

Kumar S. Ray

Soft Computing Approach to Pattern Classification and Object Recognition

A Unified Concept

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Dedicated to:
Dhira (Wife)
Aratrika (Daughter)

Preface

The basic aim of this research monograph is to develop a unified approach to supervised pattern classification and model-based occluded object recognition. To perform this task we essentially consider soft computing tools, viz., fuzzy relational calculus (FRC), genetic algorithm (GA), and multilayer perceptron (MLP). The supervised approach to pattern classification and model-based approach to occluded object recognition are treated in one framework which is based on either conventional interpretation or new interpretation of multidimensional fuzzy implication (MFI) and a novel notion of fuzzy pattern vector (FPV). A completely independent design methodology has been developed on a unified framework which has been thoroughly tested on several synthetic and real life data. In the field of soft computing such application-oriented design study is unique in nature. The monograph essentially mimics the cognitive process of human decision making. It carries a message of *perceptual integrity in representational diversity*.

The monograph is very much useful to the researchers in the area of pattern classification and computer vision. It is useful for the academics as well as for the professional computer scientists of different research and development centers of industry. The monograph has a combined flavor of theory and practice.

The monograph is basically a collection of research contributions of Prof. Kumar S. Ray at Electronics and Communication Sciences Unit of Indian Statistical Institute, Kolkata.

Prof. Kumar S. Ray is grateful to Mandrita Mondal for her constant encouragement and support to complete the monograph.

Kumar S. Ray

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Chapter 1

Soft Computing Approach to Pattern Classification and Object Recognition

Abstract The basic aim of this research monograph is to develop a unified approach to supervised pattern classification (Tou and Gonzalez, *Pattern Recognition Principles*. Addison-Wesley, Reading, 1974) and model based occluded object recognition (Koch and Kashyap, *IEEE Trans Pattern Anal Machine Intell.* 9(4):483–494, 1987; Ray and Dutta Mazumder, *Pattern Recogn Lett* 9:351–360, 1989). To perform this task we essentially consider soft computing tools, viz., fuzzy relational calculus (FRC) (Pedrycz, *Fuzzy Sets Syst* 16:163–174, 1985, *Pattern Recogn* 23(1/2):121–146, 1990), genetic algorithm (GA) (Goldberg, *Genetic Algorithms in Search, Optimization and Machine Learning*. Addison Wesley, Reading, 1989; Michalewicz, *Genetic Algorithm + Data Structures = Evolution Programs*, Springer, New York, 1994) and multilayer perceptron (MLP) (Pao, *Adaptive pattern recognition and neural networks*. Addison Wesley, Reading, 1989). The supervised approach to pattern classification and model based approach to occluded object recognition are treated in one framework which is based on either conventional interpretation or new interpretation of multidimensional fuzzy implication (MFI) (Sugeno and Takagi, *Fuzzy Sets Syst* 9:313–325, 1983; Tsukamoto, *Advance in Fuzzy Set Theory and Applications*. North-Holland, Amsterdam, 137–149, 1979) and a novel notion of fuzzy pattern vector (FPV). In the context of representation of knowledge about patterns and/or objects we try to generalize the concept of feature vector by fuzzy feature vector. Readers are advised to read Appendix-A before they go into the details of classification (recognition) concept based on soft computing tools.

1.1 Introduction

As the primary concern of soft computing approach to pattern classification (object recognition) (Zadeh 1977; Pedrycz 1990) is to mimic the cognitive process of human reasoning for classification (recognition), we try to imitate the way human beings perceive (Newell and Simon 1972) different classes of objects (patterns) based on some rough (inexact) information of certain parameters (features). For instance, human being can easily distinguish between a poor person and a rich person just by looking at the individual's standard of living which cannot be measured explicitly by any specific scale but can be indirectly estimated by considering the area where the person lives, the kind of food he/she takes, the kind of education his/her family takes, the kind of clothes he/she wears, the kind of commodities he/she uses, etc. And such information (knowledge) can be represented by a rule (law of implication) as given below.

R_1 : if standard of living of a person is high then the person is rich,

R_2 : if standard of living of a person is low then the person is poor;

where standard of living high, standard of living low, etc., are linguistic terms and rich, poor are different classes. Each primary linguistic term (i.e. high/low, etc.) is associated with a term set which is finite and where each primary term in the term set is defined on the same universe of discourse. The said universe of discourse is partitioned (in an overlapped manner) by the finite elements of the term set. For instance, if we place the standard of living between 0 and 10 (by some arbitrary scale) then Fig. 1.1 explains how the elements of the term set partition the universe on which each element of the term set is defined. Each primary term of the term set is represented by a fuzzy set. Between partition there is an overlap which indicates the degree of uncertainty of the elements of the said universe to become member of either of the fuzzy set (fuzzy set of low/fuzzy set of medium/fuzzy set of high).

Now, instead of having one-dimensional implications (i.e. R_1 and R_2) we can have multidimensional implication for representing our knowledge (information). For instance,

R_3 : if (behavior of a person is smart, appearance of a person is beautiful)^t then he/she becomes a candidate for interview of a personal assistant of a firm; where t is transpose.

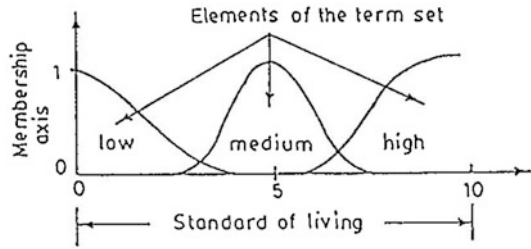
Usually, the above type of rule is read as

R_4 : if behavior of a person is smart and appearance of a person is beautiful then he/she becomes a candidate for interview of a personal assistant of a firm.

Such one-dimensional implication (i.e. R_4) is a kind of interpretation (see Eq. (A.1) of Appendix-A) of the said multi-dimensional form (i.e. R_3) of an implication.

Here, according to the one-dimensional form of an implication, we have two antecedent clauses (e.g. smart behavior and beautiful appearance) which can be represented by two fuzzy sets defined over two different universe of discourses. In the consequent part, we always have single clause which can be represented by a

Fig. 1.1 Overlapped partition of the universe of discourse by the elements of the term set



fuzzy set defined over a finite universe of different classes. The cardinality of the term set of each antecedent clause of an implication determines the number of rules that can be generated. For instance, if we have two antecedent clauses in an implication each having the cardinality 3 (say, low, medium and high), then we will have total $3 \times 3 = 9$ rules. Note, that the cardinality of a term set defined over a given universe is not unique. Depending upon the need of the problem it is determined. It simply indicates the granularity (see Fig. 1.2) by which we want to partition the given universe to facilitate our representation of perception about grouping of objects (patterns).

Depending on whether the universe of discourse is continuous or discrete, we can define the fuzzy sets of the antecedent clause of an implication by two ways. In case universe is continuous, we may go by functional definition, e.g. a bell-shaped function (see Fig. 1.1), triangle shaped function (see Fig. 1.2), trapezoid shaped function or any arbitrary shaped function. In case of discrete universe, we may go by numerical definition. In this case, the grade of membership function of a fuzzy set is represented by a one-dimensional array of numbers. The length of the array depends on the degree of discretization.

Discretization of a universe of discourse is frequently referred to as quantization. In effect quantization discretizes a universe into a certain number of segments (quantization levels). Each segment is labeled as a generic element of a discrete universe. A fuzzy set is then defined over the said discrete universe by assigning grade of membership values to each generic element of the discrete universe.

The consequent clause of an implication basically represents different classes of objects (patterns) existing over the finite range of the pattern space as shown in Fig. 1.3. The possibility of occurrence of different classes of patterns in the pattern space under a particular observation (it may be imprecise observation, like feature F_1 is high and feature F_2 is low, etc.) may be represented by a fuzzy set defined over the pattern space which is treated as universe of different classes of patterns.

Now, we try to give a more meaningful discussion on the correspondence between conventional approach to pattern classification and soft computing approach to pattern classification. Note that recognition of the occluded scene consists of model objects is based on the features of few dominant points of the occluded scene and model objects. The features of patterns and objects are unifiedly treated on a feature (pattern) space spanned by the individual feature axis. Further note that the basic concept of supervised approach to pattern classification

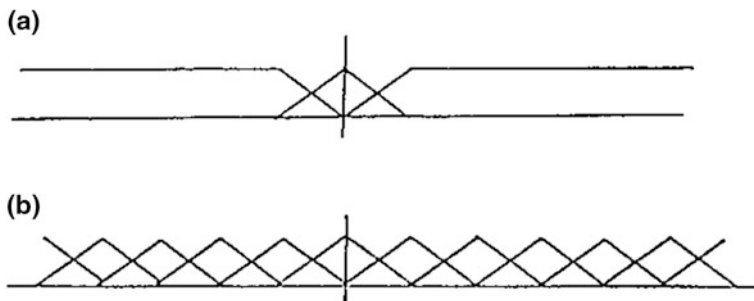


Fig. 1.2 Granularity of partition of universe, a coarse. b Fine

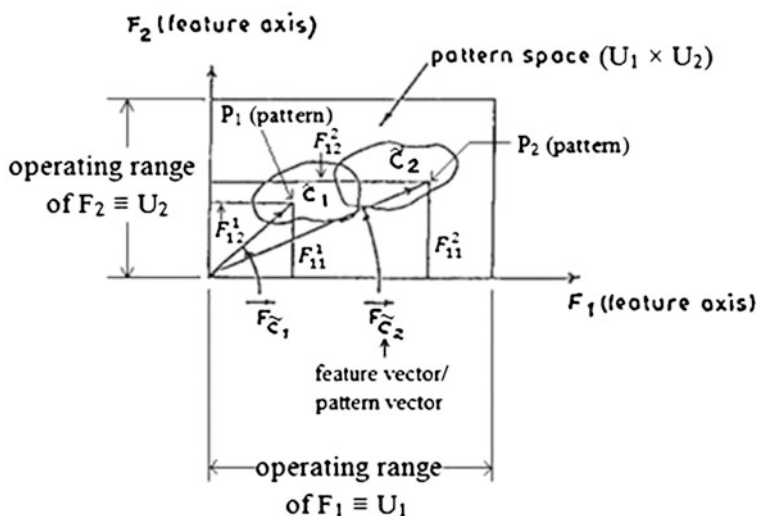


Fig. 1.3 Representation of different classes of patterns on R^2 by feature vectors/pattern vectors

and model based approach to occluded object recognition are similar in nature. Hence both the approaches are treated on a unified platform, i.e. either based on conventional interpretation of MFI or based on new interpretation of MFI (Sugeno and Takagi 1983; Tsukamoto 1979).

In the supervised approach to pattern classification and model based approach to occluded object recognition, the declarative form of knowledge representation about the features of the training patterns or model objects is performed through fuzzy IF-THEN rules. The antecedent part of the rules represent the feature values in terms of primary fuzzy sets and the consequent part of the rules represent the class belongingness of the known patterns or objects. Thus feature values of training patterns or model objects are related to different classes formed by training patterns or model objects. To estimate the said relations is the primary task of classifier design (for patterns) or recognizer design (for scene consists of model

objects). Usually, for inferencing (reasoning/decision making) from a set of fuzzy IF–THEN rules, multivalued concept of fuzzy logic is used (Mizumoto 1985; Zadeh 1970). But in this research monograph, for inferencing (reasoning/decision making) we replace logic by the concept of learning, simply because the logical approach to fuzzy reasoning depends upon suitable choice of interpretation, viz. Mamdani’s min operator, Zadeh’s arithmetic rules etc. (Mizumoto 1985; Zadeh 1970) to construct a relation between antecedent clause(s) and consequent clause. To design a classifier (for patterns) or recognizer (for occluded scene consists of model objects) we adopt the concept of learning to estimate the relation between antecedent clause(s) and consequent clause.

1.2 Passage Between Conventional Approach to Pattern Classification (Object Recognition) and Soft Computing Approach to Pattern Classification (Object Recognition)

For simplicity of discussion and/or demonstration, let us restrict ourselves to the problem of supervised pattern classification (model based object recognition) on \mathbb{R}^2 . Without lack of any generality, all the discussions and/or demonstrations will be valid for any problem of supervised pattern classification (model based object recognition) on \mathbb{R}^n , $n \geq 2$. In case of supervised pattern classification (model based object recognition) we basically have two stages; viz. learning (training) stage and testing (verification) stage. At the learning (training) stage we design the classifier (recognizer) based on the training data (model objects) set. Subsequently, at the testing (verification) stage we test the performance of the designed classifier (recognizer) based on the test data set.

In conventional approach to pattern classification (object recognition) on \mathbb{R}^2 (Tou and Gonzalez 1974; Koch and Kashyap 1987; Ray and DuttaMazumder 1989), we usually represent the training data set by two dimensional feature vector (pattern vector) having two feature axes (say F_1 and F_2). Depending upon the limit of the operating range of features, we obtain a finite range of pattern space formed by the finite length of each feature axis (see Fig. 1.3). Looking at the scatter diagram of the training data set we get an idea of how the training data are grouped together and accordingly each group of training data set is labeled by a particular class say \tilde{c}_i , $i = 1, 2, \dots$. Each data (pattern) of each class is represented by a pattern vector/feature vector as stated earlier. For instance, any data (pattern) P_i on the pattern space is represented by the tip of the following pattern vector/feature vector (see Fig. 1.3).

$$\vec{F}_{\tilde{c}_i} = \begin{bmatrix} F_{11}^i \\ F_{22}^i \end{bmatrix}, \quad i = 1, 2, \dots \quad (1.1)$$

The task of the conventional approach to pattern classification is to classify each vector $\vec{F}_{\tilde{c}_i}$ to one of the classes \tilde{c}_i on \mathbb{R}^2 .

In conventional approach, position of each pattern (say, P_i) on the finite range of the pattern space is represented by a pattern vector/feature vector $\vec{F}_{\tilde{C}_i}$ and we always try to discriminate among patterns by classifying the pattern vector/feature vector $\vec{F}_{\tilde{C}_i}$.

Therefore, from the given data set (i.e. the training data set), we always know where the patterns are located and then try to separate them by some appropriate decision function. Subsequently, we use the said decision function to classify the test pattern vector/feature vector.

But, if we go by mimicking the cognitive process of human reasoning (Newell and Simon 1972; Zadeh et al. 1975) for pattern classification (object recognition) then the problem is, from a given set of imprecise observations stated in terms of fuzzy **If-Then** rules, how to represent the patterns on the pattern space for developing suitable inferencing technique for classification. As the observations (knowledge) are given in terms of fuzzy **If-Then** rules, it is clear that at every observation (rule), features are given in the form of fuzzy set defined over the feature universe. Therefore, in such a case we cannot represent/locate individual pattern by a vector. Instead, from a given observation (rule), we can represent/locate a population of patterns in an area (in case of R^2 ; but region in general) on the pattern space by fuzzy pattern vector/feature vector (see Fig. A.1 of Appendix-A) which is a very natural representation of the antecedent parts of a multidimensional implication (i.e. R_3) represented by a fuzzy set, i.e. a multidimensional fuzzy implication (MFI) (Sugeno and Takagi 1983; Tsukamoto 1979).

Now, we stipulate the following definition;

Definition 1.1. Let \vec{F}_f be a fuzzy vector having “n” components, each of which is a fuzzy set D^i defined over the universe U_i of the feature axis F_i . The fuzzy vector \vec{F}_f is a fuzzy set in the quantized product space $U_1 \times U_2 \times \dots \times U_n$. Each element of the fuzzy set is a vector having same initial point but different terminal points which are the elements of the fuzzy point which is a fuzzy set. Each terminal point of each vector in the set carries one membership value indicating its (vector’s) degree of belongingness to the set \vec{F}_f . A fuzzy vector \vec{F}_f is represented as

$$\vec{F}_f = \left\{ \mu_{\vec{F}_f} \left(\left(\vec{V} \right) \cdot \vec{V} \right) \forall \vec{V} \in U_1 \times U_2 \times \dots \times U_n \right\}$$

where $\mu_{\vec{F}_f} : U_1 \times U_2 \times \dots \times U_n \rightarrow [0, 1]$ is the membership function of \vec{F}_f and $\mu_{\vec{F}_f}(\vec{V})$ is the grade of membership of \vec{V} in \vec{F}_f .

The process of defuzzification of the fuzzy vector \vec{F}_f is performed based on selecting the elements of the fuzzy vector \vec{F}_f (which is a fuzzy set) having highest membership values. The defuzzified version of the fuzzy vector \vec{F}_f is a fuzzy vector which is a fuzzy set. In case the defuzzified version of the fuzzy vector \vec{F}_f

represents a fuzzy set which is a fuzzy singleton then the defuzzified version of \vec{F}_f becomes the crisp version (see Example 1.1). The fuzzy set D^i as mentioned in the Definition 1.1 is an element of the term set as discussed in Fig. 1.1. The universe of the components of \vec{F}_f , i.e. the fuzzy set D^i , may be continuous/discrete and the universe of \vec{F}_f may be continuous/discrete. In case universes are discrete we should follow the numerical definition of membership functions; otherwise, we should follow the functional definition. In case the defuzzified version of \vec{F}_f reduces to the crisp vector as stated earlier the membership value at the terminal point of the vector \vec{V}_{ij} of \vec{F}_f can alternatively be interpreted as the highest possibility of \vec{V}_{ij} to hold the property of the fuzzy vector \vec{F}_f . By the term property we want to mean a particular combination of the elements of different term sets. For instance, with respect to Fig. (A.1) of Appendix-A the property assumed by fuzzy vector \vec{F}_f is (medium, medium). Like this we can have property (medium, small), (medium, big) etc. (also see Fig. 1.5). The crisp vector $\vec{V}_{15,13}$ of Fig. (A.1) of Appendix-A obtained by the process of defuzzification of the fuzzy vector \vec{F}_f has the highest possibility to hold the above said property (medium = M1, medium = M2). In case we obtain an induced fuzzy vector \vec{F}_f which is basically a cylindrical extension of a primary fuzzy set D^i over U_i , all the elements of the defuzzified version of \vec{F}_f have equal possibility values which are the highest possibility to hold the property of the element D^i of the term set over U_i . For instance, in Fig. A.2 of Appendix-A, the vectors from $\vec{V}_{15,1}$ to $\vec{V}_{15,25}$ have equal possibility values which are the highest possibility to hold the property of the fuzzy set M_1 which is an element of the term set $\{S_1, M_1, B_1\}$ defined over the universe of U_1 . The relation of the generic element represented by the line segment ($9 \leq u_{10}^2 < 10$) defined over the universe of U_2 with all the elements of the fuzzy set M_1 is written by an array $\{0.01/\vec{V}_{10,10}, 0.2/\vec{V}_{11,10}, 0.4/\vec{V}_{12,10}, 0.6/\vec{V}_{13,10}, 0.8/\vec{V}_{14,10}, 1/\vec{V}_{15,10}, 0.8/\vec{V}_{16,10}, 0.6/\vec{V}_{17,10}, 0.4/\vec{V}_{18,10}, 0.2/\vec{V}_{19,10}, 0.01/\vec{V}_{20,10}\}$. The array itself is a fuzzy set where the vectors $\vec{V}_{10,10}, \vec{V}_{11,10}, \vec{V}_{12,10}, \vec{V}_{13,10}, \vec{V}_{14,10}, \vec{V}_{15,10}, \vec{V}_{16,10}, \vec{V}_{17,10}, \vec{V}_{18,10}, \vec{V}_{19,10}, \vec{V}_{20,10}$ are the elements of the said fuzzy set (see Fig. A.2 of Appendix-A). The possibility of the vector $\vec{V}_{10,10}$ to hold the property of the vector $\vec{V}_{15,10}$ is 0.01.

The fuzzy set M_1 , which is linguistically stated as fuzzy set medium over the universe U_i of F_1 , is basically obtained by a process of fuzzification of the discrete element represented by the line segment $14 \leq u_{15}^1 < 15$. Thus a point on real line or a discrete line segment (on real line) which is the quantized version of the point is represented by a fuzzy set on real line. Intuitively speaking, the quantize point 15 represented by the line segment $14 \leq u_{15}^1 < 15$ on the range of feature axis F_1 , i.e. $0 \leq U_1 < 28$ (see Fig. A.1 and A.2 of Appendix-A) is approximately the 'midpoint'. To represent the 'midpoint' by linguistically stated fuzzy set M_1 defined

over the discrete universe of U_1 we consider some spread on both sides of the quantize point 15 which is the quantification of our perception about the said 'midpoint' 15 which is fuzzified over the discrete universe U_1 . This quantized representation of the fuzzy set medium (M_1) is not unique and may vary depending upon the perception of an individual which is not unique. But whatever may be the variation of human perception, it always follows a particular trend. Thus the integrity of perception (i.e. the unique trend in perception) is reflected on the diversity of representation where lies the real flavour of fuzziness.

Remark 1.1.

1. A fuzzy point on R^1 is a straight line. For example, consider the fuzzy set M_1 on Fig. A.1 and A.2 of Appendix-A.
2. A fuzzy point on R^2 is an area. For example, consider the $ABCD$ of Fig. A.1 of Appendix-A and area $EFGH$ of Fig. A.2 of Appendix-A.
3. A fuzzy point on R^3 is a volume.
4. A fuzzy point on R^n is a region.

Now we introduce the notion of fuzzy pattern vector/feature vector i.e. \vec{F}_f which is an analogous representation of \vec{F}_{ci} on R^2 (see Fig. 1.3). Tip of the fuzzy pattern vector/feature vector no longer represents a single pattern on R^2 ; rather it represents a population of patterns (set of patterns).

Example 1.1. Let us consider the following fuzzy feature vector/pattern vector (see Fig. A.1 of Appendix-A). The primary fuzzy set M_1 and M_2 are defined over the discrete universe of U_1 and U_2 respectively. Each discrete element of the universe $U_i \forall i$ is represented by small line segment over $U_i \forall i$.

$$\begin{aligned}
 \left[\begin{array}{l} F_1 \text{ is } M_1 \\ F_2 \text{ is } M_2 \end{array} \right] &= \vec{F}_f = \left\{ \left(\mu \vec{F}_f(\vec{V}), \vec{V} \right) \right\} \\
 &= \left\{ \mu \vec{F}_f(\vec{V}) / \vec{V} \right\} \\
 &= \sum_{i=1}^{20} \sum_{j=1}^{19} \mu \vec{F}_f(\vec{V}_{ij}) / \vec{V}_{ij} \\
 &= \left\{ \left(0.01 / \vec{V}_{10,7} \right) + \dots + \left(0.01 / \vec{V}_{10,17} \right) + \dots + \left(0.01 / \vec{V}_{10,19} \right) \right. \\
 &\quad \left. + \dots + \left(0.8 / \vec{V}_{14,12} \right) + \dots + \left(1 / \vec{V}_{15,13} \right) + \dots + \left(0.01 / \vec{V}_{20,19} \right) \right\}
 \end{aligned}$$

Where $+$ and \sum are in the set – theoretic sense and M_1 is a fuzzy set $\{0.01/(9 \leq u_{10}^1 < 10), 0.2/(10 \leq u_{11}^1 < 11), 0.4/(11 \leq u_{12}^1 < 12), 0.6/(12 \leq u_{13}^1 < 13), 0.8/(13 \leq u_{14}^1 < 14), 1/(14 \leq u_{15}^1 < 15), 0.8/(15 \leq u_{16}^1 < 16), 0.6/(16 \leq u_{17}^1 < 17), 0.4/(17 \leq u_{18}^1 < 18), 0.2/(18 \leq u_{19}^1 < 19), 0.01/(19 \leq u_{20}^1 < 20)\}$ on the features axis U_1 , M_2 is a fuzzy set $\{0.01/(6 \leq u_7^2 < 7), 0.1/(7 \leq u_8^2 < 8), 0.2/(8 \leq u_9^2 < 9), 0.4/(9 \leq u_{10}^2 < 10), 0.6/(10 \leq u_{11}^2 < 11), 0.8/(11 \leq u_{12}^2 < 12), 1/(12 \leq u_{13}^2 < 13) <$