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The Kolmogorov–Obukhov Theory of Turbulence

A Mathematical Theory of Turbulence

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To Inga, Adda and Einar.

Preface

In this book we present the recently developed statistical theory of turbulence in a form that can be appreciated by physicists, mathematicians, and engineers. The theory is grounded in probability theory and we develop from probability all the results that are necessary to understand the turbulence theory, without proofs. However, references are given to standard texts where the proofs and more background details can be found. The goals are to find estimates for structure functions of turbulence that are realized in simulation and experiments; to derive the invariant measure of turbulence both for the one-point statistics and the two-point statistics; and finally to derive the probability density function (PDF) for the statistics that are used in practice. We do not assume any mathematical background but familiarity with basic probability theory and partial differential equations obviously helps.

We will see that the Navier–Stokes equation for all but the largest scales in turbulent flow can be expressed as a stochastic Navier–Stokes equation (1.65). The stochastic forcing results from instabilities of the flow that magnifies small ambient noise and saturates its growth into large stochastic forcing. This has been modeled before by a Reynolds decomposition and by a coarse graining of the flow. The stochastic force is generic and is determined by the general principles of probability with a minimum of physical inputs. It consists of two components: additive noise and multiplicative noise and the additive component is determined by the central limit theorem and the large deviation principle. The physical input is that these two terms must produce similar scalings because they are the detailed description of the same dissipative processes. This determines the rate in the large deviation principle. The multiplicative noise multiplies the fluid velocity and models jumps (vorticity concentrations) in the velocity gradient. It is expressed by a generic Poisson process where only the rate needs to be given. This rate is determined by the spectral analysis of the (linearized) Navier–Stokes operator and the requirement, following [64], that the dimension of the most singular vorticity structure (filaments) is one. Thus the stochastic forcing is generic and determined with two mild physical inputs.

The stochastic Navier–Stokes equation can be expressed as an integral equation (2.17) and the log-Poissonian processes found by She and Leveque and explored by She and Waymire and Dubrulle are produced from the multiplicative noise by the Feynman–Kac formula. This gives a satisfying mathematical derivation of the intermittency phenomena that had earlier been derived from impirical considerations. Moreover, the integral equations show how the Navier–Stokes evolution and the log-Poissonian intermittency processes act on the dissipation processes to produce the intermittency in the dissipation. This is a mathematical derivation of the experimental observation that intermittent dissipation processes accompany intermittent velocity variations. Using the integral equation, we get an estimate on all the structure functions of the velocity differences in turbulence. The evidence from simulations and experiments is that this upper bound is reached in turbulent flow. Why the inertial cascade achieves this maximal efficiency in the energy transfer remains to be explained.

We then build on Hopf's [29] ideas to compute the invariant measure of turbulent flow. This measure can be computed because it solves a linear functional differential equation, the Kolmogorov–Hopf equation; see [56]. It turns out to be an infinite-dimensional Gaussian multiplied by a (discrete) Poisson distribution. This Poisson distribution corresponds to the intermittency and the log-Poisson processes. Then by taking the trace of the invariant measure we get the PDF of the velocity differences. We first derive the functional differential equation (PDE) for the PDF and then show that there are infinitely many PDFs, each corresponding to a particular moment because of the intermittency corrections. The PDE (3.15) for the sequence of PDFs can also be solved and the PDFs turn out to be the normalized inverse Gaussian (NIG) distributions of Barndorff–Nielsen [7]. Their parameters are easly computed and we see how to do this for both simulations and experiments.

It is interesting to notice that although the solution of the Navier–Stokes equation may not be unique or smooth the invariant measure of the velocity differences (3.12) is still well defined by Leray's [42] existence theory. Moreover, different velocities produce equivalent measures, so the statistical observables of turbulence are unique although the turbulent velocity may not be.

The theory presented in this book must be complemented by a dynamical systems theory for the large-scale structures in fluid flow and eventually one wants to work out how the small-scale flow presented here influences the large-scale dynamics. This is a material for future research, but hopefully the tools presented in this book will also be helpful in that endeavor.

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Chapter 1
The Mathematical Formulation of Fully Developed Turbulence

1.1 Introduction to Turbulence

The purpose of the research discussed in this book is to develop new mathematical tools that open the theory of turbulence up to theoretical investigations. Great strides are currently being made both in turbulence experiments and simulations, but the new mathematical development will allow theoreticians to compare with both simulations and experiments and make new predictions useful to both areas.

The ultimate goal of turbulence research is to develop methods to systematically improve the simulations of turbulent systems. Such methods have been ad hoc so far, with different techniques applied to each situations. The new theory will permit a systematic approach where the simulations and experiment can be made increasingly accurate in a stepwise fashion.

These developments will eventually have a big effect on technology permitting improvements in aircraft and car design, more efficient travel in and out of space, less pollution, more fuel efficiency, and greater efficiency of wind turbines and wave energy farms. It will help to understand weather patterns and greatly advance weather predictions. In addition it will aid a wide variety of applications of turbulence in industry and science.

In 1941 Kolmogorov and Obukhov [34, 35, 49] proposed a statistical theory of turbulence based on dimensional arguments. The main consequence and test of this theory was that the structure functions of the velocity differences of a turbulent fluid

$$E(|u(x,t)-u(x+l,t)|^p) = S_p = C_p l^{p/3}$$

should scale with the distance (lag variable) l between them, to the power $p/3$. This theory was immediately criticized by Landau for not taking into account the influence of the large flow structure on the constants C_p and later for not including the influence of the intermittency, in the velocity fluctuations, on the scaling exponents.

In 1962 Kolmogorov and Obukhov [36, 50] proposed a corrected theory where both of those issues were addressed. They also pointed out that the scaling exponents

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