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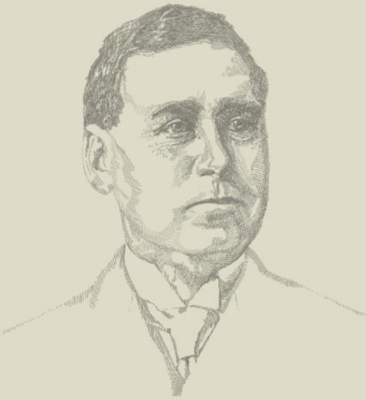

Jan Pachl

Uniform Spaces and Measures

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Uniform Spaces and Measures

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Preface

When measures are considered, by means of integration, as functionals on spaces of functions, certain continuity properties of measures and of sets of measures often play a prominent role. This is the case in topological measure theory (smoothness of measures in duality with continuous functions), abstract harmonic analysis (conditions for well-behaved convolution of measures on topological groups) and probability theory (uniform tightness and its relationship with weak convergence of probability distributions on complete metric spaces).

The main thesis of this treatise is that a number of continuity properties commonly imposed on measures are instances of a general property defined in the language of uniform spaces and that several fundamental results about measures remain valid in such a setting. The general property singles out a class of functionals called *uniform measures* on the space of bounded uniformly continuous functions on a uniform space.

Although a uniform measure is a functional on a function space, not a genuine measure (a countably additive set function on a σ-algebra), in some respects uniform measures behave like measures, and for some purposes they may be used as a substitute for measures. Moreover, in many problems, the underlying space carries a natural uniform structure, which leads to questions about functionals on spaces of uniformly continuous functions; uniform measures often feature in answers to such questions.

The basic theory of uniform measures was developed by a number of researchers in the 1960s and 1970s, but an interested reader would need considerable patience to track down scattered sources, some unpublished, written using a variety of definitions and differing notations. Recently the need for an accessible exposition became more apparent in view of new results in abstract harmonic analysis. In this monograph I offer a unified treatment of the theory of uniform measures, with a view towards such applications and others that I expect still to come.

This is primarily a reference for the theory of uniform measures and related functionals on spaces of uniformly continuous functions. It is also suitable for graduate or advanced undergraduate courses on selected topics in topology and functional analysis.

Part I is a self-contained development of the necessary results about uniform spaces. Part II is a systematic presentation of the basic theory of uniform measures, concluding with applications in the study of convolution algebras. Part III complements the basic theory with results in several related areas.

Uniform measures may be, and have been, defined in several equivalent ways. Although it adds to their usefulness, it also means that anyone attempting to cover all their major facets in a linear text is faced with a number of choices about starting points and the order of presentation. In selecting and organizing the material, my main objective has been to assemble a foundation for applying the theory in functional analysis, in a way that is likely to appeal to those interested in such applications. I have omitted or deferred some developments that would distract from the main objective. Part III includes several such developments that are intrinsically interesting and supply a broader context for the theory in Part II.

Despite a distance in time and space, this book owes much to the late Zdeněk Frolík and to the supportive environment for young mathematicians that he created in Prague in the 1970s. He made major contributions to the theory described here, and uniform measures were often discussed in his seminars. I obtained my first results about uniform measures with his advice and encouragement.

I am grateful to Henri Buchwalter, David Fremlin, Ramon van Handel and Miloš Zahradník for allowing me to use their unpublished ideas and materials. I thank Anthony Hager, Petr Holický, Matthias Neufang and Juris Steprāns for their comments and valuable suggestions, Carl Riehm for shepherding the text through the publication process, and my wife Cynthia for her patience and support.

The content incorporates ideas and techniques that I learnt from many mathematicians over the years. I appreciate their implicit contributions, even if I cannot name them all here.

The monograph was written while I was a visitor at the Fields Institute in Toronto. Contacts with colleagues at the institute and access to its facilities helped me a great deal in assembling the material that follows.

Toronto, Canada Jan Pachl

Contents

Prerequisites

The theory developed in this book builds on a number of standard results in topology, functional analysis and measure theory. The prerequisites are summarized in this chapter, along with notation conventions. Definitions of common terms and most proofs are omitted. Detailed explanations and proofs may be found in the reference works cited in each section.

The traditional QED symbol □ marks the end of a proof, or of a lemma or theorem when the proof is omitted. The square dot ■ marks the end of a definition, example or exercise. The expression $A := B$ means that A is defined to be B. The expression "iff" stands for "if and only if" in definitions.

P.1 Sets and Mappings

Reference: Jech [102].

To have a concrete setting for the notions of cardinal and ordinal numbers, I assume the ZFC set theory (Zermelo–Fraenkel with Choice). However, any other mainstream set theory with the axiom of choice would do just as well. The definitions and results in this treatise use only elementary properties of sets and do not require understanding of the intricacies of axiomatic set theories.

The cardinality of a set S is $|S|$. The cardinal successor of a cardinal κ is κ^+. The smallest infinite cardinal is $\aleph_0$. Infinite cardinals $\aleph_\alpha$ are indexed by ordinals α. The cofinality of a cardinal κ is $\mathsf{cf}(\kappa)$.

The reader should be familiar with Zorn's Lemma and other consequences of the Hausdorff Maximal Principle, as described, for example, in [109, 0.25].

A *mapping* φ from a set A to a set B, written as $\varphi\colon A \to B$, always means a total mapping; that is, $\varphi(x)$ is defined for every $x \in A$. For a fixed set S and $A \subseteq S$, the *characteristic function of A in S* is $\mathrm{I}_A\colon S \to \{0,1\}$. When A and B are sets, B^A denotes the set of all mappings from A to B.

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I use a simple version of the general lambda-calculus notation to mark domain restriction for multivariate mappings. If φ is a mapping of several variables $x, x', \dots$, then $\backslash_x \varphi$ stands for the mapping $x \mapsto \varphi(x, x', \dots)$. For example, if φ maps $X \times Y$ to Z, then the mappings $\backslash_x \varphi(x,y) \colon X \to Z$ and $\backslash_y \varphi(x,y) \colon Y \to Z$ are defined by $\backslash_x \varphi(x,y)(x_0) := \varphi(x_0, y)$, $\backslash_y \varphi(x,y)(y_0) := \varphi(x, y_0)$ for $x_0 \in X$, $y_0 \in Y$. Similarly for mappings from the product of three or more sets.

P.2 Topological Spaces and Groups

References: Császár [31], Engelking [47], Kelley [109].

I assume the reader knows the basics of general topology as covered in the references. Several results needed later are stated in this section. Many other concepts and results from the theory of metric and topological spaces are used throughout the treatise without explicit mention. I do not assume knowledge of uniform spaces. Necessary parts of uniform space theory are developed in Part I.

By definition, all topological and vector spaces are non-empty. On occasion, it will be convenient to allow non-Hausdorff topological spaces. However, all completely regular and all compact topological spaces are Hausdorff. When A is a subset of a topological space, $\overline{A}$ is the closure of A and $\operatorname{int} A$ is the interior of A.

In dealing with topology, I frequently use (directed) nets and Moore–Smith convergence [109, Ch. 2]. A *net* is a collection $\{x_\gamma\}_{\gamma \in \Gamma}$ of elements x_γ indexed by an upwards-directed partially ordered set Γ. Most of the time I omit Γ and write simply $\{x_\gamma\}_\gamma$. When $P(\gamma)$ is a property that depends on $\gamma \in \Gamma$, the statement "$P(\gamma)$ holds for almost all γ" means that there exists $\gamma_0 \in \Gamma$ such that $P(\gamma)$ holds for all $\gamma \geq \gamma_0$.

A *sequence* is a net $\{x_j\}_{j \in \omega}$ indexed by the totally ordered set $\omega := \{0, 1, 2, \dots\}$.

A net $\{x_\gamma\}_{\gamma \in \Gamma}$ in a set S is *universal in* S iff for every set $A \subseteq S$ either $x_\gamma \in A$ for almost all γ or $x_\gamma \in S \setminus A$ for almost all γ.

Lemma P.1. *Let S be a non-empty set. Every net of elements of S has a subnet that is universal in S.*

Proof. [109, 2.J]. □

Let $\mathbb{N}$ denote the set ω with the discrete topology (and later on also with the discrete uniformity).

The ordered field of real numbers is denoted by $\mathbb{R}$, and $\mathbb{R}^+ := \{r \in \mathbb{R} \mid r \geq 0\}$. Besides the algebraic operations and the order that make it an ordered field, $\mathbb{R}$ carries also the usual metric (absolute value of the difference) and the topology defined by the metric. The symbol $\mathbb{R}$ stands for the set of real numbers or the set with one or more of these structures. The meaning will be clear from the context and will not cause any confusion.

When S is a set, $\mathbb{R}^S$ is the set of all real-valued functions on S. Unless stated otherwise, $\mathbb{R}^S$ is considered with the algebraic operations and partial order defined pointwise at each point of S. When $\mathscr{F} \subseteq \mathbb{R}^S$, the expression $\mathscr{F} \nearrow h$ means that $\mathscr{F}$

is upwards directed and $\sup\{f(x) \mid f\in\mathscr{F}\} = h(x)$ for every $x\in S$. The expression $\mathscr{F} \searrow h$ means that $\mathscr{F}$ is downwards directed and $\inf\{f(x) \mid f\in\mathscr{F}\} = h(x)$ for every $x\in S$. Similarly for $f_\gamma \nearrow h$ and $f_\gamma \searrow h$ when $\{f_\gamma\}_\gamma$ is a net of functions.

If $\mathfrak{S}$ is a set of subsets of S, $\{A_\gamma\}_\gamma$ is a net of subsets of S and $A \subseteq S$, then $\mathfrak{S} \nearrow A$, $\mathfrak{S} \searrow A$, $A_\gamma \nearrow A$ and $A_\gamma \searrow A$ stand for the same expressions with sets replaced by their characteristic functions. Thus for example, $\mathfrak{S} \nearrow A$ means that $\mathfrak{S}$ is upwards directed by inclusion and $\bigcup\mathfrak{S} = A$.

The space of all continuous real-valued functions on a topological space T is $\mathsf{C}(T)$, and $\mathsf{C}_\mathsf{b}(T)$ is the space of uniformly bounded functions in $\mathsf{C}(T)$. When T is a completely regular topological space, βT is its Čech–Stone compactification and T is identified with a subspace of βT.

A set $\mathscr{W}$ of subsets of a topological space T is a *base of the topology* of T if every member of $\mathscr{W}$ is an open set and every non-empty open set in T is a union of some subset of $\mathscr{W}$. The *weight* of a topological space T is the least cardinal κ for which T has a base of cardinality κ.

Definition P.2. A *pseudometric* on a set S is a function $\Delta\colon S \times S \to \mathbb{R}^+$ such that

(M1) $\Delta(x,x) = 0$ for all $x\in S$.
(M2) $\Delta(x,y) = \Delta(y,x)$ for all $x,y\in S$.
(M3) $\Delta(x,y) + \Delta(y,z) \geq \Delta(x,z)$ for all $x,y,z\in S$.

A *metric* on S is a pseudometric such that

(M4) $\Delta(x,y) > 0$ for all $x,y\in S$ such that $x \neq y$. ∎

Every pseudometric on a set S defines a topology on S. The topology is Hausdorff if and only if the pseudometric is a metric.

When Δ is a pseudometric on S, $x\in X$ and $A,B \subseteq S$, $A \neq \emptyset \neq B$, define

$$\Delta(A,B) := \inf\{\Delta(a,b) \mid a\in A, b\in B\}$$
$$\Delta(x,A) := \Delta(A,x) := \Delta(\{x\},A).$$

The *Δ-diameter* of a non-empty set $A \subseteq S$ is a number in $\mathbb{R}^+$ or ∞, defined by

$$\Delta\text{-}\mathsf{diam}(A) := \sup\{\Delta(a,b) \mid a,b\in A\}.$$

Every pseudometric Δ on a non-empty set S defines the *associated metric space* S/Δ with metric $\Delta^\bullet$ and the canonical surjection $\chi_\Delta\colon S \to S/\Delta$, as follows. The equivalence relation $\stackrel{\bullet}{\sim}$ on S is defined by $x \stackrel{\bullet}{\sim} y$ iff $\Delta(x,y) = 0$, for $x,y\in S$. The points of the metric space S/Δ are the equivalence classes of $\stackrel{\bullet}{\sim}$, and the metric $\Delta^\bullet$ is given by $\Delta^\bullet(x^\bullet,y^\bullet) := \Delta(x,y)$ where $x^\bullet, y^\bullet \in S/\Delta$, $x,y\in S$, and $x^\bullet$ and $y^\bullet$ are the equivalence classes of x and y, respectively. The mapping χ_Δ sends every $x\in S$ to its equivalence class.

A subset A of a Hausdorff topological space T is

- *relatively compact in T* iff the closure $\overline{A}$ of A in T is compact;

- *relatively sequentially compact in T* iff every sequence in A has a subsequence that converges in T;
- *relatively countably compact in T* iff every sequence in A has a subnet that converges in T.

A Hausdorff space T is *sequentially compact* iff it is relatively sequentially compact in itself, and T is *countably compact* iff it is relatively countably compact in itself.

A Hausdorff topological space T is *paracompact* iff every open cover of T has an open locally finite refinement. Equivalently, iff every open cover has an open σ-discrete refinement [109, 5.28]. Every metrizable topological space is paracompact.

Let S be a non-empty set, $\mathscr{W}$ a set of subsets of S and $\mathscr{F} \subseteq \mathbb{R}^S$. Then $\sigma(\mathscr{W})$ is the smallest σ-algebra Σ on S such that $\Sigma \supseteq \mathscr{W}$, and $\sigma(\mathscr{F})$ is the smallest σ-algebra on S for which all functions in $\mathscr{F}$ are measurable.

When T is a topological space, the *Borel σ-algebra* $\mathsf{Bo}(T)$ on T is the smallest σ-algebra of subsets of T that includes all open sets. The sets in $\mathsf{Bo}(T)$ are called *Borel sets*. A mapping φ from T to a topological space T' is *Borel measurable* iff $\varphi^{-1}(V) \in \mathsf{Bo}(T)$ for every open set $V \subseteq T'$. Clearly $\sigma(\mathsf{C}(T)) \subseteq \mathsf{Bo}(T)$ for every topological space T. If T is metrizable, then $\sigma(\mathsf{C}(T)) = \mathsf{Bo}(T)$.

For topological groups, I follow the standard terminology as defined, for example, by Császár [31]. All topological groups are assumed to be Hausdorff.

The binary operation in groups, and more generally in semigroups, is written as $(x,y) \mapsto xy$ or $(x,y) \mapsto x \cdot y$. When G is a group, its identity element is e_G, the inverse of $x \in G$ is x^{-1}, and if $A \subseteq G$, then $A^{-1} := \{x^{-1} \mid x \in A\}$.

A pseudometric Δ on a semigroup S is *right-invariant* if $\Delta(x,y) = \Delta(xz,yz)$ for all $x,y,z \in S$.

Theorem P.3. *Let G be a topological group. For every neighbourhood V of e_G there is a right-invariant pseudometric Δ on G such that the function $x \mapsto \Delta(e_G,x)$ is continuous on G and $\Delta(e_G,x) \geq 1$ for every $x \in G \setminus V$.*

Proof. The statement follows from [97, 8.2]. □

Category theory is not a prerequisite for the material presented in this treatise. However, the reader acquainted with categorical notions will recognize many well-behaved categories and functors. In fact, C, $\mathsf{C_b}$ and β as well as other operations defined further on are functors on familiar categories and as such act not only on objects but also on morphisms.

Moreover, a number of "forgetful" functors appear throughout, such as the obvious functor from topological groups to topological spaces or from topological spaces to sets. In most cases, when no confusion results, I use the same symbol for an object and for the same object with some structure "forgotten". For example, if G is a topological group, then G denotes also the induced topological space in expressions such as $\mathsf{C_b}(G)$ and βG. Similarly, when X is a topological space or a topological group, the symbol X denotes also the set of points of X in expressions such as $x \in X$ and $Y \subseteq X$. This convention causes no confusion, as long as we are careful with expressions that include equality; for example, if G and G' are two topological groups, then $G = G'$ means that G and G' are equal as topological groups, not merely as sets or groups or topological spaces.

P.3 Topological Vector Spaces

References: Fabian et al. [48], Schaefer [164].

All vector spaces considered in this treatise are over the field $\mathbb{R}$ of real numbers, and accordingly a *linear functional* on a vector space E is a linear mapping from E to $\mathbb{R}$. It is a simple exercise to extend the results proved here to vector spaces over the field of complex numbers.

All locally convex topological vector spaces are assumed to be Hausdorff. When E is a locally convex space, E^* is its topological dual.

When α is a seminorm on a vector space E, the *pseudometric of* α is the function $(x,y) \mapsto \alpha(x-y)$, $x,y \in E$. The topology of any locally convex space is defined by a set of seminorms [164, II.4] or, equivalently, by the corresponding set of pseudometrics. The pseudometric of α is a metric if and only if α is a norm.

When f is a real-valued function on a set S, define

$$\|f\|_A := \sup\left\{ |f(x)| \;\middle|\; x \in A \right\}$$

for $\emptyset \neq A \subseteq S$ and $\|f\|_\emptyset := 0$. Thus $\|f\|_A$ is a non-negative real number or ∞.

Lemma P.4. *Let E be a vector space and let $\mathscr{F}$ be a finite set of linear functionals on E. If g is a linear functional on E such that $\bigcap\{f^{-1}(0) \mid f \in \mathscr{F}\} \subseteq g^{-1}(0)$, then g is a linear combination of the functionals in $\mathscr{F}$.*

Proof. [164, IV.1.1]. □

Theorem P.5. *Let A be closed convex non-empty subset of a locally convex space E. For every $x_0 \in E \setminus A$ there exists $f \in E^*$ such that $\sup_{x \in A} f(x) < f(x_0)$.*

Proof. [164, II.9.2]. □

Corollary P.6. *Let E be a locally convex space. A vector subspace E_0 of E is dense in E if and only if every functional in E^* that is identically 0 on E_0 is identically 0 on E.* □

Let $\mathfrak{S}$ be a set of subsets of a non-empty set S that is upwards directed by inclusion $\subseteq$. For any $\mathscr{F} \subseteq \mathbb{R}^S$, the sets $\{f \in \mathscr{F} \mid \|f - f_0\|_A < \varepsilon\}$, where $\varepsilon > 0$, $f_0 \in \mathscr{F}$, and $A \in \mathfrak{S}$, form a base for a topology on $\mathscr{F}$, called the $\mathfrak{S}$-*topology* (or the *topology of uniform convergence on the sets in* $\mathfrak{S}$). The next theorem gives a sufficient condition for the $\mathfrak{S}$-topology to be locally convex.

Theorem P.7. *Let T be a topological space and let $\mathfrak{S}$ be a set of subsets of T that is upwards directed and whose union is dense in T. If F is a vector subspace of $\mathbb{R}^T$ whose elements are continuous on T and bounded on each set in $\mathfrak{S}$, then F with the $\mathfrak{S}$-topology is a locally convex space.*

Proof. [164, III.3.2]. □

Several instances of the $\mathfrak{S}$-topology have their own names. When $\mathfrak{S}$ is the set of all finite subsets of S, the $\mathfrak{S}$-topology is called the *S-pointwise topology*. When E and F are two vector spaces in duality, the *F-weak topology* on E is the F-pointwise topology obtained by identifying E with a subspace of $\mathbb{R}^F$. When T is a topological space and $\mathfrak{S}$ is the set of all compact subsets of T, the $\mathfrak{S}$-topology is called the *compact–open topology*.

When E and F are two vector spaces in duality and E is given the F-weak topology, the dual of E is F [164, IV.1.2]. In fact, the F-weak topology is the coarsest locally convex topology on E for which its dual is F. It follows from the next theorem that there exists also the finest locally convex topology on E with dual F.

Theorem P.8 (Mackey–Arens). *Let E and F be two vector spaces in duality, and identify each element of F with the linear functional it defines on E. Let $\mathfrak{S}$ be a set of absolutely convex E-weakly compact subsets of F such that $\mathfrak{S} \nearrow F$ and $rA \in \mathfrak{S}$ for any $A \in \mathfrak{S}$, $r \in \mathbb{R}^+$. When E is endowed with the $\mathfrak{S}$-topology, the dual of E is F.*

Proof. [164, IV.3.2]. □

A net $\{x_\gamma\}_\gamma$ in a locally convex space E is called *Cauchy* iff for every neighbourhood V of 0 in E there exists $y \in E$ such that $x_\gamma \in V + y$ for almost all γ. The space E is *complete* if every Cauchy net in E converges in E. This definition agrees with the more general notion of a complete uniform space, defined in Sect. 1.1. The space E is *sequentially complete* if every Cauchy sequence in E converges in E.

Theorem P.9 (Hahn–Banach). *Let E be a vector space and E_0 a vector subspace of E. Let α be a seminorm on E and f_0 a linear functional on E_0, and assume that $|f_0(x)| \le \alpha(x)$ for all $x \in E_0$. Then there exists a linear functional f on E that extends f_0 and such that $|f(x)| \le \alpha(x)$ for all $x \in E$.*

Proof. [164, II.3.2]. □

Corollary P.10. *If α is a seminorm on a vector space E, then*

$$\alpha(x) = \sup\{f(x) \mid f \text{ is a linear functional on } E \text{ and } |f(y)| \le \alpha(y) \text{ for all } y \in E\}$$

for every $x \in E$. □

A subset B of a locally convex space E is *bounded* iff for every neighbourhood V of 0 there exists $r \in \mathbb{R}$ such that $B \subseteq rV$. In the next theorem, the condition $\sup_{b \in B} |b(x)| < \infty$ for every $x \in E$ means that B is bounded in the E-weak topology.

Theorem P.11 (Banach–Steinhaus). *Let E be a Banach space with norm $\|\cdot\|$. If $B \subseteq E^*$ and $\sup_{b\in B}|b(x)| < \infty$ for every $x \in E$, then $\sup_{b\in B}\|b\| < \infty$.*

Proof. [48, 3.15]. □

Theorem P.12. *Let E be a complete locally convex space and f a linear functional on E^*. If the restriction of f to every equicontinuous subset of E^* is E-weakly continuous, then f is E-weakly continuous on E^*.*

Proof. This follows from Corollary 2 in [164, IV.6.2]. □

Theorem P.13. *Let E be a locally convex space, and let B be a bounded subset of E whose closed convex hull is complete. These properties of B are equivalent:*

(i) B is relatively E^*-weakly compact in E.
(ii) For every equicontinuous sequence $\{f_i\}_i$ in E^* and for every sequence $\{x_j\}_j$ in B, if the two double limits $\lim_i \lim_j f_i(x_j)$ and $\lim_j \lim_i f_i(x_j)$ exist, then they are equal.

Proof. [164, IV.11.2]. □

Theorem P.14 (Ascoli). *Let T be a compact space, and let $\mathscr{F}$ be a $\|\cdot\|_T$ bounded subset of $\mathsf{C_b}(T)$. The set $\mathscr{F}$ is relatively compact in $\mathsf{C_b}(T)$ with the $\|\cdot\|_T$ topology if and only if $\mathscr{F}$ is equicontinuous on T.*

Proof. [100, III.37]; [109, 7.17]. □

For any non-empty set S, the space $\ell_\infty(S)$ of all bounded real-valued functions on S with the vector operations defined pointwise and the sup norm $\|\cdot\|_S$ is a Banach space. The space $\ell_1(S)$ of the functions $d \in \mathbb{R}^S$ for which $\|d\|_1 := \sum_{x\in S} |d(x)|$ is finite, with the vector operations defined pointwise and the $\|\cdot\|_1$ norm, is also a Banach space, and $\ell_1(S)^* = \ell_\infty(S)$. Clearly every function in $\ell_1(S)$ vanishes outside of a countable subset of S. Write $\ell_1 := \ell_1(\omega)$ and $\ell_\infty := \ell_\infty(\omega)$. Since ℓ_∞ is the dual of ℓ_1, the ℓ_∞-weak topology is simply the weak topology of the Banach space ℓ_1.

Theorem P.15. *1. The space ℓ_1 is ℓ_∞-weakly sequentially complete.*
2. Every ℓ_∞-weakly convergent sequence in ℓ_1 converges in the $\|\cdot\|_1$ norm.
3. Every relatively ℓ_∞-weakly countably compact set in ℓ_1 is relatively $\|\cdot\|_1$ norm compact in ℓ_1.
4. The $\|\cdot\|_1$ unit ball in ℓ_1 is not metrizable in the ℓ_∞-weak topology.

Proof. Parts 1 and 2 are in [48, 5.19]. Part 3 follows from Part 2 and [48, 4.47]. Part 4 follows from [48, 3.28]. □

When Σ is a σ-algebra of subsets of a non-empty set S, $\ell_\infty(S,\Sigma)$ denotes the space of bounded real-valued Σ-measurable functions on S.

Definition P.16. Let Δ be a pseudometric on a non-empty set S and $f\colon S \to \mathbb{R}$. The function f is 1-*Lipschitz for* Δ, or Δ-1-*Lipschitz*, iff $|f(x) - f(y)| \le \Delta(x,y)$ for all $x, y \in S$. For any function $h\colon S \to \mathbb{R}^+$, define

$$\mathsf{BLip}(\varDelta,h) := \{f \in \mathbb{R}^S \mid |f| \le h \text{ and } f \text{ is 1-Lipschitz for } \varDelta\},$$
$$\mathsf{Lip}(\varDelta,h) := \bigcup_{j\in\omega} j\,\mathsf{BLip}(\varDelta,h).$$

Write $\mathsf{BLip}_\mathsf{b}(\varDelta) := \mathsf{BLip}(\varDelta,1)$ and $\mathsf{Lip}_\mathsf{b}(\varDelta) := \mathsf{Lip}(\varDelta,1)$. ■

The set $\mathsf{BLip}(\varDelta,h)$ is compact in the S-pointwise topology. Every $\mathsf{Lip}(\varDelta,h)$ is a Banach space with the norm that assigns

$$\inf\left\{r \in \mathbb{R}^+ \;\middle|\; |f| \le rh \text{ and } |f(x)-f(y)| \le r\varDelta(x,y) \text{ for all } x,y \in S\right\}$$

to $f \in \mathsf{Lip}(\varDelta,h)$, and $\mathsf{BLip}(\varDelta,h)$ is its closed unit ball. Weaver [178] surveys the theory of such *Lipschitz spaces*.

Lemma P.17. *Let $\varDelta$ be a pseudometric on a non-empty set S, and let $h\colon S \to \mathbb{R}^+$ be a $\varDelta$-1-Lipschitz function. Let $\chi_\varDelta : S \to S/\varDelta$ be the canonical surjection onto the associated metric space $S/\varDelta$ with metric $\varDelta^\bullet$. Then there is a unique $g\colon S/\varDelta \to \mathbb{R}^+$ such that $h = g \circ \chi_\varDelta$. The function g is $\varDelta^\bullet$-1-Lipschitz and*

$$\mathsf{BLip}(\varDelta,h) = \{f \circ \chi_\varDelta \mid f \in \mathsf{BLip}(\varDelta^\bullet, g)\},$$
$$\mathsf{Lip}(\varDelta,h) = \{f \circ \chi_\varDelta \mid f \in \mathsf{Lip}(\varDelta^\bullet, g)\},$$
$$\mathsf{BLip}_\mathsf{b}(\varDelta) = \{f \circ \chi_\varDelta \mid f \in \mathsf{BLip}_\mathsf{b}(\varDelta^\bullet)\},$$
$$\mathsf{Lip}_\mathsf{b}(\varDelta) = \{f \circ \chi_\varDelta \mid f \in \mathsf{Lip}_\mathsf{b}(\varDelta^\bullet)\}.$$

Proof. If $x,y \in S$, $x \overset{\bullet}{\sim} y$, then $|h(x)-h(y)| \le \varDelta(x,y) = 0$; hence there is a function $g\colon S/\varDelta \to \mathbb{R}^+$ such that $h = g \circ \chi_\varDelta$. The function g is unique because $\chi_\varDelta$ is surjective, and it is $\varDelta^\bullet$-1-Lipschitz because

$$|g(x^\bullet) - g(y^\bullet)| = |h(x)-h(y)| \le \varDelta(x,y) = \varDelta^\bullet(x^\bullet, y^\bullet).$$

The same argument proves the second statement in the lemma. □

P.4 Riesz Spaces

Reference: Fremlin [59, Ch.35].

When F is a partially ordered vector space, $H^+ := \{h \in H \mid h \ge 0\}$ for any $H \subseteq F$. The *positive cone* of F is F^+.

Definition P.18. Let F with the partial order $\le$ be a Riesz space (=vector lattice).

1. For $f \in F$, write $f^+ := f \vee 0$, $f^- := (-f) \vee 0$ and $|f| := f \vee -f$.

2. A vector subspace E of F is *solid in F* iff $g \in E$ whenever $g \in F$ and $|g| \leq |f|$ for some $f \in E$.
3. A vector subspace E of F is a *band in F* iff E is solid in F and $\sup H \in E$ whenever $H \subseteq E$ is a non-empty upwards directed set such that $\sup H$ exists in F.
4. The dual partial order $\leq$ on the space of real-valued linear functionals on F is defined by

$$\mathfrak{m} \leq \mathfrak{n} \text{ iff } \mathfrak{m}(f) \leq \mathfrak{n}(f) \text{ for every } f \in F^+,$$

for functionals $\mathfrak{m}$ and $\mathfrak{n}$.
5. Let $F^\sim$ denote the *order-bounded dual* of F with the dual partial order. ■

Lemma P.19. *If F is a Riesz space, then $F^\sim$ is a Riesz space. If $\mathfrak{m}, \mathfrak{n} \in F^\sim$ and $f \in F^+$, then*

$$\mathfrak{m}^+(f) = \sup\{\mathfrak{m}(g) \mid g \in F \text{ and } 0 \leq g \leq f\},$$
$$|\mathfrak{m}|(f) = \sup\{\mathfrak{m}(g) \mid g \in F \text{ and } |g| \leq f\},$$
$$\mathfrak{m} \vee \mathfrak{n} = \mathfrak{m} + (\mathfrak{n} - \mathfrak{m})^+.$$

Proof. The first two identities are in 356B of [59]. The last identity holds in every Riesz space [59, 352D]. □

Lemma P.20. *Let F be a Riesz space, $h \in F^+$, $\mathfrak{m} \in F^\sim$ and $\varepsilon > 0$. Then there exists $g \in F^+$ such that $\mathfrak{m}^+(f) < \mathfrak{m}(g \wedge f) + \varepsilon$ whenever $f \in F^+$, $f \leq h$.*

Proof. By Lemma P.19, there is $g \in F^+$, $g \leq h$, such that $\mathfrak{m}^+(h) < \mathfrak{m}(g) + \varepsilon$. Take any $f \in F^+$ such that $f \leq h$. Then $0 \leq g - g \wedge f \leq h - f$; therefore

$$\mathfrak{m}(g) - \mathfrak{m}(g \wedge f) = \mathfrak{m}(g - g \wedge f) \leq \mathfrak{m}^+(g - g \wedge f) \leq \mathfrak{m}^+(h) - \mathfrak{m}^+(f)$$

and $\mathfrak{m}^+(f) \leq \mathfrak{m}(g \wedge f) + \mathfrak{m}^+(h) - \mathfrak{m}(g) < \mathfrak{m}(g \wedge f) + \varepsilon$. □

Let S be a non-empty set. The space $\mathbb{R}^S$ with the vector operations and partial order defined pointwise at every $x \in S$ is a Riesz space. The space $\ell_\infty(S)$ is a Riesz subspace of $\mathbb{R}^S$; moreover, $\ell_\infty(S)$ is a Banach lattice with the norm $\|\cdot\|_S$.

The space $\ell_1(S)$ with the vector operations and partial order defined pointwise at every $x \in S$ is also a Riesz space, and it is a Banach lattice with the norm $\|\cdot\|_1$.

Theorem P.21. *Let S be a non-empty set and let F be a Riesz subspace of the Riesz space $\ell_\infty(S)$ containing constant functions.*

1. *The order-bounded dual $F^\sim$ of F is also its normed-space dual F^*. With the dual norm $\|\cdot\|$, $F^\sim$ is a Banach lattice.*
2. *$\|\mathfrak{m}\| = |\mathfrak{m}|(1)$ for every $\mathfrak{m} \in F^\sim$.*
3. *Any $\|\cdot\|$-closed subspace of $F^\sim$ that is solid in $F^\sim$ is a band in $F^\sim$.*

Proof. Parts 1 and 2 follow from 356N(a,b) in [59]. Part 3 follows from 356N(a), 354N and 354E(a) in [59]. □

Lemma P.22. *Let S be a non-empty set and let F be a Riesz subspace of the Riesz space $\ell_\infty(S)$ containing constant functions. If $\mathfrak{m} \in F^\sim$ and $\{\mathfrak{m}_\gamma\}_\gamma$ is a net in $F^\sim$ such that* $\lim_\gamma \mathfrak{m}_\gamma(f) = \mathfrak{m}(f)$ *for every $f \in F$ and* $\lim_\gamma \|\mathfrak{m}_\gamma\| \le \|\mathfrak{m}\|$, *then*

$$\lim_\gamma \mathfrak{m}_\gamma^+(f) = \mathfrak{m}^+(f)$$

$$\lim_\gamma \mathfrak{m}_\gamma^-(f) = \mathfrak{m}^-(f)$$

for every $f \in F$.

Proof. The proof is adapted from [19, 3.3]. Since the closed $\|\cdot\|$ balls in $F^\sim$ are F-weakly compact, it is enough to prove the conclusion when the nets $\{\mathfrak{m}_\gamma^+\}_\gamma$ and $\{\mathfrak{m}_\gamma^-\}_\gamma$ are F-weakly convergent. Let $\mathfrak{n}, \mathfrak{n}' \in F^\sim$ be such that $\lim_\gamma \mathfrak{m}_\gamma^+(f) = \mathfrak{n}(f)$ and $\lim_\gamma \mathfrak{m}_\gamma^-(f) = \mathfrak{n}'(f)$ for every $f \in F$.

Clearly $\mathfrak{m} = \mathfrak{m}^+ - \mathfrak{m}^- = \mathfrak{n} - \mathfrak{n}'$. By Lemma P.19, if $f \in F^+$, then

$$\mathfrak{m}^+(f) = \sup\left\{\lim_\gamma \mathfrak{m}_\gamma(g) \mid g \in F \text{ and } 0 \le g \le f\right\} \le \lim_\gamma \mathfrak{m}_\gamma^+(f) = \mathfrak{n}(f)$$

which means that also $\mathfrak{m}^-(f) \le \mathfrak{n}'(f)$. Then

$$\mathfrak{n}(1) + \mathfrak{n}'(1) = \lim_\gamma |\mathfrak{m}_\gamma|(1) = \lim_\gamma \|\mathfrak{m}_\gamma\| \le \|\mathfrak{m}\| = \mathfrak{m}^+(1) + \mathfrak{m}^-(1),$$

hence $\mathfrak{m}^+(1) = \mathfrak{n}(1)$ and $\mathfrak{m}^-(1) = \mathfrak{n}'(1)$.

Take any $f \in F$ such that $0 \le f \le 1$. Then

$$\mathfrak{n}(f) = \mathfrak{n}(1) - \mathfrak{n}(1-f) \le \mathfrak{m}^+(1) - \mathfrak{m}^+(1-f) = \mathfrak{m}^+(f) \le \mathfrak{n}(f),$$

so that $\mathfrak{n}(f) = \mathfrak{m}^+(f)$ for all such f and hence for all $f \in F$. □

P.5 Measures

References: Fremlin [60].

The basic reference for the results in this section is Volume 4 of Fremlin's Measure Theory [60], but I use a slightly modified notation and terminology. The main difference is that I use the term *measure* for Fremlin's *countably additive functional* on a σ-algebra (also known as a *finite signed measure*).

A *measure* is a σ-additive real-valued function on a σ-algebra Σ. For any measure $\mu : \Sigma \to \mathbb{R}$, its *positive part* μ^+, *negative part* μ^- and *total variation* $|\mu|$ are defined by