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Steven G. Krantz
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The Implicit Function Theorem

History, Theory, and Applications

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To the memory of Kennan Tayler Smith (1926–2000)

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Preface

The implicit function theorem is, along with its close cousin the inverse function theorem, one of the most important, and one of the oldest, paradigms in modern mathematics. One can see the germ of the idea for the implicit function theorem in the writings of Isaac Newton (1642–1727), and Gottfried Leibniz's (1646–1716) work explicitly contains an instance of implicit differentiation. While Joseph Louis Lagrange (1736–1813) found a theorem that is essentially a version of the inverse function theorem, it was Augustin-Louis Cauchy (1789–1857) who approached the implicit function theorem with mathematical rigor and it is he who is generally acknowledged as the discoverer of the theorem. In Chapter 2, we will give details of the contributions of Newton, Lagrange, and Cauchy to the development of the implicit function theorem.

The form of the implicit function theorem has evolved. The theorem first was formulated in terms of complex analysis and complex power series. As interest in, and understanding of, real analysis grew, the real-variable form of the theorem emerged. First the implicit function theorem was formulated for functions of two real variables, and the hypothesis corresponding to the Jacobian matrix being nonsingular was simply that one partial derivative was nonvanishing. Finally, Ulisse Dini (1845–1918) generalized the real-variable version of the implicit function theorem to the context of functions of any number of real variables. As mathematicians understood the theorem better, alternative proofs emerged, and the associated modern techniques have allowed a wealth of generalizations of the implicit function theorem to be developed.

Today we understand the implicit function theorem to be an *ansatz*, or a way of looking at problems. There are implicit function theorems, inverse function theorems, rank theorems, and many other variants. These theorems are valid on

Euclidean spaces, manifolds, Banach spaces, and even more general settings. Roughly speaking, the implicit function theorem is a device for solving equations, and these equations can live in many different settings.

In addition, the theorem is valid in many categories. The textbook formulation of the implicit function theorem is for C^k functions. But in fact the result is true for $C^{k,\alpha}$ functions, Lipschitz functions, real analytic functions, holomorphic functions, functions in Gevrey classes, and for many other classes as well. The literature is rather opaque when it comes to these important variants, and a part of the present work will be to set the record straight.

Certainly one of the most powerful forms of the implicit function theorem is that which is attributed to John Nash (1928–) and Jürgen Moser (1928–1999). This device is actually an infinite iteration scheme of implicit function theorems. It was first used by John Nash to prove his celebrated imbedding theorem for Riemannian manifolds. Jürgen Moser isolated the technique and turned it into a powerful tool that is now part of partial differential equations, functional analysis, several complex variables, and many other fields as well. This text will culminate with a version of the Nash–Moser theorem, complete with proof.

This book is one both of theory and practice. We intend to present a great many variants of the implicit function theorem, complete with proofs. Even the important implicit function theorem for real analytic functions is rather difficult to pry out of the literature. We intend this book to be a convenient reference for all such questions, but we also intend to provide a compendium of examples and of techniques. There are applications to algebra, differential geometry, manifold theory, differential topology, functional analysis, fixed point theory, partial differential equations, and to many other branches of mathematics. One learns mathematics (in part) by watching others do it. We hope to set a suitable example for those wishing to learn the implicit function theorem.

The book should be of interest to advanced undergraduates, graduate students, and professional mathematicians. Prerequisites are few. It is not necessary that the reader be already acquainted with the implicit function theorem. Indeed, the first chapter provides motivation and examples that should make clear the form and function of the implicit function theorem. A bit of knowledge of multivariable calculus will allow the reader to tackle the elementary proofs of the implicit function theorem given in Chapter 3. Rudiments of real and functional analysis are needed for the third proof in Chapter 3 which uses the Contraction Mapping Fixed Point Principle. Some knowledge of complex analysis is required for a complete reading of the historical material—this seems to be unavoidable since the earliest rigorous work on the implicit function theorem was formulated in the context of complex variables. In many cases a willing suspension of disbelief and a bit of determination will serve as a thorough grounding in the basics.

There are many sophisticated applications of implicit function theorems, particularly the Nash–Moser theorem, in modern mathematics. The imbedding theorem for Riemannian manifolds, the imbedding theorem for CR manifolds, and the deformation theory of complex structures are just a few of them. Richard Hamilton's masterful survey paper (see the Bibliography) indicates several more applications from different parts of mathematics. While each of these is a lovely *tour de force* of modern analytical technique, it is also the case that each requires considerable technical background. In order to keep the present volume as self-contained as possible, we have decided not to include any of these modern applications; instead we have provided exclusively classical applications of the implicit function theorem. For a basic book on the subject, we have found this choice to be most propitious.

We intend this book to be a useful resource for scientists of all types. We have exerted a considerable effort to make the bibliography extensive (if not complete). Therefore topics that can only be touched on here can be amplified with further reading. Although there are no formal exercises, the extensive remarks provide grist for further thought and calculation. We trust that our exposition will imbue our readers with some of the same fascination that led to the writing of this book.

There are a number of people whom we are pleased to thank for their helpful comments and contributions: David Barrett, Michael Crandall, John P. D'Angelo, Gerald B. Folland, Judith Grabiner, Robert E. Greene, Lars Hörmander, Seth Howell, Kang-Tae Kim, Laszlo Lempert, Maurizio Letizia, Richard Rochberg, Walter Rudin, Steven Weintraub, Dean Wills, Hung-Hsi Wu. Robert Burckel cast his critical eye on every page of our manuscript and the result is a much cleaner and more accurate book. Librarian Barbara Luszczynska performed yeoman service in helping us to track down references. This book is better because of the friendly assistance of all these good people; but, of course, all remaining failings are the province of the authors.

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1

Introduction to the Implicit Function Theorem

1.1 Implicit Functions

To the beginning student of calculus, a function is given by an analytic expression such as

$$f(x) = x^3 + 2x^2 - x - 3, \tag{1.1}$$

$$g(y) = \sqrt{y^2 + 1}, \tag{1.2}$$

or

$$h(t) = \cos(2\pi t). \tag{1.3}$$

In fact, 250 years ago this was the approach taken by Léonard Euler (1707–1783) when he wrote (see Euler [EB 88]):

> *A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities.*

Almost immediately, one finds the notion of "function as given by a formula" to be too limited for the purposes of calculus. For example, the locus of

$$y^5 + 16y - 32x^3 + 32x = 0 \tag{1.4}$$

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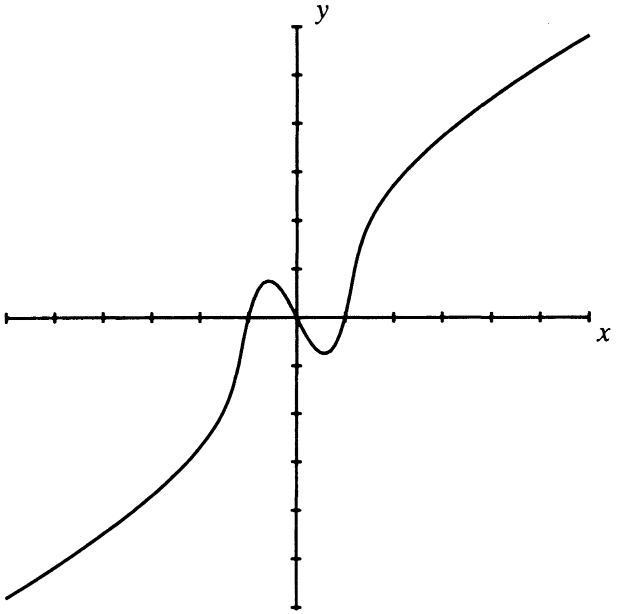

Figure 1.1. The Locus of Points Satisfying (1.4)

defines the nice subset of $\mathbf{R}^2$ that is sketched in Figure 1.1. The figure leads us to suspect that the locus is the graph of y as a function of x, but no formula for that function exists.

In contrast to the naive definition of functions as formulas, the modern, set-theoretic definition of a function is formulated in terms of the graph of the function. Precisely, a *function* with *domain* X and *codomain* or *range* Y is a subset, let us call it f, of the cartesian product

$$X \times Y = \{(x, y) : x \in X,\ y \in Y\}$$

having the properties that (i) for each $x \in X$ there is an element $(x, y) \in f$, and (ii) if $(x, y) \in f$ and $(x, \widetilde{y}) \in f$, then $y = \widetilde{y}$. In case these two properties hold, the choice of $x \in X$ determines the unique $y \in Y$ for which $(x, y) \in f$; because of this uniqueness, we find it a convenient shorthand to write

$$y = f(x)$$

to mean that $(x, y) \in f$.

Example 1.1.1 The locus defined by (1.4) has the property that, for each choice of $x \in \mathbf{R}$, there is a unique $y \in \mathbf{R}$ such that the pair (x, y) satisfies the equation. Thus there is a function, f, in the modern sense, such that the graph $y = f(x)$ is the locus of (1.4).

To confirm this assertion, we fix a value of $x \in \mathbf{R}$ and consider the left-hand side of (1.4) as a function of y alone. That is, we will examine the behavior of

$$F(y) = y^5 + 16y - 32x^3 + 32x$$

with x fixed.

Since the powers of y in $F(y)$ are odd, we have $\lim_{y \to -\infty} F(y) = -\infty$ and $\lim_{y \to +\infty} F(y) = +\infty$. Also we have

$$F'(y) = 5y^4 + 16 > 0 ,$$

so $F(y)$ is strictly increasing as y increases. By the intermediate value theorem, we see that $F(y)$ attains the value 0 for a unique value of y. That value of y is the value of $f(x)$ for the fixed value of x under consideration. □

Note that it is not clear from (1.4) by itself that y is a function of x. Only by doing the extra work in the example can we be certain that y really is uniquely defined as a function of x. Because it is not immediately clear from the defining equation that a function has been given, we say that the function is defined *implicitly* by (1.4). In contrast, when we see

$$y = f(x) \tag{1.5}$$

written, we then take it as a hypothesis that $f(x)$ is a function of x; no additional verification is required, even when in the right-hand side the function is simply a symbolic representation as in (1.5) rather than a formula as in (1.1), (1.2), and (1.3). To distinguish them from implicitly defined functions, the functions in (1.1), (1.2), (1.3), and (1.5) are called (in this book) *explicit* functions.

1.2 An Informal Version of the Implicit Function Theorem

Thinking heuristically, one usually expects that one equation in one variable

$$F(x) = c ,$$

c a constant, will be sufficient to determine the value of x (though the existence of more than one, but only finitely many, solutions would come as no surprise).[1] When there are two variables, one expects that it will take two simultaneous equations

$$\begin{aligned} F(x, y) &= c , \\ G(x, y) &= d , \end{aligned}$$

[1] What we are doing is informally describing the notion of "degrees of freedom" that is commonly used in physics.