words, if i and j belong to the same subgroup, and consider the following *comfort function*:

$$c(\sigma) = \sum_{i<j} g_{ij} \sigma_i \sigma_j = \sum_{i \sim j} g_{ij} - \sum_{i \not\sim j} g_{ij}. \tag{1.1}$$

The Dean's problem is then to maximize this function over all configurations σ in Σ_N. The interpretation of this objective is clear, since maximizing the comfort function means that we would like to keep positive interactions as much as possible within the same groups and separate negative interactions into different groups. It would be interesting to try to understand how this maximum behaves in a typical situation and one natural way to give this question some meaning is to model the interaction parameters (g_{ij}) as random variables. The simplest choice is to let the interactions be independent among pairs and have the standard Gaussian distribution (we will see in Sect. 3.8 that, in some sense, the choice of the distribution is not really important). This is one common way to introduce the Sherrington–Kirkpatrick (SK) model.

As a formal definition, we will consider a Gaussian process indexed by $\sigma \in \Sigma_N$,

$$H_N(\sigma) = \frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} g_{ij} \sigma_i \sigma_j, \tag{1.2}$$

where the random variables g_{ij} for $1 \le i, j \le N$ are i.i.d. standard Gaussian. This process is called the *Hamiltonian* of the SK model and our goal is to study its maximum $\max_{\sigma \in \Sigma_N} H_N(\sigma)$ as the size of the system N goes to infinity or, as the physicists would say, in the *thermodynamic limit*. Notice that, compared with the comfort function above, the Hamiltonian includes N^2 terms indexed by all vectors (i, j). From a mathematical point of view this will make no difference whatsoever, but from the point of view of notation this choice will be more convenient. The normalization by $\sqrt{N}$ is, in some sense, also done for convenience of notation. With this normalization, the covariance of the Gaussian process can be written as

$$\mathbb{E} H_N(\sigma^1) H_N(\sigma^2) = \frac{1}{N} \sum_{i,j=1}^{N} \sigma_i^1 \sigma_j^1 \sigma_i^2 \sigma_j^2 = N \Big(\frac{1}{N} \sum_{i=1}^{N} \sigma_i^1 \sigma_i^2 \Big)^2 = N R_{1,2}^2 \tag{1.3}$$

where the normalized scalar product

$$R_{1,2} = \frac{1}{N}\sum_{i=1}^{N} \sigma_i^1 \sigma_i^2 \tag{1.4}$$

is called *the overlap* of two configurations $\sigma^1, \sigma^2 \in \Sigma_N$. The fact that the covariance of the Gaussian process (1.2) is a function of the scalar product, or overlap, between points in Σ_N will have very important consequences and one could say that this property is what makes the SK model so special. For example, this means that the distribution of the maximum $\max_{\sigma\in\Sigma_N} H_N(\sigma)$ would not be affected if we replaced the index set Σ_N by its image under any orthogonal transformation on $\mathbb{R}^N$, since the scalar products and the covariance would be left unchanged. The consequences of this will gradually become clear.

Going back to the maximum of $H_N(\sigma)$, one would expect it to be of order N, on average, since this is what happens in the case of 2^N independent Gaussian random variables with variance of order N, as in Eq. (1.3). However, we are not just interested in the order of the maximum, but in the precise asymptotics

$$\lim_{N\to\infty} \frac{1}{N} \mathbb{E} \max_{\sigma\in\Sigma_N} H_N(\sigma). \tag{1.5}$$

One standard approach to this random optimization problem is to think of it as the zero-temperature case of a general family of problems at positive temperature and, instead of dealing with the maximum in Eq. (1.5) directly, first to try to compute its "smooth approximation"

$$\lim_{N\to\infty} \frac{1}{N\beta} \mathbb{E} \log \sum_{\sigma\in\Sigma_N} \exp \beta H_N(\sigma) \tag{1.6}$$

for every *inverse temperature parameter* $\beta > 0$. To connect these two quantities, let us write

$$\begin{aligned} \frac{1}{N}\mathbb{E}\max_{\sigma\in\Sigma_N} H_N(\sigma) &\le \frac{1}{N\beta}\mathbb{E}\log\sum_{\sigma\in\Sigma_N}\exp\beta H_N(\sigma) \\ &\le \frac{\log 2}{\beta} + \frac{1}{N}\mathbb{E}\max_{\sigma\in\Sigma_N} H_N(\sigma), \end{aligned} \tag{1.7}$$

where the lower bound follows by keeping only the largest term in the sum inside the logarithm and the upper bound follows by replacing each term by the largest one. This shows that Eqs. (1.5) and (1.6) differ by at most $\beta^{-1}\log 2$ and Eq. (1.6) approximates Eq. (1.5) when the inverse temperature parameter β goes to infinity. Let us denote

$$F_N(\beta) = \frac{1}{N}\mathbb{E}\log Z_N(\beta), \tag{1.8}$$

where $Z_N(\beta)$ is defined by

$$Z_N(\beta) = \sum_{\sigma\in\Sigma_N} \exp\beta H_N(\sigma). \tag{1.9}$$

The quantity $Z_N(\beta)$ is called the *partition function* and $F_N(\beta)$ is called the *free energy* of the model. We will prove in the next section that the limit

$$F(\beta) = \lim_{N\to\infty} F_N(\beta)$$

exists, which means that Eq. (1.6) is equal to $\beta^{-1}F(\beta)$. It is easy to see, by Hölder's inequality, that

$$\beta^{-1}(F_N(\beta) - \log 2) = \frac{1}{N\beta}\mathbb{E}\log\frac{1}{2^N}\sum_{\sigma\in\Sigma_N}\exp\beta H_N(\sigma)$$

is increasing in β and, therefore, so is $\beta^{-1}(F(\beta)-\log 2)$, which implies that the limit $\lim_{\beta\to\infty}\beta^{-1}F(\beta)$ exists. It then follows from Eq. (1.7) that

$$\lim_{N\to\infty}\frac{1}{N}\mathbb{E}\max_{\sigma\in\Sigma_N} H_N(\sigma) = \lim_{\beta\to\infty}\frac{F(\beta)}{\beta}. \tag{1.10}$$

The problem of computing Eq. (1.5) was reduced to the problem of computing the limit $F(\beta)$ of the free energy $F_N(\beta)$ at every positive temperature. The formula for $F(\beta)$ was discovered by the physicist Giorgio Parisi in 1979 and a proof of the Parisi formula will be one of the main results presented in this book.

In statistical mechanics, the notion of the free energy $F_N(\beta)$ is closely related to another notion—the *Gibbs measure* of the model—which is a random probability measure on Σ_N defined by

$$G_N(\sigma) = \frac{\exp\beta H_N(\sigma)}{Z_N(\beta)}. \tag{1.11}$$

The problem of computing the free energy in the thermodynamic limit turns out to be closely related to the problem of understanding the asymptotic structure of the Gibbs measure G_N, in a certain sense. On a purely intuitive level, it is clear that the Gibbs measure Eq. (1.11) assigns more weight to the configurations σ corresponding to the larger values of the Hamiltonian $H_N(\sigma)$ and, when the inverse temperature parameter β goes to infinity, in the limit, G_N concentrates on the optimal configurations corresponding to the maximum $\max_{\sigma\in\Sigma_N} H_N(\sigma)$. Therefore, a set where G_N carries most of its weight, in some sense, corresponds to a set where H_N takes large values and understanding the structure of this set can be helpful in the computation of the limit in Eq. (1.5), or the limit of the free energy $F_N(\beta)$. This is another way to see that the model at small but positive temperature can be viewed as an approximation of the model at zero temperature, the original optimization problem. In the remainder of the chapter, we will try to explain this connection between the free energy and Gibbs measure from several different points of view, while at the same time introducing various techniques that will be used throughout the book.

Besides the classical Sherrington–Kirkpatrick model, we will also consider its generalization, the so-called *mixed p-spin model*, corresponding to the Hamiltonian

$$H_N(\sigma) = \sum_{p\geq 1}\beta_p H_{N,p}(\sigma), \tag{1.12}$$