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Leonid P. Lebedev
Iosif I. Vorovich
Michael J. Cloud

Functional Analysis in Mechanics

Second Edition

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Michael J. Cloud

Functional Analysis in Mechanics

Second Edition

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Leonid P. Lebedev
Department of Mathematics
National University of Colombia
Bogotá, Colombia

Iosif I. Vorovich†
Department of Mathematics & Mechanics
Rostov State University
Rostov-on-Don, Russia

Michael J. Cloud
Department of Electrical Engineering
Lawrence Technological University
Southfield, MI, USA

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Preface

In Russia, a university Mechanics department will typically exist within a "Mathematical Faculty." Such a department is not an engineering department in the western sense, but is something intermediate between a mathematics department and an engineering department. It will offer courses on calculus, linear algebra, analysis, differential geometry, differential equations, and so on, along with extensive courses on analytical mechanics, the strength of materials, continuum mechanics, elasticity, fluid mechanics, and more specialized subjects.

When the first author of this book was a student of the second author, functional analysis was not in the curriculum for mechanicists. In 1971, Professor Vorovich offered a short course on functional analysis to a broad audience consisting of mathematicians and mechanicists, students and professors. It included a simple and minimal introduction to the theory of Banach and Hilbert spaces that opened the door to understanding (with some difficulty on the part of the non-mathematicians) certain interesting applications in mechanics. The mathematicians were surprised at how abstract theorems could be applied to mechanics and, moreover, that these theorems could actually be rooted in mechanics. It was emphasized that strain energy is not only a physical notion, but a measure by which a norm and inner product can be imposed on the set of deformations or velocities of a body. This idea was developed by Vorovich in his doctoral dissertation on the nonlinear theory of elastic shallow shells in 1957, but was imbedded in just a few long examples. Many subsequent publications of the idea were made in *Doklady AN USSR*, the central scientific journal of the USSR, where the results were presented without proof.

Later, Professor Vorovich's lectures were extended and became the basis for a regular course at Rostov State University and many other institutions across the USSR. The course contents, including the applications considered, continued to evolve, but the present book preserves the main ideas of the original course. As the course was just one semester in length, it contained only a minimal subset of the abstract theory that enabled students to understand the applications. The first edition of this book contained some abstract material that was not presented in the course. This second edition includes more extended coverage of the classical, abstract portions of functional analysis, as well as additional mechanics problems. Taken together, the first three chapters now constitute a regular text on applied functional analysis. This potential use of the book is supported by a significantly extended set of exercises with hints and solutions.

The Introduction (pages 1–7) is an unnumbered chapter with single-digit internal numbering of its equations, examples, and theorems. Chapters 1 through 4 employ a three-digit scheme where the first two numbers are the chapter and section numbers (hence Remark 1.3.2 is the second labeled remark in Section 1.3). The Introduction closes with some additional remarks on notation.

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Bogota, Colombia *Leonid Lebedev*

Okemos, Michigan, USA *Michael Cloud*

August 2012

Contents

Introduction 1

1 Metric, Banach, and Hilbert Spaces 9
1.1 Preliminaries 9
1.2 Hölder's Inequality and Minkowski's Inequality 14
1.3 Metric Spaces of Functions 18
1.4 Some Relations for the Metrics in $\tilde{L}^p(\Omega)$ and ℓ^p 22
1.5 Metrics in Energy Spaces 24
1.6 Sets in a Metric Space 28
1.7 Convergence in a Metric Space 29
1.8 Completeness 30
1.9 Completion Theorem 31
1.10 Lebesgue Integrals and the Space $L^p(\Omega)$ 34
1.11 Banach Spaces 39
1.12 Norms on n-Dimensional Spaces 42
1.13 Other Examples of Banach Spaces 44
1.14 Hilbert Spaces 48
1.15 Factor Spaces 50
1.16 Separability 53
1.17 Compactness, Hausdorff Criterion 58
1.18 Arzelà's Theorem and Its Applications 61
1.19 Theory of Approximation in a Normed Space 66
1.20 Decomposition Theorem, Riesz Representation 70
1.21 Linear Operators and Functionals 72
1.22 Space of Linear Continuous Operators 76
1.23 Uniform Boundedness Theorem 79
1.24 Banach–Steinhaus Principle 82
1.25 Closed Operators and the Closed Graph Theorem 83
1.26 Inverse Operator 89
1.27 Lax–Milgram Theorem 92
1.28 Open Mapping Theorem 93

1.29 Dual Spaces 94
1.30 Hahn–Banach Theorem 98
1.31 Consequences of the Hahn–Banach Theorem 102
1.32 Contraction Mapping Principle 103
1.33 Topology, Weak and Weak* Topologies 109
1.34 Weak Topology in a Normed Space X 111
1.35 Conclusion and Further Reading 113

2 Mechanics Problems from the Functional Analysis Viewpoint 115
2.1 Introduction to Sobolev Spaces 115
2.2 Operator of Imbedding 119
2.3 Some Energy Spaces 120
2.4 Generalized Solutions in Mechanics 135
2.5 Existence of Energy Solutions to Some Mechanics Problems 141
2.6 Operator Formulation of an Eigenvalue Problem 142
2.7 Problem of Elastico-Plasticity; Small Deformations 144
2.8 Bases and Complete Systems; Fourier Series 151
2.9 Weak Convergence in a Hilbert Space 157
2.10 Ritz and Bubnov–Galerkin Methods 166
2.11 Curvilinear Coordinates, Nonhomogeneous Boundary Conditions 168
2.12 Bramble–Hilbert Lemma and Its Applications 171

3 Some Spectral Problems of Mechanics 177
3.1 Introduction 177
3.2 Adjoint Operator 178
3.3 Compact Operators 185
3.4 Compact Operators in Hilbert Space 190
3.5 Functions Taking Values in a Banach Space 192
3.6 Spectrum of a Linear Operator 195
3.7 Resolvent Set of a Closed Linear Operator 199
3.8 Spectrum of a Compact Operator in Hilbert Space 201
3.9 Analytic Nature of the Resolvent of a Compact Linear Operator . . . 209
3.10 Spectrum of a Holomorphic Compact Operator Function 211
3.11 Self-Adjoint Compact Linear Operator in Hilbert Space 213
3.12 Some Applications of Spectral Theory 219
3.13 Courant's Minimax Principle 222

4 Elements of Nonlinear Functional Analysis 225
4.1 Fréchet and Gâteaux Derivatives 225
4.2 Liapunov–Schmidt Method 231
4.3 Critical Points of a Functional 233
4.4 Von Kármán Equations of a Plate 236
4.5 Buckling of a Thin Elastic Shell 243
4.6 Nonlinear Equilibrium Problem for an Elastic Shallow Shell 251

4.7 Degree Theory 255
4.8 Steady-State Flow of a Viscous Liquid 258

Summary of Inequalities and Imbeddings 265
A.1 Inequalities 265
A.2 Imbeddings 268

Hints for Selected Problems 269

References 299

In Memoriam: Iosif I. Vorovich 301

Index 303

Introduction

Long ago it was traditional to apply mathematics only to mechanics and physics. Now it difficult to find an area of knowledge in which mathematics is not used as a tool to create new models and to simulate them. This is due mainly to the fantastic ability of computers to process models having thousands of parameters.

Fortunately, mathematics tends to produce methods of great generality. Functional analysis, in particular, allows us to approach different mathematical facts and methods from a unified point of view. Let us consider some examples.

Example 1. A system of linear algebraic equations

$$x_i = \sum_{j=1}^{n} a_{ij} x_j + c_i \qquad (i = 1, \ldots, n) \tag{1}$$

can be solved by an iterative scheme of the form

$$\begin{aligned} x_i^{(0)} &= c_i \ , \\ x_i^{(k+1)} &= \sum_{j=1}^{n} a_{ij} x_j^{(k)} + c_i \qquad (i = 1, \ldots, n, \quad k = 0, 1, 2, \ldots) \ . \end{aligned}$$

To establish convergence, let us consider the difference

$$x_i^{(k+1)} - x_i^{(k)} = \sum_{j=1}^{n} a_{ij} [x_j^{(k)} - x_j^{(k-1)}] \ .$$

We have

$$\max_{1 \le i \le n} |x_i^{(k+1)} - x_i^{(k)}| \le \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}| \, |x_j^{(k)} - x_j^{(k-1)}| \le q \cdot \max_{1 \le j \le n} |x_j^{(k)} - x_j^{(k-1)}|$$

where

$$q = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}| \ .$$

If $q < 1$, then convergence is ensured and a solution to (1) is $(z_1, \ldots, z_n)$ where

$$z_i = \lim_{k \to \infty} x_i^{(k)} \qquad (i = 1, \ldots, n) \ .$$

Now consider a system of integral equations

$$x_i(t) = \sum_{j=1}^{n} \int_0^1 a_{ij}(t, s) x_j(s) \, ds + c_i(t) \qquad (i = 1, \ldots, n) \tag{2}$$

where $c_i(t)$ and $a_{ij}(t, s)$ are given continuous functions on the interval $[0, 1]$ and the square $[0, 1] \times [0, 1]$, respectively. The iterative scheme

$$\begin{aligned} x_i^{(0)}(t) &= c_i(t), \\ x_i^{(k+1)}(t) &= \sum_{j=1}^{n} \int_0^1 a_{ij}(t, s) x_j^{(k)}(s) \, ds + c_i(t) \qquad (i = 1, \ldots, n) \end{aligned}$$

produces functions $x_i^{(k)}(t)$ that satisfy

$$x_i^{(k+1)}(t) - x_i^{(k)}(t) = \sum_{j=1}^{n} \int_0^1 a_{ij}(t, s) [x_j^{(k)}(s) - x_j^{(k-1)}(s)] \, ds \ .$$

We have

$$|x_i^{(k+1)}(t) - x_i^{(k)}(t)| \le \sum_{j=1}^{n} \int_0^1 |a_{ij}(t, s)| \, |x_j^{(k)}(s) - x_j^{(k-1)}(s)| \, ds$$

so that

$$\max_{\substack{1 \le i \le n \\ 0 \le t \le 1}} |x_i^{(k+1)}(t) - x_i^{(k)}(t)| \le q \cdot \max_{\substack{1 \le j \le n \\ 0 \le s \le 1}} |x_j^{(k)}(s) - x_j^{(k-1)}(s)|$$

where

$$q = \max_{\substack{1 \le i \le n \\ 0 \le t \le 1}} \sum_{j=1}^{n} \int_0^1 |a_{ij}(t, s)| \, ds \ .$$

If $q < 1$, the component sequences $\{x_i^{(k)}(t)\}$ $(i = 1, \ldots, n)$ are uniformly convergent on $[0, 1]$. Hence a solution to (2) is $(z_1(t), \ldots, z_n(t))$ where

$$z_i(t) = \lim_{k \to \infty} x_i^{(k)}(t) \qquad (i = 1, \ldots, n) \ .$$

The obvious similarity between the treatments of (1) and (2) suggests that a general approach might cover these and other cases of interest. Later we will present the generalization, known as Banach's contraction mapping principle.

Example 2. In what follows, we deal mainly with spaces of infinite dimension. For example, the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \tag{3}$$

describes the vibrations $u = u(x, t)$ of a stretched string. Let the ends of the string be fixed:

$$u(0, t) = u(1, t) = 0 .$$

It is natural to seek a solution with finite potential and kinetic energies, i.e., with

$$\int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 dx < \infty , \qquad \int_0^1 \left(\frac{\partial u}{\partial t}\right)^2 dx < \infty .$$

We could seek a solution in the form of a Fourier series

$$u(x, t) = \sum_{k,m} A_{km} \sin \pi k x \sin \pi m t .$$

This solution is evidently described by a countable set of numbers A_{km}, which can be regarded as the components of a vector having infinitely many components. The set of such "vectors" clearly constitutes a space that is infinite dimensional.

One difficulty in dealing with an infinite dimensional space is that the Bolzano–Weierstrass principle (that any bounded infinite sequence contains a convergent subsequence) breaks down. For example, we cannot select a convergent subsequence from the bounded sequence of functions $\{\sin kx\}$.

Example 3. In contemporary mathematical physics, generalized solutions are typical. Without going into too much detail, we may briefly consider the problem of a beam with clamped ends bending under a load $q(x)$. A corresponding boundary value problem is

$$(B(x)y''(x))'' - q(x) = 0 , \qquad y(0) = y'(0) = y(l) = y'(l) = 0 , \tag{4}$$

where $B(x)$ and l are the stiffness and length, respectively, of the beam. This formulation supposes $y = y(x)$ to possess derivatives up to fourth order.

The same boundary value problem can be posed differently through the use of variational principles. It can be shown that the total potential energy functional of the beam, defined by

$$I(y) = \frac{1}{2} \int_0^l [B(y'')^2 - 2q(x)y]\, dx ,$$

assumes a minimum value at an equilibrium state of the beam (here all the functions $y(x)$ under consideration must satisfy the boundary conditions stated in (4)). The first variation of $I(y)$,

$$\delta I(y, \varphi) = \int_0^l [B(x)y''(x)\varphi''(x) - q(x)\varphi(x)]\, dx ,$$

vanishes for any sufficiently smooth function $\varphi(x)$ satisfying the boundary conditions

$$\varphi(0) = \varphi'(0) = \varphi(l) = \varphi'(l) = 0 \tag{5}$$

if $y(x)$ satisfies (4); that is,

$$\delta I(y, \varphi) = 0 \; . \tag{6}$$

A function $y(x)$ is called a generalized solution to the problem (4) if equation (6) holds for any sufficiently smooth function $\varphi(x)$ satisfying the conditions (5). So a generalized solution, which can have no more than two derivatives on the interval $[0, l]$, satisfies the equilibrium equation in the sense of Lagrange's variational principle. For a moving system, we can introduce generalized solutions using Hamilton's variational principle. All we require of such a solution is that we can calculate its potential and kinetic energy.

Since the smoothness restrictions for generalized solutions are milder than those for classical solutions, the above approach extends the circle of problems we may investigate and solve numerically. In particular, problems with non-smooth loads occur in applications. The generalized approach also arises naturally when we study convergence of the finite element method (FEM)— a powerful tool in mathematical physics. In the FEM, a generalized problem setup is called a *weak setup*. We prefer the term *energy setup*, as this type of setup can be obtained from the well-known energetic variational principles of mechanics.

At this point we hope the reader has begun to picture functional analysis as a powerful tool in applications. It emerged as a generalization of certain aspects of classical linear algebra and mathematical analysis to infinite dimensional cases, incorporating portions of the calculus of variations and topology. With an aim toward applications in the various mathematical sciences, it also inherited ideas from physics and, in particular, mechanics. In Chapter 1 we will present a more systematic study of its fundamentals.

We would like to add a point for the practitioner who is new to functional analysis and doubtful of its utility. When an engineer deals with some mechanical structure, he carries a mental image of how the structure deforms or behaves dynamically under a given range of loads. Rather than regarding the solution to his boundary value problem as a collection of pointwise calculated values, he concerns himself with how the solution *as a whole object* changes under certain changes in the loads. This is very similar to how one considers a mechanical problem by the methods of functional analysis. In this approach, a load is defined pointwise but is considered as a whole object — an element F of a functional space. Next, a unique characteristic of the structure's response to this load is identified; suppose it is the displacement field u corresponding to the load F. While u is a function (or vector function) depending on the space coordinates and possibly on time, it can also be regarded as a whole object — another element of some functional space. The relation between F and u can be written quite simply as

$$u = u(F) \; , \tag{7}$$

provided it is remembered that this is not a pointwise dependence: here, u in total depends on F as the entire set of loads acting on the structure. This viewpoint on the correspondence is more or less similar to the way an engineer imagines structural behavior. On one hand, (7) is a convenient representation for the solution of the problem. On the other hand, it should be understood that this representation, even for problems of linear mechanics, is much more complicated than those encountered in the finite dimensional problems of linear algebra. So it is useful to know which properties of a problem in linear algebra can be carried over for use with the continuous problems of linear mechanics and, just as importantly, which ones cannot. Such questions, pertaining to general relations of the type (7), may be pursued through the tools of functional analysis.

Let us close this introduction by presenting three important theorems from classical analysis. They are all named after Karl Weierstrass (1815–1897) and find frequent application in the book. Recall that in $\mathbb{R}^k$, the term *compact set* refers to a closed and bounded set.

Theorem 1. Let $f(\mathbf{x})$ be a function continuous on a compact set $\Omega \subset \mathbb{R}^k$. Then $f(\mathbf{x})$ is bounded on Ω and attains its supremum and infimum on Ω. That is,

1. there exists a constant c such that $|f(\mathbf{x})| \le c$ for all $\mathbf{x} \in \Omega$, and
2. there exist points $\mathbf{x}_* \in \Omega$ and $\mathbf{x}^* \in \Omega$ such that

$$f(\mathbf{x}_*) = \inf_{\mathbf{x}\in\Omega} f(\mathbf{x}) , \qquad f(\mathbf{x}^*) = \sup_{\mathbf{x}\in\Omega} f(\mathbf{x}) .$$

Hence the values $\max_{\mathbf{x}\in\Omega} f(\mathbf{x})$ and $\min_{\mathbf{x}\in\Omega} f(\mathbf{x})$ exist.

Theorem 2. Suppose a sequence $\{f_n(\mathbf{x})\}$ of functions continuous on a compact set $\Omega \subset \mathbb{R}^k$ converges uniformly; that is, for any $\varepsilon > 0$ there is an integer $N = N(\varepsilon)$ such that

$$|f_{n+m}(\mathbf{x}) - f_n(\mathbf{x})| < \varepsilon$$

for any $n > N$, $m > 0$, and $\mathbf{x} \in \Omega$. Then the limit function

$$f(\mathbf{x}) = \lim_{n\to\infty} f_n(\mathbf{x})$$

exists and is continuous on Ω.

Theorem 3. Let $f(\mathbf{x})$ be a function continuous on a compact set $\Omega \subset \mathbb{R}^k$. For any $\varepsilon > 0$ there is a polynomial $P_n(\mathbf{x})$ of the nth degree such that

$$|f(\mathbf{x}) - P_n(\mathbf{x})| < \varepsilon$$

for any $\mathbf{x} \in \Omega$.

Some familiarity with the main inequalities of analysis is assumed. These are summarized in an appendix beginning on p. 265.

Some Remarks on Notation

In this book we employ standard set and logic symbols such as

$\{x\colon P\}$	set of all x having property P
$\overline{S}$	closure of a set S
$\in$	set membership
$\subseteq, \subset$	subset relation, proper subset relation
$\cup$	union
$\cap$	intersection
$A \setminus B$	set difference
$\mathbb{R}$	set of real numbers
$\mathbb{Q}$	set of rational numbers
$\Longrightarrow$	logical implication
$\Longleftrightarrow$	logical equivalence (if and only if)

A proof, remark, or problem is punctuated with a right-justified empty square: $\square$

We denote sequences using braces. The notation $\{x_n\} \subset S$ means that $\{x_n\}$ is a sequence with $x_n \in S$ for each n. When the range of indices must be specified more precisely we use notation such as $\{x_n\}_{n=0}^{\infty}$ or x_n $(n = 0, 1, 2, \ldots)$.

Subsequences play an important role in our development. They are denoted using compound subscripts, with a couple of distinct conventions employed where convenient.

1. Given a sequence $\{x_n\}$, we may choose an element x_{i_1}, a second element x_{i_2} (with $i_2 > i_1$), and so forth, thereby selecting a subsequence denoted by $\{x_{i_k}\}$:

   $$\{x_{i_k}\} \equiv x_{i_1}, x_{i_2}, x_{i_3}, \ldots.$$

   Note that $i_k \geq k$ for each k. In such cases, for simplicity, we often renumber the resulting subsequence immediately and rename it as $\{x_n\}$.
2. In other cases, however, we begin with a sequence $\{x_n\}$ and "sift" it repeatedly with the aim of constructing a *diagonal sequence*. In such cases the notation $\{x_{i_1}\}$ will be used for the subsequence that results from the first sifting,

   $$\{x_{i_1}\} = x_{1_1}, x_{2_1}, x_{3_1}, \ldots,$$

   the notation $\{x_{i_2}\}$ will be used for the subsequence that results from the second sifting,

   $$\{x_{i_2}\} = x_{1_2}, x_{2_2}, x_{3_2}, \ldots,$$

   and so forth. The nth sifting step $(n = 1, 2, 3, \ldots)$ is done in such a way that the resulting subsequence satisfies a certain property, $\mathcal{P}_n$ say. Because the diagonal sequence

$$\{x_{n_n}\} = x_{1_1}, x_{2_2}, x_{3_3}, \ldots$$

can be considered as a subsequence of any of the sifted subsequences, it satisfies all of the properties $\mathcal{P}_n$ for $n = 1, 2, 3, \ldots$.

In contrast to authors who use $C(\Omega)$ to denote the set of functions continuous on a domain Ω, we reserve the symbol $C(\Omega)$ for the *space* of functions continuous on Ω with the sup metric. As Ω is a closed and bounded set throughout most of our discussion, the max metric is usually appropriate. The symbol $\overline{\Omega}$, for the closure of Ω, appears in just a few spots where Ω is permitted to be open for comparison with the treatments of certain topics in other books. On those occasions when we impose another type of metric on the set of continuous functions on Ω, we refrain from using the symbol $C(\Omega)$ for the resulting metric space. Even in such cases, however, we may still use statements such as $f \in C(\Omega)$ to indicate that f is continuous on Ω (while making no reference to the usual max metric on the space $C(\Omega)$).

The norm symbol $\|\cdot\|$ is formally introduced on p. 39, but is used a few times prior to that on the assumption that the reader has encountered it in more elementary courses. The same holds for the terms *operator* and *functional*, introduced formally on p. 72.

We avoid the operator notation

$$f\colon D(f) \subseteq X \to Y\,,$$

preferring to say that f *acts from* X *to* Y (or, if $Y = X$, that f *acts in* X). By this we do not necessarily mean that f is defined on all of X.

When space permits, we write out partial derivatives as

$$\frac{\partial u}{\partial x}\,, \quad \frac{\partial^2 u}{\partial x^2}\,, \quad \text{and so on.}$$

Otherwise we make use of the more compact subscript notation u_x, u_{xx}, etc.

The *Kronecker delta* symbol

$$\delta_{mn} = \begin{cases} 1\,, & m = n\,, \\ 0\,, & m \neq n\,, \end{cases}$$

is used when convenient.

Chapter 1
Metric, Banach, and Hilbert Spaces

1.1 Preliminaries

Consider a set of particles $P_1, \ldots, P_n$. To locate these particles in the space $\mathbb{R}^3$, we need a reference system. Let the Cartesian coordinates of particle P_i be (ξ_i, η_i, ζ_i). Identifying (ξ_1, η_1, ζ_1) with the triple (x_1, x_2, x_3), (ξ_2, η_2, ζ_2) with (x_4, x_5, x_6), and so on, we obtain a vector $\mathbf{x}$ of the Euclidean space $\mathbb{R}^{3n}$ with coordinates $(x_1, x_2, \ldots, x_{3n})$. Then $\mathbf{x}$ determines the positions of all particles in the set.

To distinguish different configurations $\mathbf{x}$ and $\mathbf{y}$ of the system, we can introduce a distance from $\mathbf{x}$ to $\mathbf{y}$:

$$d_E(\mathbf{x}, \mathbf{y}) = \left[\sum_{i=1}^{3n} (x_i - y_i)^2 \right]^{1/2} .$$

This is the *Euclidean distance* (or *metric*) of $\mathbb{R}^{3n}$. Alternatively, we could characterize the distance from $\mathbf{x}$ to $\mathbf{y}$ using the function

$$d_S(\mathbf{x}, \mathbf{y}) = \max_{1 \le i \le 3n} |x_i - y_i| .$$

It is easily seen that d_E and d_S each satisfy the following properties, known as the *metric axioms*:

D1. $d(\mathbf{x}, \mathbf{y}) \ge 0$;

D2. $d(\mathbf{x}, \mathbf{y}) = 0$ if and only if $\mathbf{x} = \mathbf{y}$;

D3. $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$;

D4. $d(\mathbf{x}, \mathbf{y}) \le d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y})$ for any third vector $\mathbf{z}$.

Any real-valued function $d(\mathbf{x}, \mathbf{y})$ defined for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{3n}$ is called a *metric* on $\mathbb{R}^{3n}$ if it satisfies the properties D1–D4. Property D1 is the *axiom of positiveness*, property D3 is the *axiom of symmetry*, and property D4 is the *triangle inequality*.

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Problem 1.1.1. Let a real-valued function $d(\mathbf{x}, \mathbf{y})$ be defined for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Show that if d satisfies D2, D3, and D4, then it also satisfies D1. Confirm that this does not depend on the nature of the elements $\mathbf{x}$ and $\mathbf{y}$.[1] □

Remark 1.1.1. It follows from Problem 1.1.1 that the metric axioms can be restricted to just D2, D3, and D4. □

The metrics d_E and d_S are *equivalent* relative to sequence convergence in $\mathbb{R}^{3n}$, since there exist positive constants m_1 and m_2 independent of $\mathbf{x}$ and $\mathbf{y}$ such that

$$0 < m_1 \le \frac{d_E(\mathbf{x}, \mathbf{y})}{d_S(\mathbf{x}, \mathbf{y})} \le m_2 < \infty \tag{1.1.1}$$

whenever $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{3n}$ and $\mathbf{x} \ne \mathbf{y}$. So

$$\lim_{k\to\infty} d_E(\mathbf{x}_k, \mathbf{x}) = 0 \iff \lim_{k\to\infty} d_S(\mathbf{x}_k, \mathbf{x}) = 0 \ .$$

Remark 1.1.2. In what follows, we shall use notation such as "m_i" for those constants whose exact values are not important. An example occurred in (1.1.1). □

Inequality (1.1.1) shows that, in a certain way, $d_E(\mathbf{x}, \mathbf{y})$ and $d_S(\mathbf{x}, \mathbf{y})$ have the same standing as metrics on $\mathbb{R}^{3n}$. We can introduce other functions on $\mathbb{R}^{3n}$ satisfying axioms D1–D4, e.g.,

$$d_p(\mathbf{x}, \mathbf{y}) = \left(\sum_{i=1}^{3n} |x_i - y_i|^p \right)^{1/p} \qquad (p = \text{constant} \ge 1)$$

and

$$d_k(\mathbf{x}, \mathbf{y}) = \left(\sum_{i=1}^{3n} k_i \, |x_i - y_i|^2 \right)^{1/2} \qquad (k_i > 0) \ .$$

Problem 1.1.2. Show that any two of the metrics introduced above are equivalent on $\mathbb{R}^n$. Note that two metrics $d_1(\mathbf{x}, \mathbf{y})$ and $d_2(\mathbf{x}, \mathbf{y})$ on $\mathbb{R}^n$ are equivalent if there exist m_1 and m_2 such that

$$0 < m_1 \le \frac{d_1(\mathbf{x}, \mathbf{y})}{d_2(\mathbf{x}, \mathbf{y})} \le m_2 < \infty \tag{1.1.2}$$

for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ with $\mathbf{x} \ne \mathbf{y}$. □

In general, the equivalence of a pair of metrics is defined so that any sequence convergent relative to one metric is also convergent relative to the other one. The existence of constants $m_k > 0$ satisfying (1.1.2) is sufficient but not necessary; the reader may wish to find a pair of equivalent metrics on $\mathbb{R}^n$ for which (1.1.2) does not hold.

The notion of metric generalizes the notion of distance in $\mathbb{R}^3$. It can be applied to particle locations, velocities, accelerations, and masses in order to distinguish between different states of a given system or between different systems of particles.

[1] Hints for many of the problems are contained in an appendix beginning on p. 269.

The same holds for any mechanical system described by a finite number of parameters (such as forces or temperatures).

Let us extend the idea of distance to continuum problems. Take a string, with fixed ends, of length π. For a loaded string, we can use the Fourier expansion

$$u(s) = \sum_{k=1}^{\infty} x_k \sin ks \tag{1.1.3}$$

to describe the resulting deflection $u(s)$. Any state of the string can be identified with a vector $\mathbf{x}$ having infinitely many coordinates $x_1, x_2, \ldots$. The dimension of the space S of all such vectors is obviously not finite.

We can modify the metric of $\mathbb{R}^n$ to determine the distance from $\mathbf{x}$ to $\mathbf{y}$ in S. The necessary changes are evident; we can use

$$d(\mathbf{x}, \mathbf{y}) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^p \right)^{1/p} \qquad (p \geq 1)$$

or

$$d(\mathbf{x}, \mathbf{y}) = \sup_i |x_i - y_i| \, .$$

The resulting distances satisfy D1–D4 and are metrics. The analogy between $\mathbb{R}^n$ and S is not perfect, however. Consider the distance from $\mathbf{0} = (0, 0, 0, \ldots)$ to $\mathbf{x}_0 = (1, 1/2, 1/3, \ldots)$ using the metrics

$$d_1(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{\infty} |x_i - y_i| \quad \text{and} \quad d_2(\mathbf{x}, \mathbf{y}) = \sup_i |x_i - y_i| \, .$$

Since

$$d_1(\mathbf{x}_0, \mathbf{0}) = \sum_{i=1}^{\infty} 1/i \quad \text{and} \quad d_2(\mathbf{x}_0, \mathbf{0}) = 1$$

and the series diverges, we do not have

$$d_1(\mathbf{x}_0, \mathbf{0}) / d_2(\mathbf{x}_0, \mathbf{0}) \leq m_2 < \infty \, .$$

So an inequality of the form (1.1.2) is not satisfied. Moreover, as convergence in one metric does not necessitate convergence in the other, the two metrics are not equivalent. This example also shows why a metric must be *well-defined*, i.e., uniquely defined and finite for any pair of elements $\mathbf{x}, \mathbf{y}$. In the case of S above, we can only introduce a metric on some subset of S over which the metric is well-defined. This yields various metric spaces of infinite dimensional vectors. For general sets, this idea is realized in the following

Definition 1.1.1. Let X be a nonempty set of elements (often called *points*). A function $d(x, y)$, uniquely defined and finite for all $x, y \in X$, is a *metric* on X if it satisfies the metric axioms D1–D4. We refer to the pair (X, d) as a *metric space*.

Although a metric space consists of a set X and a metric d, certain standard metric spaces (for which the metric is understood) can safely be denoted by the underlying set X alone.

Remark 1.1.3. In the following pages, we shall not distinguish between metric spaces based on the same set of elements if their metrics are equivalent. If d_1 and d_2 are equivalent metrics, we shall not distinguish between (X, d_1) and (X, d_2). However, if d_1 and d_2 are not equivalent, then (X, d_1) and (X, d_2) are different metric spaces. We also saw that metric spaces could be formed from the elements of S (the set of infinite dimensional vectors) only by restriction to certain subsets of S. So S is not a metric space, but merely a (linear) set of infinite dimensional vectors whose linear subsets (subspaces), together with their metrics, can constitute various metric spaces of vectors having infinitely many coordinates. □

In the definition of metric space the nature of the elements of the space is unimportant. The elements could be abstract objects, even ordinary objects such as chairs or tables — it is merely necessary that we can introduce, for each pair of elements of the set, a function satisfying the metric axioms. In mathematical physics, metric spaces of functions are often employed. These are the spaces to which solutions of certain equations or the parameters of a problem must belong. During the rigorous investigation of such problems, restrictions are always imposed on the properties of the solutions sought. This is due not only to a desire for rigor and formalism; a mathematical problem can have several solutions, certain parts of which may contradict our ideas about the nature of the process described by the problem. Additional restrictions based on the physical nature of the problem allow us to select physically reasonable solutions. One way to impose such restrictions is to require that the solution belong to a metric space. Thus the choice of space in which one seeks a solution can be crucial for addressing a realistic problem. Depending on this choice, we may or may not have existence and uniqueness of a solution, etc. Metric spaces in mathematical physics are usually linear and infinite dimensional.

We now list some metric spaces of infinite dimensional vectors

$$\mathbf{x} = (x_1, x_2, \ldots) \ .$$

A vector with infinitely many components may be regarded as a sequence $\mathbf{x} = \{x_i\}$, and the following spaces are often called *sequence spaces*.

1. *The metric space m.* The space m is the set of all bounded sequences $\mathbf{x}$. For $\mathbf{x} = (x_1, x_2, \ldots)$ and $\mathbf{y} = (y_1, y_2, \ldots)$, the metric is given by

$$d(\mathbf{x}, \mathbf{y}) = \sup_i |x_i - y_i| \ . \tag{1.1.4}$$

2. *The metric space ℓ^p.* The space ℓ^p $(p \geq 1)$ consists of the set of all sequences

$$\mathbf{x} = (x_1, x_2, \ldots) \quad \text{for which} \quad \sum_{i=1}^{\infty} |x_i|^p < \infty \ ,$$

along with the metric

$$d_p(\mathbf{x},\mathbf{y}) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^p \right)^{1/p} . \tag{1.1.5}$$

3. *The metric space c.* The space c is the linear subspace of m that consists of all convergent sequences; the metric is (1.1.4).

4. *The metric space c_0.* The space c_0 is the subspace of c consisting of all sequences converging to 0; again, the metric is (1.1.4).

The metrics of these spaces were introduced by analogy with metrics on $\mathbb{R}^n$. Similarly we can introduce (with no change in notation) spaces of infinite dimensional vectors with complex components. Of course, any proposed metric space must be checked for compliance with Definition 1.1.1. For the spaces m, c, c_0, and ℓ^1 this task is almost trivial. For ℓ^p with $p > 1$ (and even for $\mathbb{R}^n$ with the corresponding metrics) the task is not so simple and will be carried out later.

We now consider another class of metrics: the energy metrics.

5. *The energy space for a string.* The potential energy of a string is proportional to

$$\int_0^{\pi} \left(\frac{\partial u}{\partial s} \right)^2 ds = \pi \sum_{k=1}^{\infty} k^2 x_k^2 ,$$

x_k being defined by the representation (1.1.3). We can compare two states

$$u(s) = \sum_{k=1}^{\infty} x_k \sin ks \quad \text{and} \quad v(s) = \sum_{k=1}^{\infty} y_k \sin ks$$

of the string via the metric

$$d(u,v) \equiv d(\mathbf{x},\mathbf{y}) = \left[\sum_{k=1}^{\infty} k^2 (x_k - y_k)^2 \right]^{1/2} . \tag{1.1.6}$$

The energy space for the string is the set of all sequences of Fourier coefficients x_k such that

$$\sum_{k=1}^{\infty} k^2 x_k^2 < \infty .$$

The metric is given by (1.1.6).

Problem 1.1.3. Show that (1.1.4) and (1.1.6) are indeed metrics on their respective sets. *Preliminary hint*: For (1.1.4) the task is simple. Now consider (1.1.6). For each element of the energy space for a string, defined by Fourier coefficients x_k, we introduce another infinite vector $\mathbf{z}$ with components $z_k = kx_k$. By the definition of the energy metric, it induces the metric of ℓ^2 over the set of all such $\mathbf{z}$. Because ℓ^2 is a metric space, so is the energy space of Fourier coefficients for the string. □

Energy spaces are advantageous when applied to mechanics problems, as we shall see later.

6. *The metric space of straight lines.* The notion of metric space is abstract, and the elements of the underlying set need not be vectors. Consider the set M of all straight lines in the plane which do not pass through the coordinate origin. A straight line L is given by the equation

$$x\cos\alpha + y\sin\alpha - p = 0\ .$$

Let us show that

$$d(L_1, L_2) = \left[(p_1 - p_2)^2 + 4\sin^2\frac{\alpha_1 - \alpha_2}{2}\right]^{1/2}$$

is a metric on M. Axioms D1 and D3 are obviously satisfied. Consider D2. Certainly $d(L_1, L_2) = 0$ whenever $L_1 = L_2$. Conversely, $d(L_1, L_2) = 0$ implies both

$$p_1 = p_2 \quad \text{and} \quad \sin(\alpha_1 - \alpha_2)/2 = 0\ ;$$

the latter condition gives

$$\alpha_1 - \alpha_2 = 2\pi n \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

hence $L_1 = L_2$. Finally, consider D4. Since

$$4\sin^2\frac{\alpha_1 - \alpha_2}{2} = (\sin\alpha_1 - \sin\alpha_2)^2 + (\cos\alpha_1 - \cos\alpha_2)^2$$

we have

$$d(L_1, L_2) = \left[(p_1 - p_2)^2 + (\sin\alpha_1 - \sin\alpha_2)^2 + (\cos\alpha_1 - \cos\alpha_2)^2\right]^{1/2}.$$

Let $(p_i, \sin\alpha_i, \cos\alpha_i)$ be the coordinates of a point A_i in three-dimensional Euclidean space. Noting that $d(L_i, L_j)$ equals the Euclidean distance from A_i to A_j in $\mathbb{R}^3$, we see that D4 is also satisfied. However, the metric $d(L_1, L_2)$ lacks an immediate geometrical interpretation.

To prove that ℓ^p with $p \geq 1$ is a metric space, we require the inequalities of the following section.

1.2 Hölder's Inequality and Minkowski's Inequality

We start with *Hölder's inequality* for sums,

$$\left|\sum_{k=1}^{n} x_k y_k\right| \leq \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p} \left(\sum_{k=1}^{n} |y_k|^q\right)^{1/q}, \tag{1.2.1}$$

where $p > 1$ and

$$1/p + 1/q = 1 . \tag{1.2.2}$$

The constants p and q satisfying (1.2.2) are called *conjugate exponents*. Introducing the notation

$$\|\mathbf{x}\|_p = \left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p}$$

(later we verify that this expression is a norm[2] of $\mathbf{x}$ on $\mathbb{R}^n$) we rewrite (1.2.1) as

$$\left| \sum_{k=1}^{n} x_k y_k \right| \le \|\mathbf{x}\|_p \, \|\mathbf{y}\|_q \ .$$

We begin to prove Hölder's inequality.

Proof. Because

$$\left| \sum_{k=1}^{n} x_k y_k \right| \le \sum_{k=1}^{n} |x_k y_k| \ ,$$

Hölder's inequality follows from the relation

$$\sum_{k=1}^{n} |x_k y_k| \le \|\mathbf{x}\|_p \, \|\mathbf{y}\|_q \ . \tag{1.2.3}$$

If either $\mathbf{x}$ or $\mathbf{y}$ is zero, then (1.2.3) is trivial. So we suppose they are both nonzero and prove the equivalent result

$$\sum_{k=1}^{n} \frac{|x_k|}{\|\mathbf{x}\|_p} \frac{|y_k|}{\|\mathbf{y}\|_q} \le 1 \ .$$

Now we need an elementary inequality for positive numbers a and b:

$$a^{1/p} b^{1/q} \le \frac{a}{p} + \frac{b}{q} \quad \text{where} \quad \frac{1}{p} + \frac{1}{q} = 1 \ , \tag{1.2.4}$$

which we prove later. Let us take

$$a = \frac{|x_k|^p}{\|\mathbf{x}\|_p^p} \ , \qquad b = \frac{|y_k|^q}{\|\mathbf{y}\|_q^q} \ .$$

We get

$$\frac{|x_k|}{\|\mathbf{x}\|_p} \cdot \frac{|y_k|}{\|\mathbf{y}\|_q} \le \frac{|x_k|^p}{p\,\|\mathbf{x}\|_p^p} + \frac{|y_k|^q}{q\,\|\mathbf{y}\|_q^q} \qquad (k = 1, \ldots, n) \ .$$

Summing these over k, we have

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[2] As stated in the Introduction, the norm symbol $\|\cdot\|$ is formally introduced on p. 39. It is used here on the assumption that the reader has seen it in more elementary courses.

$$\sum_{k=1}^{n} \frac{|x_k|\,|y_k|}{||\mathbf{x}||_p\,||\mathbf{y}||_q} \le \frac{\sum_{k=1}^{n}|x_k|^p}{p\sum_{k=1}^{n}|x_k|^p} + \frac{\sum_{k=1}^{n}|y_k|^q}{q\sum_{k=1}^{n}|y_k|^q} = \frac{1}{p} + \frac{1}{q} = 1\ .$$

To complete the proof, we show (1.2.4). Taking the derivative of the function

$$f(t) = \frac{t}{p} + \frac{1}{q} - t^{1/p}$$

we see that the minimum point of f for $t \ge 0$ is $t = 1$. Moreover, $f(1) = 0$, so for $t \ge 0$ we have

$$t^{1/p} \le \frac{t}{p} + \frac{1}{q}\ .$$

Putting $t = a/b$, we get

$$\frac{a^{1/p}}{b^{1/p}} \le \frac{a}{pb} + \frac{1}{q}\ .$$

Multiplying this by b and noting that $1/q = 1 - 1/p$, we get (1.2.4). □

Provided the series

$$\sum_{k=1}^{\infty} |x_k|^p \quad \text{and} \quad \sum_{k=1}^{\infty} |y_k|^q$$

both converge, the limit passage as $n \to \infty$ in (1.2.3) shows that the series

$$\sum_{k=1}^{\infty} x_k y_k$$

converges absolutely. This yields Hölder's inequality for series:

$$\left|\sum_{k=1}^{\infty} x_k y_k\right| \le \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p} \left(\sum_{k=1}^{\infty} |y_k|^q\right)^{1/q}. \tag{1.2.5}$$

Now we prove *Minkowski's inequality*

$$\left(\sum_{k=1}^{n} |x_k + y_k|^p\right)^{1/p} \le \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p} + \left(\sum_{k=1}^{n} |y_k|^p\right)^{1/p} \tag{1.2.6}$$

for $p \ge 1$, which can be rewritten as

$$||\mathbf{x} + \mathbf{y}||_p \le ||\mathbf{x}||_p + ||\mathbf{y}||_p\ .$$

This constitutes the triangle inequality for the norm $||\cdot||_p$ in $\mathbb{R}^n$. For $p = 1$ the result is trivial so we consider it for $p > 1$. Multiplying both sides of (1.2.6) by the quantity

$$\left(\sum_{k=1}^{n}|x_k+y_k|^p\right)^{1/q},$$

we get an equivalent inequality

$$\sum_{k=1}^{n}|x_k+y_k|^p \le \left[\left(\sum_{k=1}^{n}|x_k|^p\right)^{1/p}+\left(\sum_{k=1}^{n}|y_k|^p\right)^{1/p}\right]\left(\sum_{k=1}^{n}|x_k+y_k|^p\right)^{1/q}, \tag{1.2.7}$$

which we prove. We start with a simple inequality

$$\begin{aligned}\sum_{k=1}^{n}|x_k+y_k|^p &= \sum_{k=1}^{n}|x_k+y_k|\,|x_k+y_k|^{p-1}\\ &\le \sum_{k=1}^{n}|x_k|\,|x_k+y_k|^{p-1}+\sum_{k=1}^{n}|y_k|\,|x_k+y_k|^{p-1}\,. \end{aligned}\tag{1.2.8}$$

Using (1.2.1), we get

$$\sum_{k=1}^{n}|x_k|\,|x_k+y_k|^{p-1} \le \left(\sum_{k=1}^{n}|x_k|^p\right)^{1/p}\left(\sum_{k=1}^{n}|x_k+y_k|^{q(p-1)}\right)^{1/q}.$$

But $q=p/(p-1)$ and we have

$$\sum_{k=1}^{n}|x_k|\,|x_k+y_k|^{p-1} \le \left(\sum_{k=1}^{n}|x_k|^p\right)^{1/p}\left(\sum_{k=1}^{n}|x_k+y_k|^{p}\right)^{1/q}.$$

Similarly,

$$\sum_{k=1}^{n}|y_k|\,|x_k+y_k|^{p-1} \le \left(\sum_{k=1}^{n}|y_k|^p\right)^{1/p}\left(\sum_{k=1}^{n}|x_k+y_k|^{p}\right)^{1/q}.$$

Substituting these into (1.2.8) we get (1.2.7) and hence (1.2.6). If the series

$$\sum_{k=1}^{\infty}|x_k|^p \quad \text{and} \quad \sum_{k=1}^{\infty}|y_k|^p$$

converge, then the limit as $n\to\infty$ produces Minkowski's inequality for series:

$$\left(\sum_{k=1}^{\infty}|x_k+y_k|^p\right)^{1/p} \le \left(\sum_{k=1}^{\infty}|x_k|^p\right)^{1/p}+\left(\sum_{k=1}^{\infty}|y_k|^p\right)^{1/p}. \tag{1.2.9}$$

Now we can prove

Theorem 1.2.1. For $p\ge 1$, the space ℓ^p, consisting of the vectors $\mathbf{x}$ such that $\|\mathbf{x}\|_p<\infty$, is a metric space.

Proof. The case $p=1$ is trivial so let $p>1$. Clearly $\|\mathbf{x}\|_p$, which is the distance from $\mathbf{x}$ to the zero vector, is finite by hypothesis. Verification of axioms D1–D3 is trivial. If $\mathbf{x} = (x_1, x_2, \ldots)$, $\mathbf{y} = (y_1, y_2, \ldots)$, and $\mathbf{z} = (z_1, z_2, \ldots)$, then by Minkowski's inequality

$$d_p(\mathbf{x}, \mathbf{y}) \equiv \left(\sum_{k=1}^{\infty} |x_k - y_k|^p \right)^{1/p} = \left(\sum_{k=1}^{\infty} |(x_k - z_k) + (z_k - y_k)|^p \right)^{1/p}$$
$$\leq \left(\sum_{k=1}^{\infty} |x_k - z_k|^p \right)^{1/p} + \left(\sum_{k=1}^{\infty} |z_k - y_k|^p \right)^{1/p} = d_p(\mathbf{x}, \mathbf{z}) + d_p(\mathbf{z}, \mathbf{y})$$

and D4 is satisfied. □

Problem 1.2.1. Let $h_1, h_2, \ldots$ be positive numbers. Show that the space of infinite sequences $\mathbf{x} = (x_1, x_2, \ldots)$ satisfying

$$\|\mathbf{x}\|_{w,p} = \left(\sum_{k=1}^{\infty} h_k \, |x_k|^p \right)^{1/p} < \infty \, ,$$

with the weighted metric $d_{w,p}(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_{w,p}$, is a metric space. □

For the spaces of infinite dimensional vectors considered above, a vector $\mathbf{x}$ belongs to a space if and only if $d(\mathbf{x}, \mathbf{0}) < \infty$. So we should establish this inequality. The task is more complicated for function spaces. Another note concerns terminology. Although for $\mathbf{x} = (x_1, x_2, \ldots)$ we use the terms "sequence" and "infinite dimensional vector" interchangeably, implicitly assuming the x_k to be the components of the vector $\mathbf{x}$, we will never carry out the change of basis that is so central to ordinary linear algebra. This is because the issue of basis is not as simple in infinite dimensional spaces as it is in $\mathbb{R}^n$.

1.3 Metric Spaces of Functions

Functions of two or three variables serve to describe the behavior or change in state of a body in space. Displacements, velocities, loads, and temperatures are all functions of position. The notion of metric space is an appropriate tool for distinguishing between states of a body. In continuum mechanics we deal mostly with real-valued continuous or differentiable functions.

Let Ω be a closed and bounded (i.e., compact) domain in $\mathbb{R}^n$. A natural measure of the deviation between two continuous functions $f(\mathbf{x})$ and $g(\mathbf{x})$, $\mathbf{x} \in \Omega$, is the *max metric* given by

$$d(f, g) = \max_{\mathbf{x} \in \Omega} |f(\mathbf{x}) - g(\mathbf{x})| \, . \tag{1.3.1}$$

Clearly $d(f, g)$ satisfies axioms D1–D3 on p. 9. Let us verify D4. Since the function $|f(\mathbf{x}) - g(\mathbf{x})|$ is continuous on Ω, there is a point $\mathbf{x}_0 \in \Omega$ such that

$$d(f, g) = \max_{\mathbf{x} \in \Omega} |f(\mathbf{x}) - g(\mathbf{x})| = |f(\mathbf{x}_0) - g(\mathbf{x}_0)|$$

(recall Theorem 1 on p. 5). For any function $h(\mathbf{x})$ which is continuous on Ω, we get

$$d(f,g) = |f(\mathbf{x}_0) - g(\mathbf{x}_0)| \le |f(\mathbf{x}_0) - h(\mathbf{x}_0)| + |h(\mathbf{x}_0) - g(\mathbf{x}_0)| \le d(f,h) + d(h,g)\ .$$

Thus $d(f,g)$ in (1.3.1) is a metric.

Definition 1.3.1. Let Ω be a closed and bounded domain in $\mathbb{R}^n$. We denote by $C(\Omega)$ the metric space consisting of the set of all continuous functions on Ω supplied with the metric (1.3.1).

Remark 1.3.1. When $\Omega = [a,b]$, a finite interval along the real line, we will denote $C(\Omega)$ by $C(a,b)$ instead of $C([a,b])$ or $C[a,b]$. □

We can also introduce the metric space $C(V)$ of the bounded functions that are continuous on an open set $V \subseteq \mathbb{R}^n$. Because Weierstrass' theorem is not valid on V, however, we use

$$d(f,g) = \sup_{\mathbf{x}\in V} |f(\mathbf{x}) - g(\mathbf{x})|\ . \tag{1.3.2}$$

The reader may easily verify the metric axioms in this case. The definition of the metric space of continuous functions can also be extended to the case in which the functions are continuous with respect to an abstract argument belonging to some topological space V. The metric is defined by (1.3.2).

To account for the derivatives of functions, we use a metric such as

$$d(f,g) = \sum_{|\alpha|\le k} \max_{\mathbf{x}\in\Omega} |D^\alpha f(\mathbf{x}) - D^\alpha g(\mathbf{x})| \tag{1.3.3}$$

where

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}\ , \qquad |\alpha| = \alpha_1 + \cdots + \alpha_n\ , \tag{1.3.4}$$

and α is regarded as an n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$. The notation introduced in (1.3.4) is often called *multi-index notation*.

Problem 1.3.1. Verify the metric axioms for the space $C^{(k)}(\Omega)$ of all continuous functions on a closed and bounded domain Ω whose derivatives up to order k are continuous on Ω, with the metric (1.3.3). □

Problem 1.3.2. Interpret the notation (1.3.4) for the case $n = 3$ and $\alpha = (1,0,2)$. □

On the set of all continuous functions on Ω, let us consider another metric:

$$d(f,g) = \left(\int_\Omega |f(\mathbf{x}) - g(\mathbf{x})|^p\, d\Omega\right)^{1/p} \qquad (p \ge 1) \tag{1.3.5}$$

where Ω is a Jordan measurable compact domain in $\mathbb{R}^n$. Jordan measurability, which ensures that the Riemann integral exists for any function under consideration, guarantees that (1.3.5) is well-defined for any functions f, g that are continuous on Ω. In