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Mathematical Theory and Numerical Analysis

Second Edition

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Plasticity

Mathematical Theory and Numerical Analysis

Second Edition

Springer

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To our wives
Huidi Tang and Shaada
and our children
Elizabeth, Michael, and Jordi

Preface to the Second Edition

In this second edition, we include new material on single-crystal plasticity and on various models of strain-gradient plasticity. The relevant mechanics and variational aspects of these models are presented, and attention is given to selected numerical analyses. We have also taken the opportunity to revise various parts of the first edition. For instance, Section 10.2 has been completely rewritten. The set of references has been updated and expanded, and a number of minor refinements have also been made.

We thank our many friends, colleagues and family members whose interest, guidance, and encouragement made this work possible. One of us (B.D.R.) is grateful to Morton Gurtin for his contributions, whether through collaborative work or the extensive discussions on the topic of gradient plasticity. We are grateful for the support from the Simons Foundation (to W.H.) and from the National Research Foundation through the South African Research Chair in Computational Mechanics (to B.D.R.). We thank Andrew McBride for his comments on drafts of the manuscript. Tim Povall prepared a number of new figures, and together with Andrew McBride also revised the figures in the first edition. This assistance is gratefully acknowledged. It is a pleasure to acknowledge the skillful assistance from the staff at Springer, especially Achi Dosanjh and Donna Chernyk, and members of the Springer TeX support team.

W.H. B.D.R.
Iowa City Cape Town
August 2012

Preface to the First Edition

The basis for the modern theory of elastoplasticity was laid in the nineteenth-century, by TRESCA, ST. VENANT, LÉVY, and BAUSCHINGER. Further major advances followed in the early part of this century, the chief contributors during this period being PRANDTL, VON MISES, and REUSS. This early phase in the history of elastoplasticity was characterized by the introduction and development of the concepts of irreversible behavior, yield criteria, hardening and perfect plasticity, and of rate or incremental constitutive equations for the plastic strain.

Greater clarity in the mathematical framework for elastoplasticity theory came with the contributions of PRAGER, DRUCKER, and HILL, during the period just after the Second World War. Convexity of yield surfaces, and all its ramifications, was a central theme in this phase of the development of the theory.

The mathematical community, meanwhile, witnessed a burst of progress in the theory of partial differential equations and variational inequalities from the early 1960s onwards. The timing of this set of developments was particularly fortuitous for plasticity, given the fairly mature state of the subject, and the realization that the natural framework for the study of initial boundary value problems in elastoplasticity was that of variational inequalities. This confluence of subjects emanating from mechanics and mathematics resulted in yet further theoretical developments, the outstanding examples being the articles by MOREAU, and the monographs by DUVAUT AND J.-L. LIONS, and TEMAM. In this manner the stage was set for comprehensive investigations of the well-posedness of problems in elastoplasticity, while the simultaneous rapid growth in interest in numerical methods ensured that equal attention was given to issues such as the development of solution algorithms, and their convergences.

The interaction between elastoplasticity and mathematics has spawned among many engineering scientists an interest in gaining a better understanding of the modern mathematical developments in the subject. In the same way, given the richness of plasticity in interesting and important mathematical problems, many mathematicians, either students or mature researchers, have developed an interest in understanding the mechanical and engineering basis of the subject, and its connections with the mathematical theory. While there are many textbooks and monographs on plasticity that deal with the mechanics of the subject, they are written mainly for a readership in the engineering sciences; there does not appear to us to have existed an extended account of elastoplasticity which would serve these dual needs of both engineering scientists and mathematicians. It is our hope that this monograph will go some way towards filling that gap.

We present in this work three logically connected aspects of the theory of elastic-plastic solids: the constitutive theory, the variational formulations of the related initial boundary value problems, and the numerical analysis of these problems. These three aspects determine the three parts into which the monograph is divided.

The constitutive theory, which is the subject of Part I, begins with a motivation grounded in physical experience, whereafter the constitutive theory of classical elastoplastic media is developed. This theory is then cast in a convex analytic setting, after some salient results from convex analysis have been reviewed. The term "classical" refers in this work to that theory of elastic-plastic material behavior which is based on the notion of convex yield surfaces, and the normality law. Furthermore, only the small strain, quasi-static theory is treated. Much of what is covered in Part I will be familiar to those working on plasticity, though the greater insights offered by exploiting the tools of convex analysis may be new to some researchers. On the other hand, mathematicians unfamiliar with plasticity theory will find in this first part an introduction that is self-contained and accessible.

Part II of the monograph is concerned with the variational problems in elastoplasticity. Two major problems are identified and treated: the primal problem, of which the displacement and internal variables are the primary unknowns; and the dual problem, of which the main unknowns are the generalized stresses.

Finally, Part III is devoted to a treatment of the approximation of the variational problems presented in the previous part. We focus on finite element approximations in space, and both semi- and fully discrete problems. In addition to deriving error estimates for these approximations, attention is given to the behavior of those solution algorithms that are in common use.

Wherever possible we provide background materials of sufficient depth to make this work as self-contained as possible. Thus, Part I contains a review of topics in continuum mechanics, thermodynamics, linear elasticity, and convex analytic setting of elastoplasticity. In Part II we include a treatment of those topics from functional analysis and function spaces that are relevant to a discussion of the well-posedness of vatriational problems. And Part III begins with an overview of the mathematics of finite elements.

In writing this work we have drawn heavily on the results of our joint collaboration in the past few years. We have also consulted, and made liberal use of the works of many: we mention in particular the major contributions of G. Duvaut and J.-L. Lions, C. Johnson, J.B. Martin, H. Matthies, and J.C. Simo. While we acknowledge this debt with gratitude, the responsibility for any inaccuracies or erroneous interpretations that might exist in this work, rests with its authors.

We thank our many friends, colleagues and family members whose interest, guidance, and encouragement made this work possible.

W.H. B.D.R.
Iowa City Cape Town

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Part I

Continuum Mechanics and Elastoplasticity Theory

1

Preliminaries

1.1 Introduction

The theory of elastoplastic media is now a mature branch of solid and structural mechanics, having experienced significant development during the latter half of the twentieth century. In particular, the classical theory, which deals with small-strain elastoplasticity problems, has a firm mathematical basis, and from this basis further developments, both mathematical and computational, have evolved. Small-strain elastoplasticity is well understood, and the understanding of its governing equations can be said to be almost complete. Likewise, theoretical, computational, and algorithmic work on approximations in the spatial and time domains are at a stage at which approximations of desired accuracy can be achieved with confidence.

The finite-strain theory has evolved along parallel lines, although it is considerably more complex and is subject to a number of alternative treatments. The form taken by the governing equations is nevertheless reasonably settled, and there has been considerable progress in mathematical analyses of the problem. Computationally, great strides have been made in in the last four decades, and it is now possible to solve highly complex problems with the aid of the computer.

This monograph focuses on theoretical aspects of the small-strain theory of elastoplasticity with hardening assumptions. It is intended to provide a reasonably comprehensive and unified treatment of the mathematical theory and numerical analysis, exploiting in particular the great advantages to be gained by placing the theory in a convex-analytic framework.

The monograph is divided into three parts. The first part contains the first four chapters and provides a detailed introduction to plasticity, in which the mechanics of elastoplastic behavior is emphasized. The equations describing elastoplastic behavior are subsequently recast in the language and setting of convex analysis. In particular, the flow law can be written in terms of either the dissipation function or the yield function. Thus, it is possible to present

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the flow law in two alternative yet equivalent forms, which are dual to each other.

The conventional or classical problem of elastoplasticity is treated first, with formulations of the problem for both the polycrystalline and single-crystal cases. Thereafter, attention shifts to various non-local or strain-gradient extensions of these problems. Strain-gradient plasticity has received increasing attention over the last 25 years, primarily because of its importance in modelling size-dependent effects that the conventional theory is unable to capture.

The second part of the monograph is taken up with mathematical considerations of the elastoplasticity problem. It begins with some preparations on basic knowledge from functional analysis and weak formulations of boundary value problems. These are the contents of Chapters 5 and 6. Depending on the form of the flow law used, we obtain two formulations for the elastoplasticity problem: the primal variational formulation, which uses the dissipation function to describe the flow law, and the dual variational formulation, which uses the yield function to describe the flow law. The two forms are equivalent. The main task of the second part is a thorough mathematical treatment of the well-posedness of the two alternative formulations of the small-strain problem. The primal variational problem is analyzed in Chapter 7, and the dual variational problem in Chapter 8.

Numerical analysis of the elastoplasticity problem is the topic of the third part. For the convenience of the reader, we introduce the basic ideas of the finite element method and some typical finite element interpolation results in Chapter 9. We then review some standard results in the error analysis for finite element approximations of boundary value problems for differential equations and inequalities. This is followed by error analysis of various semidiscrete and fully discrete approximations for both the primal and dual variational problems. We also discuss convergence properties of a number of solution algorithms commonly used in practice.

Plasticity is a vast research area, and it is impossible to touch on every aspect of this area in a single volume. Thus, several important topics are not included in this monograph, for example, applications of elastoplasticity theory to the analysis of engineering structures, which have been covered in many books on elastoplasticity directed at the engineering community (see, for example, Martin [115] and Chen and Han [29]).

In this book, we deal exclusively with hardening elastoplasticity and strain-gradient plasticity. The reader will find in Temam [176] a comprehensive mathematical treatment of the elastic, perfectly-plastic problem. A complete treatment of the conventional perfectly plastic problem has recently been presented by Dal Maso et al. [37].

Details of the implementation and behavior of specific algorithms are omitted, as are other topics, such as viscoplasticity, and matters pertaining to the finite-strain problem. These topics are given a comprehensive treatment in the monograph by Simo and Hughes [168], the extended survey by Simo [166], and the recent book by de Souza Neto, Perić and Owen [39]. These works, and many of the references cited in them, contain a wealth of numerical examples.

The list of the references at the end of the book includes only those that are more relevant to the present exposition, and we do not attempt to make the list complete.

This work summarizes some recent results on mathematical analysis and numerical analysis of the elastoplasticity problem. We hope that it will be useful to those readers who wish to know more about recent developments in the analysis of the elastoplasticity problem and to those who are preparing to carry out research in the area of plasticity. For the convenience of the reader, we include brief introductions to various mathematical materials that should be sufficient for reading the book. In this way, it will not be necessary to have any extensive prior knowledge of advanced mathematical topics, such as functional analysis and convex analysis. Nevertheless, some degree of maturity in mathematics and some knowledge of mechanics are expected from the reader. We hope that the book will also be helpful to those whose main interests lie in the solution of plasticity problems in engineering practice. We are convinced that attempts at solving practical problems in this area—as, indeed, is the case in many other areas—would benefit from a background in the theoretical aspects of the subject.

1.2 Some Historical Remarks

Early works on plasticity. It is generally agreed that the origin of plasticity dates back to a series of papers by Tresca from 1864 to 1872 (see [179]) on the extrusion of metals. In this work the first yield condition was proposed: the condition, known subsequently as the Tresca yield criterion, stated that a metal yields when the maximal shear stress attains a critical value. In the same time period, St. Venant [10] introduced basic constitutive relations for rigid, perfectly plastic materials in plane stress, and suggested that the principal axes of the strain increment coincide with the principal axes of stress. Lévy [107] derived the general equations in three dimensions. In 1886, Bauschinger [12] observed the effect that now carries his name: a previous plastic strain with a certain sign diminishes the resistance of the material with respect to the next plastic strain with the opposite sign. In a landmark paper in 1913, von Mises [180] derived the general equations for plasticity, accompanied by his well-known pressure-insensitive yield criterion (J_2-theory, or octahedral shear stress yield condition).

In 1924, Prandtl [143] extended the St. Venant–Lévy–von Mises equations for the plane continuum problem to include the elastic component of strain, and Reuss [157] in 1930 carried out their extension to three dimensions. In 1928, von Mises [181] generalized his previous work for a rigid, perfectly plastic solid to include a general yield function and discussed the relation between the direction of plastic strain increment and the smooth yield surface, thus introducing formally the concept of using the yield function as a plastic potential in the incremental stress–strain relations of the flow theory.

Compared to perfect plasticity, the development of incremental constitutive relations for hardening materials proceeded more slowly. In 1928, Prandtl [144] attempted to formulate general relations for hardening behavior. In 1938, Melan [123] generalized the foregoing concepts of perfect plasticity by giving incremental relations for hardening solids with smooth yield surfaces, and discussing uniqueness results for elastoplastic incremental problems for both perfectly plastic and hardening materials, based on some limiting assumptions.

Since 1940, the theory of plasticity has seen more rapid development. In 1949, Prager [141] obtained a general framework for the plastic constitutive relations for hardening materials with smooth yield functions and recognized the relationship between the convexity of the yield surface plus the normality law and the uniqueness of the associated boundary value problem. Drucker [42], in 1951, proposed his material stability postulate. With this concept, the plastic stress–strain relations together with many related fundamental aspects of the subject may be treated in a unified manner. In 1953, Koiter [102] generalized the plastic stress–strain relations for nonsmooth yield surfaces and obtained some uniqueness and variational results. He introduced the device of using more than one yield function in the stress–strain relations, the plastic strain increment receiving a contribution from each active yield surface and falling within the normal cone to the yield surface. For further details, see [103].

A detailed description of the early development of plasticity theory and a comprehensive list of references on plasticity published before 1980 can be found in Życzkowski [193], which also contains a wealth of discussions on various aspects of plasticity.

Recent mathematical and numerical analyses of problems in plasticity. Mathematical and numerical aspects of the quasistatic problem in elastoplasticity have been the subject of sustained attention since the 1970s. The first systematic mathematical study of the boundary value problems of elastoplasticity is due to Duvaut and Lions [43], who considered the problem for an elastic perfectly plastic material and formulated the problem as a variational inequality. Moreau [130, 131] considered the same issues, but from a more geometric viewpoint. Johnson [90] subsequently extended the analysis in [43] by approaching the problem in two stages; in the first stage the velocity variable is eliminated and the problem becomes a variational inequality posed on a time-dependent convex set. The second stage involves the solution for the velocity variable.

The theory for perfectly plastic materials was advanced greatly through the introduction and investigation of the space $BD(\Omega)$ of functions of bounded deformation [120, 122, 177, 178]. This space is essential for a proper study of the perfectly plastic problem, since discontinuity surfaces (sliplines) may be accommodated within this framework; the framework of Sobolev spaces, on the other hand, is not appropriate. A comprehensive summary account of the mathematical theory of perfect plasticity in the framework of the space $BD(\Omega)$ can be found in [176], which is, however, confined to the total strain, or holonomic, problem, an approximate model in which a one-to-one relationship between stress and strain is assumed. A complete treatment of the evolution problem for perfect plasticity has been presented by Dal Maso et al. [37].

Analysis of the elastoplastic problem with hardening, on the other hand, can be achieved within the framework of Sobolev spaces. There are two alternative formulations of the problem, depending on the form of the flow law. One formulation makes use of the yield function in the plastic flow law and will be called the dual formulation in this work, for reasons that will become clear in Chapter 4. An alternative approach is to express the plastic flow law in terms of the dissipation function, which leads to the primal formulation of the problem. The primal and dual formulations are extensions, respectively, of the displacement and stress problems in linearized elasticity.

The first analysis of the dual formulation of the hardening problem is due to Johnson [92], who gave an existence and uniqueness result. A detailed analysis of the primal formulation of the hardening problem was presented by Han, Reddy, and Schroeder [80]. The unknowns are the displacement and internal variables, while the problem takes the form of a variational inequality of the mixed kind: it is an inequality both because of the presence of a nondifferentiable functional in the formulation *and* because the problem is posed on a closed convex cone in a Hilbert space.

In the last decade there have been significant developments in the treatment of the problem of elastoplasticity for finite strains. The works [25, 113, 126, 127] are most relevant, and take as a point of departure an energetic approach developed by Mielke for rate-independent problems [125].

Strain-gradient theories of plasticity have received sustained attention since the early 1980s. The importance of these theories lie in their ability to model scale-dependent effects, a feature that is absent in classical plastcity theories, which do not possess a natural length scale. A general survey of work in strain-gradient plasticity up to the lat 1990s may be found in [53]. The monograph [69] contains a detailed treatment of models of gradient plasticity due to Gurtin and co-authors, as well as of an early theory due to Aifantis [2]. The mathematical and numerical analyses in our work of strain-gradient plasticity will focus on the model developed by Aifantis, and more extensively on those developed by Gurtin and co-authors [67, 68].

Analyses of finite element approximations of the elastoplastic problem have enjoyed steady attention. Johnson [91] considered a formulation of the elastic, perfectly-plastic problem in which stress is the primary variable and derived error estimates for the fully discrete (that is, discrete in both time and space) problem. In a later work, Johnson [93] analyzed fully discrete finite element approximations of the elastoplasticity problem with hardening, in the context of a mixed formulation in which stress and velocity are the variables. Related work can also be found in Hlaváček [85], and summary accounts in Hlaváček, Haslinger, Nečas, and Lovíšek [86], and Korneev and Langer [104].

The dual formulation is a popular approach in practice for the hardening problem; see, for example, the comprehensive treatments of computational aspects of the problem by Simo [166], and by Simo and Hughes [168]. However, while there exist some results on stability, consistency, and convergence for certain numerical approximation schemes, the whole picture is by no means complete.

In comparison, numerical analysis of the primal formulation of the hardening problem did not receive attention until recently. Various schemes for approximating the primal formulation of the hardening problem were analyzed for the first time in [80]. Related treatments include those by Carstensen and co-authors [22, 23, 24]. These focus on the primal problem and give attention to issues such as improved estimates and adaptive mesh refinement.

A more classical approach to the analysis and numerical analysis of the hardening plasticity problem is taken by Bonnetier [16], and by Li and Babuška [108, 109]. First, spatial discretization is carried out using finite elements, and the resulting semidiscrete problem is written as a system of highly nonlinear ordinary differential equations. Then it is shown that as the finite element mesh size approaches zero, the solution of the semidiscrete problem converges, and the limit is the solution of the plasticity problem.

The work of Han and Reddy [78] provides a comprehensive treatment of the mathematical and numerical analysis of the elastoplasticity problem with hardening. The question of existence and uniqueness of solutions is addressed for both primal and dual formulations. Various approximation schemes for each formulation are studied. The schemes considered include semidiscrete approximations in which either the spatial domain or the time domain is discretized, and fully discrete approximations where discretization is carried out with respect to both space and time. Error estimates for these approximations are derived, not only for the approximate stress, but also for the approximate displacement; in comparison, the study in [93] is confined to one involving the stress and velocity, and results on convergence are presented only for the stress.

The resulting discrete systems are nonlinear and large. Various solution algorithms are used in practice to solve these systems. Some popular solution algorithms are discussed in [78], and for the first time convergence of some of the solution algorithms is proved rigorously.

The present monograph is an expanded and updated version of our previous work [78].

Recent numerical analyses of problems in strain-gradient plasticity include the works [154, 184].

1.3 Notation

Throughout this work we will use the popular mathematical symbol "$\forall$" to stand for "for any" or "for all". Also, we will use ":=" for equality by definition. The letter c will denote a generic constant independent of certain quantities (which are clear from the context). The value of c may differ at different places.

Vectors, tensors. Some pertinent results from vector and tensor analysis are summarized here for convenience. More comprehensive sources can be found in the literature (see, for example, Lemaitre and Chaboche [106]).

We will use boldface italic letters to denote vectors and tensors. We adopt the summation convention for repeated indices, unless stated otherwise. Most often, vectors are denoted by lowercase boldface italic letters, and second-order tensors by lowercase boldface Greek letters. Fourth-order tensors are usually denoted by uppercase boldface italic letters.

Our discussion applies to Euclidean space $\mathbb{R}^d$ of any dimension d (in practice, $d = 1, 2, 3$). However, for definiteness of exposition and because of its importance in applications, we will give the presentation in the context of three-dimensional space. Thus, we will make use of a Cartesian coordinate system with an orthonormal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ that is chosen *once and for all.* Where it is necessary to show components of a vector or a tensor, these will always be relative to the orthonormal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$.

A second-order tensor $\boldsymbol{\tau}$ is a linear operator mapping vectors to vectors and may be identified with a matrix. For any vector $\boldsymbol{a}$, $\boldsymbol{\tau a}$ represents a vector such that the action of $\boldsymbol{\tau}$ on $\boldsymbol{a}$ is linear; that is, $\boldsymbol{\tau}(\alpha \boldsymbol{a} + \beta \boldsymbol{b}) = \alpha \boldsymbol{\tau a} + \beta \boldsymbol{\tau b}$ for any scalars α, β, and any vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. We will always use a_i, $1 \le i \le 3$, to denote the components of the vector $\boldsymbol{a}$, and τ_{ij}, $1 \le i, j \le 3$, the components of the second-order tensor $\boldsymbol{\tau}$. With the basis defined, the action of the second-order tensor $\boldsymbol{\tau}$ on the vector $\boldsymbol{a}$ may be represented in the form

$$\boldsymbol{\tau a} = \tau_{ij} a_j \boldsymbol{e}_i.$$

The scalar products of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, and of two second-order tensors (or matrices) $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$, are denoted by $\boldsymbol{a} \cdot \boldsymbol{b}$ and $\boldsymbol{\sigma} : \boldsymbol{\tau}$ and have the component representations

$$\boldsymbol{a} \cdot \boldsymbol{b} := a_i b_i, \quad \boldsymbol{\sigma} : \boldsymbol{\tau} := \sigma_{ij} \tau_{ij}.$$

The magnitudes of the vector $\boldsymbol{a}$ and the second-order tensor $\boldsymbol{\tau}$ are defined by

$$|\boldsymbol{a}| := (\boldsymbol{a} \cdot \boldsymbol{a})^{\frac{1}{2}}, \quad |\boldsymbol{\tau}| := (\boldsymbol{\tau} : \boldsymbol{\tau})^{\frac{1}{2}}.$$

The vector product $\boldsymbol{c} = \boldsymbol{a} \wedge \boldsymbol{b}$ of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is a vector $\boldsymbol{c}$ with components defined by

$$c_i := \epsilon_{ijk} a_j b_k,$$

where ϵ_{ijk} is the permutation symbol: $\epsilon_{ijk} = +1$ for (i, j, k) a cyclic permutation of $(1, 2, 3)$, -1 for (i, j, k) an anticyclic permutation, and is zero otherwise.

The tensor product $\boldsymbol{a} \otimes \boldsymbol{b}$ of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is a second-order tensor defined by the relation

$$(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{c} := (\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a} \quad \forall\, \boldsymbol{c}.$$

Viewed as a matrix, we have the representation

$$\boldsymbol{a} \otimes \boldsymbol{b} = \boldsymbol{a}\boldsymbol{b}^T.$$

Thus the tensor product $\boldsymbol{a} \otimes \boldsymbol{b}$ has the components $a_i b_j$. The nine quantities $\boldsymbol{e}_i \otimes \boldsymbol{e}_j$, $1 \leq i, j \leq 3$, form a basis for the space of the second-order tensors, and any such tensor $\boldsymbol{\tau}$ can be represented in the form

$$\boldsymbol{\tau} = \tau_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j.$$

Since we will be working with a fixed basis, there is little point in making a formal distinction between the tensor $\boldsymbol{\tau}$ and the 3×3 matrix of its components, so that unless otherwise stated, $\boldsymbol{\tau}$ will represent the tensor *and* the matrix of its components. With this understanding, it is merely necessary to point out that all the usual matrix operations such as addition, transposition, multiplication, inversion, and so on, apply to tensors, and the standard notation is used for these operations. Thus, for example, $\boldsymbol{\tau}^T$ and $\boldsymbol{\tau}^{-1}$ are, respectively, the transpose and inverse of the tensor (or matrix) $\boldsymbol{\tau}$.

We will use $\mathbb{S}^3$ to denote the space of all the symmetric 3×3 matrices (or second-order symmetric tensors). We will use $\mathbb{S}^3_0$ to denote the subspace of $\mathbb{S}^3$ with vanishing trace; that is,

$$\mathbb{S}^3_0 := \{\boldsymbol{\tau} \in \mathbb{S}^3 : \operatorname{tr} \boldsymbol{\tau} = 0\},$$

where as usual, $\operatorname{tr} \boldsymbol{\tau} = \tau_{ii}$ is the trace of $\boldsymbol{\tau}$.

One special and important second-order tensor is the *identity* $\boldsymbol{I}$, which is defined by the relation $\boldsymbol{I}\boldsymbol{a} = \boldsymbol{a}$ for any vector $\boldsymbol{a}$. The components of the identity tensor $\boldsymbol{I}$ are the Kronecker delta

$$\delta_{ij} := \begin{cases} 1 & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

Every second-order tensor $\boldsymbol{\tau}$ may be additively decomposed into a deviatoric part $\boldsymbol{\tau}^D$ and a spherical part $\boldsymbol{\tau}^S$; these are defined by

$$\boldsymbol{\tau}^S := \tfrac{1}{3}\,(\operatorname{tr} \boldsymbol{\tau})\boldsymbol{I}, \quad \boldsymbol{\tau}^D := \boldsymbol{\tau} - \tfrac{1}{3}\,(\operatorname{tr} \boldsymbol{\tau})\boldsymbol{I},$$

so that

$$\boldsymbol{\tau} = \boldsymbol{\tau}^D + \boldsymbol{\tau}^S.$$

For spatial domains of dimension d, a second-order tensor $\boldsymbol{\tau}$ is identified with a $d \times d$ matrix, and the formulae for its deviatoric and spherical parts are modified to

$$\boldsymbol{\tau}^S := \tfrac{1}{d}\,(\mathrm{tr}\,\boldsymbol{\tau})\boldsymbol{I}, \quad \boldsymbol{\tau}^D := \boldsymbol{\tau} - \tfrac{1}{d}\,(\mathrm{tr}\,\boldsymbol{\tau})\boldsymbol{I}.$$

For planar problems, for example, $d = 2$.

The only higher-order tensors that will occur are those of third or fourth order. Third-order tensors will be needed in the discussion of strain-gradient plasticity. For two third order tensors $\mathbb{K} = (K_{ijk})$ and $\mathbb{L} = (L_{ijk})$, their scalar product is defined as

$$\mathbb{K} \,\vdots\, \mathbb{L} := K_{ijk} L_{ijk}.$$

The length of $\mathbb{K}$ is then

$$|\mathbb{K}| := (\mathbb{K} \,\vdots\, \mathbb{K})^{1/2}.$$

The fourth-order tensors will appear as tensors of material moduli and will be denoted by uppercase boldface italic letters. A fourth-order tensor $\boldsymbol{C}$ may be defined as a linear operator mapping the space of second-order tensors into itself. The action of a fourth-order tensor $\boldsymbol{C}$ on a second-order tensor $\boldsymbol{\tau}$ is denoted by $\boldsymbol{C\tau}$ and is the second-order tensor with components $C_{ijkl}\tau_{kl}$, where C_{ijkl} are the components of $\boldsymbol{C}$ relative to the canonical orthonormal basis $\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l$, $1 \le i, j, k, l \le 3$. An important special fourth-order tensor is the identity tensor $\boldsymbol{I}$, which satisfies $\boldsymbol{I\tau} = \boldsymbol{\tau}$ for any symmetric second-order tensors $\boldsymbol{\tau}$. This identity tensor has the component representation

$$I_{ijkl} = \tfrac{1}{2}\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).$$

We use the same symbol $\boldsymbol{I}$ for both the second-order and fourth-order identity tensors.

From time to time, we will need to consider collections of quantities of different kind, and we will use sans serif fonts (for example, S) or boldface lowercase Greek letters to denote such collections. For example, in Section 3.2, we will use the notion of a generalized stress Σ, defined to be $(\boldsymbol{\sigma}, \boldsymbol{\chi})$, where $\boldsymbol{\sigma}$ is the stress tensor, $\boldsymbol{\chi} = (\boldsymbol{\chi}_i)_{i=1}^m$ is a set of internal variables $\boldsymbol{\chi}_1, \cdots, \boldsymbol{\chi}_m$ which are scalars or tensors. Correspondingly, we will need the notion of a generalized plastic strain $\mathsf{P} := (\boldsymbol{p}, \boldsymbol{\xi})$ with the plastic strain tensor $\boldsymbol{p}$ and $\boldsymbol{\xi} = (\boldsymbol{\xi}_i)_{i=1}^m$. The inner product of Σ and P is indicated by the symbol $\diamond$:

$$\Sigma \diamond \mathsf{P} := \boldsymbol{\sigma} : \boldsymbol{p} + \boldsymbol{\chi}_i : \boldsymbol{\xi}_i.$$

Here, for $\boldsymbol{\chi}_i$ and $\boldsymbol{\xi}_i$ scalars, $\boldsymbol{\chi}_i : \boldsymbol{\xi}_i$ is the ordinary multiplication of $\boldsymbol{\chi}_i$ and $\boldsymbol{\xi}_i$.

Invariants of second-order tensors (or 3×3 matrices). The problem of finding a scalar λ and a nonzero vector $\boldsymbol{q}$ with

$$\boldsymbol{\tau q} = \lambda \boldsymbol{q}$$

leads to the eigenvalue problem of solving the characteristic equation

$$\det\,(\lambda \boldsymbol{I} - \boldsymbol{\tau}) = 0.$$

This equation can be written equivalently as

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0,$$

where $I_1(\boldsymbol{\tau})$, $I_2(\boldsymbol{\tau})$, and $I_3(\boldsymbol{\tau})$ are the *principal invariants* of $\boldsymbol{\tau}$. The invariants are defined by

$$\begin{aligned}
I_1 &:= \operatorname{tr}\boldsymbol{\tau} = \tau_{ii} = \lambda_1 + \lambda_2 + \lambda_3,\\
I_2 &:= \tfrac{1}{2}\left[(\operatorname{tr}\boldsymbol{\tau})^2 - \operatorname{tr}\boldsymbol{\tau}^2\right] = \tfrac{1}{2}(\tau_{ii}\tau_{jj} - \tau_{ij}\tau_{ji}) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1,\\
I_3 &:= \det\boldsymbol{\tau} = \lambda_1\lambda_2\lambda_3.
\end{aligned}$$

Here, λ_1, λ_2, and λ_3, the eigenvalues of $\boldsymbol{\tau}$, are the roots of the characteristic equation (a multiple root is counted repeatedly according to its multiplicity).

We denote by

$$\boldsymbol{\iota}(\boldsymbol{\tau}) := (I_1(\boldsymbol{\tau}),\, I_2(\boldsymbol{\tau}),\, I_3(\boldsymbol{\tau}))$$

the set of three invariants of $\boldsymbol{\tau}$. The eigenvalues λ_i of a matrix $\boldsymbol{\tau}$ are often denoted by τ_i (note the single index) and are called the *principal components* of $\boldsymbol{\tau}$.

Scalar, vector, and tensor fields. The gradient of a scalar field $\phi(\boldsymbol{x})$ is denoted by $\nabla\phi$ and is the vector defined by

$$\nabla\phi := \frac{\partial\phi}{\partial x_i}\boldsymbol{e}_i.$$

The divergence $\operatorname{div}\boldsymbol{u}$ and gradient $\nabla\boldsymbol{u}$ of a vector field $\boldsymbol{u}(\boldsymbol{x})$ are respectively a scalar and a second-order tensor field, defined by

$$\begin{aligned}
\operatorname{div}\boldsymbol{u} &:= \frac{\partial u_i}{\partial x_i},\\
\nabla\boldsymbol{u} &:= \frac{\partial u_i}{\partial x_j}\boldsymbol{e}_i \otimes \boldsymbol{e}_j.
\end{aligned}$$

Thus the components of $\nabla\boldsymbol{u}$ are $\partial u_i/\partial x_j$. The transpose of $\nabla\boldsymbol{u}$ is denoted by $(\nabla\boldsymbol{u})^T$ and is the second-order tensor with components $\partial u_j/\partial x_i$. The divergence $\operatorname{div}\boldsymbol{\tau}$ of a second-order tensor $\boldsymbol{\tau}$ is a vector defined by

$$\operatorname{div}\boldsymbol{\tau} := \frac{\partial\tau_{ij}}{\partial x_j}\boldsymbol{e}_i.$$

For a scalar-valued function $f(\boldsymbol{u})$ of a vector variable $\boldsymbol{u} = (u_1, u_2, u_3)^T$, its derivative with respect to $\boldsymbol{u}$ can be identified with a vector,

$$\frac{\partial f(\boldsymbol{u})}{\partial\boldsymbol{u}} = \frac{\partial f(\boldsymbol{u})}{\partial u_i}\,\boldsymbol{e}_i.$$

For a scalar-valued function $f(\boldsymbol{\tau})$ of a second-order tensor $\boldsymbol{\tau} = (\tau_{ij})$, the derivative with respect to $\boldsymbol{\tau}$ is a second-order tensor,

$$\frac{\partial f(\boldsymbol{\tau})}{\partial\boldsymbol{\tau}} = \frac{\partial f(\boldsymbol{\tau})}{\partial\tau_{ij}}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j.$$

If $\boldsymbol{f}(\boldsymbol{\tau})$ is a matrix-valued function of a second-order tensor variable $\boldsymbol{\tau}$, then its derivative with respect to $\boldsymbol{\tau}$ is a fourth-order tensor with components

$$\frac{\partial \boldsymbol{f}(\boldsymbol{\tau})}{\partial \tau_{ij}} = \frac{\partial f_{kl}(\boldsymbol{\tau})}{\partial \tau_{ij}}\, \boldsymbol{e}_k \otimes \boldsymbol{e}_l.$$

For a time-dependent quantity z, we will use $\dot{z}$ to denote the partial derivative of z with respect to the temporal variable t.

Landau's notation for orders of magnitude. We will use the "big oh" (O) and "little oh" (o) symbols in the following senses. Given two functions $f(t)$ and $g(t)$ of a real variable t, we say that $f(t)$ is of a lower order of magnitude than $g(t)$ as $t \to 0+$ and write

$$f(t) = o(g(t)), \quad t \to 0+,$$

if

$$\lim_{t \to 0+} \frac{f(t)}{g(t)} = 0.$$

We say that $f(t)$ is dominated by $g(t)$ as $t \to 0+$, and write

$$f(t) = O(g(t)), \quad t \to 0+,$$

if for some positive constants c and δ,

$$|f(t)| \le c\,|g(t)|, \quad t \in (0, \delta).$$

These definitions can be easily adapted to cover other similar expressions, such as

$$f(t) = o(g(t)), \quad t \to 0,$$

or

$$x_n = O(y_n), \quad n \to \infty,$$

for two sequences of numbers $\{x_n\}$ and $\{y_n\}$.

2

Continuum Mechanics and Linearized Elasticity

We will be concerned with bodies that at the macroscopic level may be regarded as being composed of material that is continuously distributed. By this it is meant, first, that such a body occupies a region of three-dimensional space that may be identified with $\mathbb{R}^3$. The region occupied by the body will of course vary with time as the body deforms. It is convenient, then, for the purpose of keeping track of the evolution of the body's behavior to locate any point in the body by its position vector $\boldsymbol{x}$ with respect to some previously chosen origin $\mathbf{0}$, at a fixed time. For simplicity we will take this to be at the time $t = 0$, and we will assume that the body is undeformed and unstressed in this state, unless stated otherwise. The region occupied by the body at the time $t = 0$ is denoted by Ω, and is called the *reference configuration*. To emphasize the identification between points in the region Ω and points in the undeformed body we will often refer to a point $\boldsymbol{x} \in \Omega$ as a *material point*. If we go one step further and place a set of Cartesian axes with the origin $\mathbf{0}$, then the position vector $\boldsymbol{x}$ has components x_i $(i = 1, 2, 3)$ with respect to the orthonormal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ associated with this set of axes. The situation is illustrated in Figure 2.1, in which Ω_t is the current configuration, the region occupied by the body at the current time t.

The objective will be to obtain a complete description of the motion and deformations of the body, for given loading conditions, within the framework of continuum mechanics. There is an extensive literature on continuum mechanics; the texts [28, 31, 69] are examples of works that may be consulted for further details.

Second, it is assumed that both the properties and the behavior of such a body can be described in terms of functions of position $\boldsymbol{x}$ in the body and time t. Thus, for example, we may associate with the body a scalar temperature distribution θ that varies within the body and with the passage of time, so that the value of the temperature of a material point $\boldsymbol{x}$ at time t is represented by the function $\theta(\boldsymbol{x}, t)$, or equivalently by $\theta(x_1, x_2, x_3, t)$.

It will be necessary at some stage to stipulate the properties assumed or expected of functions defined on the body. For the time being there is no need

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