Miguel Sánchez
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Recent Trends in Lorentzian Geometry

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Recent Trends in Lorentzian Geometry

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Preface

Since the beginning of general relativity, Lorentzian geometry has provided the language and background for such a theory, as well as a mathematical arena where prospective extensions of Riemannian techniques could be sounded out. However, the variety and depth of its developments have consolidated Lorentzian geometry as a branch of differential geometry which is interesting by itself, as it provides applications to many parts of mathematical physics and yields an appealing framework where many mathematical techniques merge: geometric analysis, functional analysis, partial differential equations, Lie groups, and so on.

Ten years ago, a biennial series of meetings focused on Lorentzian Geometry was born in the town of Benalmádena (Spain). In the sixth edition, celebrated in Granada, September 2012, around 120 researchers of 18 countries gathered to discuss on the new trends of this geometry. In fact, the progress along this decade has attracted a renewed interest for many researchers: long-standing open problems have been solved, outstanding Lorentzian spaces and groups have been classified, new applications to mathematical relativity and high energy physics have been found, and further connections with other geometries have been developed.

In this volume, a sampler of the recent progress in Lorentzian geometry is presented. Topics such as geodesics, constant mean curvature submanifolds, trapped surfaces, gravitational collapse, classifications of manifolds with relevant symmetries, connections with Finsler geometry, and applications to mathematical physics are included. The contributions to this volume give a general perspective on these topics and provide new substantial results in some of them.

Let us give a very short overview of the contents.

The first five contributions constitute a block devoted to several problems on notable surfaces (maximal, constant mean curvature, umbilical, trapped) in Lorentzian manifolds. They are studied from different viewpoints, which include connections with other classical parts of differential geometry and mathematical relativity. More precisely, Fujimori, Kawakami, Kokubu, Rossman, Umehara, and Yamada introduce and develop an original notion of extended hyperbolic metric (i.e., a hyperbolic metric with a certain kind of singularities on a Riemann surface). Surfaces endowed with such metrics will be related to surfaces of constant mean curvature one in de Sitter space $\mathbb{S}^3_1$. This relation is developed specifically for catenoids in $\mathbb{S}^3_1$, and the classification of such catenoids will provide a classification of the corresponding moduli space of hyperbolic metrics. Albujer and Alías revisit both the classical Calabi–Bernstein theorem (i.e., the only entire maximal graphs in Lorentz-Minkowski space $\mathbb{L}^3$ are the space-like planes) and quite a few of its extensions. Very recent generalizations to a product spacetime $M^2 \times \mathbb{L}^1$ are specially considered. In particular, a local approach based on a parabolicity criterium is introduced so that a new proof to Calabi–Bernstein result is achieved. Senovilla focuses on umbilical space-like 2-surfaces in a Lorentzian manifold of dimension four. He introduces the notion of *ortho-umbilicity* and provides an original criterion to characterize total umbilicity in terms of the conmutativity of two independent Weingarten operators. Some consequences are analyzed, and extensions to arbitrary signatures and higher dimensions are also discussed. Mars focuses on marginally outer trapped surfaces (MOTS), which play an important role in gravitational theory as indicators of strong gravitational fields and, eventually, of black hole boundaries (*event horizons*). They share some of the properties of minimal hypersurfaces, in particular, the existence of a useful notion of *stability*. The implications of stability on the topology of MOTS, its interplay with spacetime symmetries, and, then, the stability of Killing horizons are carefully analyzed. As a further development, Jaramillo analyzes the existence of a set of inequalities involving the area, angular momentum, and charges of stably outermost marginally trapped surfaces in a generic spacetime under natural hypotheses. These inequalities provide lower bounds for the area of spatial sections that offers quasi-local models of black hole horizons. As in Mars contribution, the extension to a Lorentzian setting of tools employed in minimal surfaces in Riemannian contexts is emphasized.

The next three contributions are devoted to properties of geodesics and gravitational collapse. Caponio makes a thorough analysis of the notion of convexity for a hypersurface in a semi-Riemannian manifold. Classical Bishop's Riemannian result stating that infinitesimal convexity is equivalent to local convexity is reviewed, and its failure for the Lorentzian case is remarked. The analogous problem for the Finsler case has been solved very recently, and the author shows that the techniques for this more general case are also applicable to the semi-Riemannian problem. Applications to geodesic connectivity and further questions are also discussed. Variational methods and techniques of global analysis in manifolds are used by Bartolo, Candela, and Flores in order to study geodesics in spacetimes. After the successful results in the last two decades about the existence of connecting geodesics in causally well-behaved spacetimes, the authors focus on Gödel-type spacetimes. The results on this case are reviewed, and further improvements are obtained. Giambó and Magli analyze the geometry of isotropic fluids under gravitational collapse. Under general assumptions defining the fluid model, the null geodesics and causal structure, as well as the possible formation of horizons and nature of singularities, are discussed, with special attention to the case of bariotropic fluids obeying a linear equation of state.

The next block of three contributions is related to the recently developed connection between the class of (conformally) standard stationary spacetimes and the class of Finsler manifolds of Randers type. Javaloyes gives a general overview of such a stationary-to-Randers correspondence. This includes relations already developed at the levels of light-like geodesics, causality or causal boundaries on the stationary side, with different Finslerian elements (geodesics, convexity/completeness, and Busemann boundaries, respectively), as well as further prospective relations on topics such as isometry groups and curvature. In this framework, Matveev solves a natural question on arbitrary Finsler manifolds. More specifically, he characterizes when a Finsler metric F can be made complete by using a trivial projective change ($F \to F + df$). This question is inspired in a result that can be proved for the class of Randers metrics as a direct consequence of the stationary-to-Randers correspondence. Flores and Herrera contribution has several aims. Firstly, they review both the new redefinition of the classical *causal boundary* of a spacetime and the tools for its computation. These include, on the one hand, Penrose's conformal boundary and, on the other, connections with several boundaries in differential geometry (Cauchy, Gromov, Busemann), which have been developed for Finsler manifolds recently. Secondly, by using such tools, the causal boundary of the stationary part of Kerr spacetime is computed explicitly here.

The last four contributions study different aspects of symmetries of Lorentzian manifolds. Lichtenfelz, Piccione, and Zeghibs contribution provides a critical survey on some topics about the isometry group of a Lorentzian manifold. They revisit carefully the subtleties to endow this group with a Lie group structure. Then, recent results on (compact or not) Lie groups acting on a compact Lorentzian manifold are reviewed. Honda and Tsukada progress towards the local classification of conformally flat homogeneous Lorentzian manifolds. Such a complete classification is available in the Riemannian setting, but in the Lorentzian one has been obtained only in dimension three. As the Schouten tensor determines the curvature in this case, the authors focus on its algebraic structure and obtain the classification for all cases with nontrivial Jordan form, except when a triple root of the minimal polynomial exists. Díaz-Ramos gives an updated review on polar and hyperpolar actions on symmetric spaces, including the discussion of the differences between the compact and the noncompact cases. The study is carried out at the Riemannian level, and the Lorentzian one appears as a prospective challenge. Finally, Gilkey and Nikcevic, after surveying some known results in geometric realizability (including the semi-Riemannian and para-Hermitian settings), provide a new result on Kähler and para-Kähler Weyl structures. Specifically, a decomposition of the corresponding space of curvature tensors (which stresses the differences between dimension 4 and higher) is obtained. Then, every (para-)Kähler algebraic structure is shown to be geometrically realized by a (para-)Kähler manifold.

Summing up, these contributions, as a whole, provide a progress and an updated guide for many of the most interesting topics in present-day research on Lorentzian geometry. Thus, we would like to acknowledge the careful work of all the contributors, as well as of the anonymous referees. We would also like to thank

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Granada, Spain

Miguel Sánchez
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Hyperbolic Metrics on Riemann Surfaces and Space-Like CMC-1 Surfaces in de Sitter 3-Space

Shoichi Fujimori, Yu Kawakami, Masatoshi Kokubu, Wayne Rossman, Masaaki Umehara, and Kotaro Yamada

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Introduction

In this chapter, we give a new notion called the *extended hyperbolic metric* on a Riemann surface (the precise definition is given in Sect. 2), which is a canonical generalization of hyperbolic metrics (i.e. metrics of constant curvature -1): Connecting two hyperbolic planes by identifying their ideal boundaries, we get the *hyperbolic* 2*-sphere* S^2_H, that is, S^2_H is a Riemann sphere $(S^2 =)\mathbf{C}\cup\{\infty\}$ with the metric $d\sigma_P^2 := 4|dz|^2/(1-|z|^2)^2$, where $|dz|^2 := dzd\bar{z}$ and z is the canonical complex coordinate of $\mathbf{C}$. We call the metric $d\sigma_P^2$ on S^2_H the *spherical Poincaré metric*. An arbitrarily given extended hyperbolic metric on a Riemann surface can be realized as the pull-back metric of the spherical Poincaré metric $d\sigma_P^2$ on S^2_H by its developing map. Such an object has been discussed mainly as a projective structure with $\mathrm{SL}(2,\mathbf{R})$-monodromy (cf. Goldman [10] and Gallo–Kapovich–Marden [8]) in the study of Teichmüller spaces, but it seems that singularities of such metrics have not been precisely examined yet. (It should be remarked that $\mathrm{SL}(2,\mathbf{R})$ is conjugate to $\mathrm{SU}(1,1)$ in $\mathrm{SL}(2,\mathbf{C})$.) In fact, an extended hyperbolic metric $d\sigma^2$ may have singularities called *ideal boundary points*, denoted also as ∂^∞-points, where any path accumulating to that ideal boundary point has infinite length, and also may have singularities called *proper singularities*, which are isolated singular points of the Schwarzian derivative of $d\sigma^2$ (cf. Sect. 2 and Appendix A). In Sect. 3, we give several important properties of extended hyperbolic metrics.

It is known that constant mean curvature one surfaces (resp. flat surfaces) in hyperbolic 3-space are closely related to spherical metrics (resp. flat metrics) on Riemann surfaces (cf. [18, 19] for spherical metrics, and [12, Theorem 4.4] for flat metrics). Similarly, extended hyperbolic metrics bijectively correspond to space-like surfaces of constant mean curvature one (CMC-1) in de Sitter 3-space S^3_1 with a given hyperbolic Gauss map (cf. Theorems 2.10 and 2.11). In other words, CMC-1 surfaces in de Sitter 3-space S^3_1 can be considered as geometric realizations of extended hyperbolic metrics. In fact, the singular set of a given CMC-1 surface in S^3_1 coincides with that of the associated co-orientable extended hyperbolic metric. Several important examples of CMC-1 surface in S^3_1 are given in Lee-Yang [14] and [1–3,5,6]. In Sect. 4, we classify S^3_1-catenoids, i.e. weakly complete CMC-1 faces in S^3_1 of genus zero with two regular ends whose hyperbolic Gauss map is of degree one.

In Sect. 5, we classify extended hyperbolic metrics with at most two regular singularities on S^2, using the above correspondence, where a proper singularity is called a *regular singularity* if the Schwarzian derivative of the metric has at most a pole of order 2 (cf. Sect. 2).

1 Generalized CMC-1 Faces in de Sitter 3-Space

First, we recall some fundamental facts about CMC-1 faces in de Sitter 3-space. For detailed expressions, see [1, 5].

Generalized CMC-1 Faces

Let $\mathbf{R}^4_1$ be the Lorentz-Minkowski 4-space with the metric $\langle\ ,\ \rangle$ of signature $(-,+,+,+)$. Then de Sitter 3-space is expressed as

$$S^3_1 = \left\{X \in \mathbf{R}^4_1\,;\, \langle X, X\rangle = 1\right\}$$

with metric induced from $\mathbf{R}^4_1$, which is a simply-connected Lorentzian 3-manifold with constant sectional curvature 1. We identify $\mathbf{R}^4_1$ with the set of 2×2 Hermitian matrices Herm(2) by

$$\mathbf{R}^4_1 \ni (x_0, x_1, x_2, x_3) \longleftrightarrow \begin{pmatrix} x_0 + x_3 & x_1 + \mathrm{i}x_2 \\ x_1 - \mathrm{i}x_2 & x_0 - x_3 \end{pmatrix} \in \mathrm{Herm}(2), \tag{1}$$

where $\mathrm{i} = \sqrt{-1}$. Then de Sitter 3-space is represented as

$$\begin{aligned} S^3_1 &= \{X \in \mathrm{Herm}(2)\,;\, \det X = -1\} \\ &= \{ae_3a^*\,;\, a \in \mathrm{SL}(2,\mathbf{C})\} = \mathrm{SL}(2,\mathbf{C})/\,\mathrm{SU}(1,1) \qquad \left(e_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right), \end{aligned} \tag{2}$$

where $a^* := {}^t\overline{a}$ is the transposed conjugate matrix of a, and

$$\mathrm{SU}(1,1) := \{a \in \mathrm{SL}(2,\mathbf{C})\,;\, ae_3a^* = e_3\}.$$

We consider the projection

$$\pi_S : \mathrm{SL}(2,\mathbf{C}) \ni a \longmapsto ae_3a^* \in S^3_1 = \mathrm{SL}(2,\mathbf{C})/\,\mathrm{SU}(1,1). \tag{3}$$

The group $\mathrm{SL}(2,\mathbf{C})$ acts isometrically on S^3_1 as

$$S^3_1 \ni X \longmapsto aXa^* \in S^3_1 \qquad \big(a \in \mathrm{SL}(2,\mathbf{C})\big). \tag{4}$$

In fact, $\mathrm{PSL}(2,\mathbf{C}) = \mathrm{SL}(2,\mathbf{C})/\{\pm\text{identity matrix}\}$ can be considered as the connected component of the identity of the isometry group of S^3_1.

[1] introduced the notion of CMC-1 faces in S^3_1, which corresponds to maxfaces (i.e., maximal surfaces with admissible singularities, see [20]) in the Lorentz-Minkowski 3-space $\mathbf{R}^3_1$. As a generalization of CMC-1 faces, we define generalized CMC-1 faces as follows. (Later, we show that an extended hyperbolic metric induces a generalized CMC-1 face with a given hyperbolic Gauss map, see Theorems 2.10 and 2.11.) We fix a Riemann surface M.

Definition 1.1. A C^∞-map $f : M \to S^3_1$ is called a *generalized CMC-1 face* if there exists an open dense subset W of M such that the restriction $f|_W$ of f on W gives a conformal (space like) immersion of constant mean curvature one.

A *singular point* of f is a point at which f is not an immersion. A singular point p satisfying $df(p) = 0$ is called a *branch point* of f. Moreover, f is called a *CMC*-1 *face* if f does not have any branch points. (A CMC-1 face may have singular points in general).

Remark 1.2. The above definition of CMC-1 face is simpler than the definition given in [1,5]. However, as seen in the following Proposition 1.4, the new definition is equivalent to the previous one. Similarly, the definition of 'maxface' given in [20] can be simplified as follows: *A C^∞-map $f : M \to \mathbf{R}^3_1$ is a maxface if and only if there exists an open dense subset W of M such that the restriction $f|_W$ of f on W gives a conformal (space like) maximal immersion and df has no zeros on M.* The proof is easier than for the case of CMC-1 faces.

To state the Weierstrass-type representation formula, we prepare some notions:

Definition 1.3. A pair (G,Q) of a meromorphic function G and a holomorphic 2-differential Q on M is said to be *definite* (resp. *semi-definite*) if

$$ds^2_{\#} = \left(1+|G|^2\right)^2 \left|\frac{Q}{dG}\right|^2 \tag{5}$$

is a positive definite (resp. positive semi-definite) metric on M.

We denote by $\widetilde{M}$ the universal cover of M and by

$$\pi : \widetilde{M} \longrightarrow M$$

its covering projection.

Proposition 1.4. *Let (G,Q) be a semi-definite pair on M and let $F = (F_{ij}) \colon \widetilde{M} \to \mathrm{SL}(2,\mathbf{C})$ be a holomorphic map of $\widetilde{M}$ such that*

$$(dF)F^{-1} = \Psi \qquad \left(\Psi := \begin{pmatrix} G & -G^2 \\ 1 & -G \end{pmatrix} \frac{Q}{dG}\right). \tag{6}$$

Then F is a null holomorphic map, that is, F is a holomorphic map such that $\det(dF/dz)$ vanishes identically for each local complex coordinate z on M. And

$$f := Fe_3F^* \colon \widetilde{M} \longrightarrow S^3_1 \tag{7}$$

is a generalized CMC-1 face if $|g|$ is not identically 1, where g is a meromorphic function on $\widetilde{M}$ (called the secondary Gauss map) defined by

$$g := -\frac{dF_{12}}{dF_{11}} = -\frac{dF_{22}}{dF_{21}}. \tag{8}$$

The induced metric ds^2 and the second fundamental form $\mathrm{I\!I}$ *are expressed as*

$$ds^2 = (1-|g|^2)^2 \left|\frac{Q}{dg}\right|^2, \qquad \mathrm{I\!I} = Q + \overline{Q} + ds^2, \tag{9}$$

respectively. (The meromorphic function G and the holomorphic 2-differential Q are called the hyperbolic Gauss map and the Hopf differential, respectively.)

Moreover, f is a CMC-1 face if and only if F is an immersion, that is, (G,Q) is a definite pair on M.

Conversely, any non-totally-umbilical generalized CMC-1 face defined on $\widetilde{M}$ with hyperbolic Gauss map G and Hopf differential Q is obtained in this manner. (Later, we give a necessary and sufficient condition for f to be single-valued on M; see Proposition 1.9.)

Proof. When the pair (G,Q) is definite, the assertion has been proved in [6, Proposition 4.2]. So we assume (G,Q) is semi-definite. Since the solution F of the ordinary differential Equation (6) is analytic, the local existence of F implies the existence of the solution on $\widetilde{M}$. Thus, it is sufficient to show that Ψ (as in Eq. (6)) is a holomorphic matrix-valued 1-form on M. In fact, under the assumption that Ψ is holomorphic, one can directly check that $f = Fe_3F^*$ is a conformal CMC-1 immersion at p if $|g(p)| \neq 1$ and $ds^2_\#$ as in (5) is positive definite at p (cf. [6, Proposition 4.2]).

We fix a point $p \in M$ arbitrarily. If G is holomorphic at p, then the boundedness of $ds^2_\#$ at p implies that Q/dG is holomorphic at p, and so is Ψ. So we consider the case that G has a pole at p. Then $\hat{G} = 1/G$ is holomorphic at p. Since

$$ds^2_\# = (1+|\hat{G}|^2)^2 \left|\frac{Q}{d\hat{G}}\right|^2$$

holds, we can conclude that $Q/d\hat{G}$ is holomorphic at p. Moreover, we have the following expression

$$\Psi = \begin{pmatrix} -\hat{G} & 1 \\ -\hat{G}^2 & \hat{G} \end{pmatrix} \frac{Q}{d\hat{G}},$$

which implies that Ψ is holomorphic at p also in this case. Here, we have shown that Ψ is holomorphic even when (G,Q) is semi-definite. Moreover, it holds that

$$(\partial f)f^{-1} = \partial(Fe_3F^*)((F^*)^{-1}e_3F^{-1}) = (dF)F^{-1} = \Psi, \tag{10}$$

which implies that f is a CMC-1 face if and only if (G,Q) is a definite pair on M, unless Q vanishes identically making f totally umbilical, where we consider S^3_1 a set of 2×2 matrices as in Eq. (2) and f a matrix-valued function. Now, one can prove all of the remaining assertions except the converse part by imitating the proof of [6, Proposition 4.2].

So we now prove the converse assertion: Let $f: M \to S^3_1$ be a generalized CMC-1 face. By definition, there exists an open dense subset W of M such that $f|_W$ is a conformal space-like CMC-1 immersion. It is well-known that f can be lifted to a null holomorphic immersion $F: \widetilde{W} \to \mathrm{SL}(2,\mathbf{C})$, where $\widetilde{W}$ is the universal covering of W. By Eq. (10), the identity

$$(\partial f)f^{-1} = (dF)F^{-1}$$

holds on W. Since W is dense, the $\mathrm{SL}(2,\mathbf{C})$-valued 1-form $(\partial f)f^{-1}$ is holomorphic on M. Then there exists a holomorphic map $\widetilde{F}: \widetilde{M} \to \mathrm{SL}(2,\mathbf{C})$ such that $(d\widetilde{F})\widetilde{F}^{-1}$ is equal to $(\partial f)f^{-1}$. Since df vanishes if ∂f does as well, $\widetilde{F}$ is an immersion if and only if df never vanishes on M. □

A generalized CMC-1 face is totally umbilical if and only if its image lies in an S^3_1-horosphere (cf. [5]). To avoid this exceptional case, we assume that the Hopf differential Q of the generalized CMC-1 face does not vanish identically, in Sect. 1–3.

Definition 1.5. For a generalized CMC-1 face f obtained as in Proposition 1.4, the null holomorphic map F in Eq. (7) is called a *null holomorphic lift* of f. The metric $ds^2_{\#}$ as in Eq. (5) is called the *dual metric* of f.

Remark 1.6. Let f be a generalized CMC-1 face obtained from given (G,Q) using Proposition 1.4. Then

1. The Eq. (6) should be regarded as an equation on the universal cover $\widetilde{M}$ (see the appendix in [6]). However, for simplicity, we use the notation here.
2. For each $a \in \mathrm{SL}(2,\mathbf{C})$, $f_a := afa^*$ gives a generalized CMC-1 face which is congruent to f [cf. Eq. (4)]. For a null holomorphic lift F of f, $F_a := aF$ is a null holomorphic lift of f_a. In particular, the hyperbolic Gauss map and the Hopf differential of f_a are $a \star G$ and Q, respectively, where

$$a \star G := \frac{a_{11}G + a_{12}}{a_{21}G + a_{22}} \qquad \bigl(a = (a_{ij})\bigr)$$

denotes the Möbius transformation (cf. [18]).
3. The choice of a null holomorphic lift F of f has the ambiguity $F \mapsto Fb^{-1}$ for $b \in \mathrm{SU}(1,1)$. Under this change, the secondary Gauss map g is transformed as

$$b \star g = \frac{b_{11}g + b_{12}}{b_{21}g + b_{22}} \qquad \bigl(b = (b_{ij})\bigr).$$

4. If $f: M \to S^3_1$ is a generalized CMC-1 face, the singular set of f is given by

$$\Sigma_f := \{\pi(p) \in M\,;\, |g(p)| = 1\ (p \in \widetilde{M})\} \cup \{q \in M\,;\, df(q) = 0\}. \qquad (11)$$

The condition in Proposition 1.4 that $|g|$ is not identical to 1 is necessary to avoid the example all of whose points are singular points of f. Such an example is unique up to isometry, whose image is a light-like line in S^3_1 (see [5, Remark 1.3]).

5. The metric

$$d\sigma^2 := \frac{4\,dg\,d\bar{g}}{(1-|g|^2)^2}$$

has the expression (cf. [5, Remark 1.10])

$$d\sigma^2 := K ds^2 \quad (K\text{: Gaussian curvature of } ds^2),$$

and has constant Gaussian curvature -1. It is the pull-back of the spherical Poincaré metric by $g : \widetilde{M} \to \mathbf{C} \cup \{\infty\}$ (cf. the Introduction and Sect. 2). In other words, $d\sigma^2$ is an extended hyperbolic metric whose developing map is g (see Sect. 3 for details). The identity

$$d\sigma^2 ds^2 = 4Q\overline{Q} = d\sigma^2_{\#} ds^2_{\#} \tag{12}$$

holds, where

$$d\sigma^2_{\#} := \frac{4\,dG\,d\overline{G}}{(1+|G|^2)^2}. \tag{13}$$

These holomorphic data are related by

$$S(g) - S(G) = 2Q, \tag{14}$$

where

$$S(h) := S_z(h) dz^2, \qquad S_z(h) := \left(\frac{h''}{h'}\right)' - \frac{1}{2}\left(\frac{h''}{h'}\right)^2 \qquad \left({}' = \frac{d}{dz}\right). \tag{15}$$

Here z is a local complex coordinate on M and $S(\cdot)$ is the Schwarzian derivative. The Schwarzian derivative has the property that $S(a \star h) = S(h)$ for each $a \in \mathrm{SL}(2,\mathbf{C})$. The difference $S(g) - S(G)$ of the Schwarzian derivatives of two meromorphic functions g and G on a given Riemann surface M does not depend on a choice of local complex coordinate z. Fundamental properties of the Schwarzian derivative are given in the appendix of [17].

On the other hand, the null holomorphic lift $\pm F$ can be expressed using G and g by

$$F = \begin{pmatrix} G\dfrac{da}{dG} - a & G\dfrac{db}{dG} - b \\ \dfrac{da}{dG} & \dfrac{db}{dG} \end{pmatrix}, \quad a := \sqrt{\frac{dG}{dg}}, \quad b = -ga, \tag{16}$$

which is called the *Bianchi-Small formula*. (Although a (and so F also) has $\pm$-ambiguity, the CMC-1 face $f = \pi_S \circ F = Fe_3F^*$ is uniquely determined.)

Proposition 1.7. *The set of zeros of the Hopf differential Q of a given generalized CMC-1 face $f : M \to S^3_1$ coincides exactly with the union of the set of umbilics and the set of branch points of f.*

Proof. Let p be a fixed point of M. By a suitable motion of S^3_1, we may assume that the hyperbolic Gauss map G is holomorphic at p. Then, p is a branch point of f if and only if Q/dG vanishes at p, in particular $Q(p) = 0$. It is well-known that p is an umbilic point of f if and only if Q vanishes and Q/dG does not vanish at p. So we get the assertion. □

Here, we give the following additional result on branch points.

Proposition 1.8. *Let $f : M \to S^3_1$ be a generalized CMC-1 face, then the following three conditions are equivalent:*

1. *$p \in M$ is a branch point of f.*
2. *$F : \widetilde{M} \to \mathrm{SL}(2, \mathbf{C})$ is not an immersion at p.*
3. *$ds^2_\#$ degenerates at p.*

Proof. The equivalency of the first two conditions is obvious. Since $ds^2_\#$ is the Hermitian metric induced by the inverse matrix F^{-1}, so the assertion follows from the fact that F^{-1} is an immersion if and only if F is as well. □

The Monodromy Representation of Generalized CMC-1 Faces

For a semi-definite pair (G, Q) on M, there exists a representation $\rho_F : \pi_1(M) \to \mathrm{SL}(2, \mathbf{C})$ associated to the solution F of Eq. (6) as in the appendix in [6]:

$$F \circ T = F\rho_F(T), \tag{17}$$

where $T \in \pi_1(M)$ is an element of the fundamental group considered as a covering transformation of the universal cover $\widetilde{M}$. Let $f : \widetilde{M} \to S^3_1$ be a generalized CMC-1 face and $g : \widetilde{M} \to \mathbf{C} \cup \{\infty\}$ the secondary Gauss map of f as in Eq. (8). By Eq. (14), $S(g)$ is a projective connection on M (cf. Appendix A), since (G, Q) are defined on M. As a consequence, there exists a group representation

$$\rho_g : \pi_1(M) \longrightarrow \mathrm{PSL}(2, \mathbf{C}) := \mathrm{SL}(2, \mathbf{C})/\{\pm e_0\}, \qquad e_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

such that

$$g \circ T^{-1} = \rho_g(T) \star g, \tag{18}$$

where $T \in \pi_1(M)$ is a covering transformation. Since $a \star g = (-a) \star g$ for $a \in \mathrm{SL}(2, \mathbf{C})$, $a \star g$ is well-defined for $\mathrm{PSL}(2, \mathbf{C})$. Let

$$\hat{\pi} : \mathrm{SL}(2, \mathbf{C}) \longrightarrow \mathrm{PSL}(2, \mathbf{C}) \tag{19}$$

be the double covering as a group homomorphism. Then it holds that

$$\hat{\pi} \circ \rho_F = \rho_g \tag{20}$$

(see [5, (1.12) and (1.13)]). One can easily prove the following criteria for a given generalized CMC-1 face $f : \widetilde{M} \to S^3_1$ as in Proposition 1.4 to be single-valued on M:

Proposition 1.9. *Let $f : \widetilde{M} \to S^3_1$ be a generalized CMC-1 face associated to a semi-definite pair (G, Q) on M. Then the following conditions are mutually equivalent:*

1. *f is single-valued on M.*
2. *$\rho_F(\pi_1(M)) \subset \mathrm{SU}(1,1)$.*
3. *$\rho_g(\pi_1(M)) \subset \mathrm{PSU}(1,1) := \mathrm{SU}(1,1)/\{\pm e_0\}$.*

Completeness and Reducibility of CMC-1 Faces

We define completeness and weak completeness of CMC-1 faces.

Definition 1.10 ([5, Definitions 1.2 and 1.3]). A CMC-1 face $f : M \to S^3_1$ is called *complete* if there exists a symmetric 2-tensor T which vanishes outside some compact set in M, such that $ds^2 + T$ is a complete Riemannian metric on M, where ds^2 is the induced metric by f as in Eq. (9). On the other hand, f is called *weakly complete* if the metric $ds^2_\#$ in Eq. (5) is complete.

The following assertion is useful:

Fact 1.11 ([21, 22] and Okuyama–Yamanoi [15]). *Let $f : M \to S^3_1$ be a weakly complete CMC-1 face. Then f is complete if and only if there exist a closed Riemann surface $\overline{M}$ and a finite number of points $p_1, \dots, p_n \in \overline{M}$ such that M is conformally equivalent to $\overline{M} \setminus \{p_1, \dots, p_n\}$, and the set of singular points Σ_f is compact.*

Remark 1.12. This fact was proved in [22] under the assumption that the Hopf differential of f is meromorphic. The last two authors pointed out in [21, 22] that the assumption of the Hopf differential can be removed if one can establish a certain generalization of the completeness lemma in minimal surface theory. Recently, Okuyama-Yamanoi [15] accomplished this, and as a consequence Fact 1.11 is obtained.

Let f be a complete CMC-1 face. Each point p_j is called an *end* of f. A complete end p_j is said to be *regular* if the hyperbolic Gauss map G has at most a pole at p_j, which is equivalent to the order of the Hopf differential Q at p_j being at least -2.

Next, we recall the reducibility of CMC-1 faces:

Definition 1.13. A CMC-1 face f is called *irreducible* (resp. *reducible*) if the image of the representation ρ_g is not (resp. is) an abelian subgroup of $\mathrm{PSU}(1,1)$. When f is reducible, it is called 3-*reducible* if ρ_g is a trivial representation (i.e., the image of ρ_g coincides with $\{\pm e_0\}$), and f is called 1-*reducible* if the image of ρ_g is abelian and not equal to $\{\pm e_0\}$. In particular, ρ_g is 3-reducible if and only if g is single-valued on M.

In the case of CMC-1 surfaces in the hyperbolic 3-space H^3, 1-reducible (resp. 3-reducible) corresponds to the terminology H^1-reducible (resp. H^3-reducible). If f is a 1-reducible (resp. 3-reducible) CMC-1 surface in S^3_1, then f has a 1-parameter family (resp. 3-parameter family) of deformations of f preserving the hyperbolic Gauss map G and the Hopf differential Q. The numbers 1 and 3 for reducibility come from the numbers of these freedoms, as follows:

Theorem 1.14. *Let $f : M \to S^3_1$ be a CMC-1 face with a given hyperbolic Gauss map G and Hopf differential Q. Then f is uniquely determined if f is irreducible. On the other hand, if f is 3-reducible (resp. 1-reducible), then there is a 3-parameter family (resp. 1-parameter family) of CMC-1 faces (as mappings of M into S^3_1) having the same hyperbolic Gauss map G and Hopf differential Q as f.*

Proof. If we replace F by Fa for $a \in \mathrm{SL}(2,\mathbf{C})$, then f changes to $Fae_3a^*F^*$, and this preserves G and Q. On the other hand, a CMC-1 face with the same hyperbolic Gauss map G and Hopf differential Q as f is of the form $Fae_3a^*F^*$ for some $a \in \mathrm{SL}(2,\mathbf{C})$. Furthermore, $Fae_3a^*F^*$ is single-valued on M if and only if the monodromy matrix of Fa belongs to $\mathrm{SU}(1,1)$. Since

$$(Fa)\circ T = F\rho_F(T)a = (Fa)(a^{-1}\rho_F(T)a),$$

$Fae_3a^*F^*$ is single-valued on M if and only if $\hat{\pi}(a) \in \mathrm{PSL}(2,\mathbf{C})$ belongs to C_Γ, where

$$C_\Gamma := \{\sigma \in \mathrm{PSL}(2,\mathbf{C})\,;\, \sigma\Gamma\sigma^{-1} \in \mathrm{PSU}(1,1)\} \qquad \big(\Gamma := \rho_g(\pi_1(M))\big).$$

If $\sigma \in C_\Gamma$, it is obvious that $a\sigma \in C_\Gamma$ for all $a \in \mathrm{PSU}(1,1)$. Then the left quotient space $I_\Gamma := \mathrm{PSU}(1,1)\backslash C_\Gamma$ can be considered as a subset of S^3_1 which parametrizes the CMC-1 faces with given (G,Q). As shown in Appendix B, I_Γ is a point if f is irreducible. I_Γ coincides with S^3_1 if f is 3-reducible, and is a geodesic line of S^3_1 if f is 1-reducible, which proves the assertion. □

Remark 1.15. The deformation of CMC-1 faces preserving (G,Q) as in Theorem 1.14 is not an isometric deformation in general. However, it gives the same image of a CMC-1 face in special cases. For example, if $f : \mathbf{C} \to S^3_1$ is an S^3_1-horosphere [cf. Eq. (29)], then it is 3-reducible, since $\mathbf{C}$ is simply connected. In this case, $f = Fe_3F^*$ and $Fae_3a^*F^*$ $(a \in \mathrm{SL}(2,\mathbf{C}))$ are both totally umbilical, and thus they are congruent. Similarly, an S^3_1-catenoid $f : \mathbf{C}\setminus\{0\} \to S^3_1$ is reducible which admits a deformation which fixes the image of the surface (see Sect. 4).

For any real number t and $\varepsilon \in \{1,-1\}$, we set

$$E(t) := \begin{pmatrix} e^{\mathrm{i}t} & 0 \\ 0 & e^{-\mathrm{i}t} \end{pmatrix}, \quad P(\varepsilon) := \begin{pmatrix} 1+\mathrm{i}\varepsilon & -\mathrm{i}\varepsilon \\ \mathrm{i}\varepsilon & 1-\mathrm{i}\varepsilon \end{pmatrix}, \quad H(t) := \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}.$$

A matrix in $\mathrm{SU}(1,1)$ is called

- *Elliptic* if it is conjugate to $E(t)$ for some $t \in (-\pi,\pi]$ in $\mathrm{SU}(1,1)$.
- *Parabolic* if it is conjugate to $\pm P(\pm 1)$ in $\mathrm{SU}(1,1)$.
- *Hyperbolic* if it is conjugate to $\pm H(t)$ for some $t > 0$ in $\mathrm{SU}(1,1)$.

Any matrix in $\mathrm{SU}(1,1)$ is one of these three types (see the appendix in [6]). Note that the parabolic matrices $P(1)$ and $P(-1)$ are not conjugate in $\mathrm{SU}(1,1)$. Since a and $-a$ have the same properties for each $a \in \mathrm{SU}(1,1)$, the ellipticity, the parabolicity, and the hyperbolicity are also well-defined for each element of $\mathrm{PSU}(1,1)$.

Singularities

Let $f\colon M \to S^3_1$ be a CMC-1 face whose secondary Gauss map and Hopf differential are g and Q, respectively.

Fact 1.16 ([4, Theorem 3.4], [6, Lemma 2.4]). *Define two meromorphic functions by*

$$\alpha := \frac{dg}{g^2\omega}, \qquad \beta := g\frac{d\alpha}{dg} \qquad \left(\omega = \frac{Q}{dg}\right).$$

Then

- *$p \in M$ is a singular point if and only if $|g(p)| = 1$. Moreover, p is a nondegenerate singular point if and only if $dg(p) \neq 0$.*
- *f gives a front (i.e., wave front; see [12] for the definition of front) on a neighborhood of p if and only if $\mathrm{Re}\,\alpha \neq 0$ at p, where $\mathrm{Re}\,\alpha$ denotes the real part of α.*
- *p is a cuspidal edge if and only if $\mathrm{Re}\,\alpha \neq 0$ and $\mathrm{Im}\,\alpha \neq 0$ hold at p, where $\mathrm{Im}\,\alpha$ denotes the imaginary part of α.*
- *p is a swallowtail if and only if $\mathrm{Re}\,\alpha \neq 0$, $\mathrm{Im}\,\alpha = 0$ and $\mathrm{Re}\,\beta \neq 0$ hold at p.*
- *p is a cuspidal cross cap if and only if $\mathrm{Re}\,\alpha = 0$, $\mathrm{Im}\,\alpha \neq 0$ and $\mathrm{Im}\,\beta \neq 0$ hold at p.*
- *The singular set Σ_f consisting of nondegenerate singular points is a cone-like singularity if and only if Σ_f is compact and $\mathrm{Im}\,\alpha = 0$ holds on Σ_f.*

Remark 1.17. Though Lemma 2.4 in [6] gives a criteria for cone-like singularities of maximal surfaces in Lorentz-Minkowski 3-space $\mathbf{R}^3_1$, one can easily show that it is also a criteria for CMC-1 faces.

As shown in [4], cuspidal edges, swallowtails and cuspidal cross caps are the generic singularities of CMC-1 faces in the compact open C^∞-topology.

HMC-1 Surfaces as Unit Normal Vector Fields of CMC-1 Faces

At the end of this section, we discuss the behavior of unit normal vector fields of CMC-1 faces. Let $f : M \to S^3_1$ be a CMC-1 face and $F : \widetilde{M} \to \mathrm{SL}(2, \mathbf{C})$ its null holomorphic lift. Then the unit normal vector field ν has the following expression (see [1, Remark 1.2]):

$$\nu := \frac{1}{|g|^2 - 1} F \begin{pmatrix} 1 + |g|^2 & 2g \\ 2\bar{g} & 1 + |g|^2 \end{pmatrix} F^* : M \setminus \Sigma_f \longrightarrow H^3_+ \cup H^3_-, \tag{21}$$

where g is the secondary Gauss map of f, Σ_f is the set of singular points, and

$$H^3_\pm := \{X = (x_0, x_1, x_2, x_3) \in \mathbf{R}^4_1 \,;\, \langle X, X \rangle = -1, \pm x_0 > 0\}$$

are the two components of a two-sheeted hyperboloid in $\mathbf{R}^4_1$. As pointed out in [11], when the unit normal vector field ν of a CMC-1 face f meets the singular set $\Sigma_f = \{|g| = 1\}$, the image of ν moves into the other sheet of the hyperboloid $H^3_+ \cup H^3_-$. Moreover, it was shown in [11, Theorem 4.2] that ν is smooth at the singular set under a certain compactification of $H^3_+ \cup H^3_-$ as follows: The *hyperbolic* 3-*sphere* S^3_H is a 3-dimensional manifold diffeomorphic to the 3-sphere:

$$S^3_H := \mathbf{R}^3 \cup \{\infty\} \cong S^3$$

endowed with the metric $4|dx|^2/(1 - |x|^2)^2$ on $S^3 \setminus \{\text{the equator}\}$, where $x := (x_1, x_2, x_3) \in \mathbf{R}^3 \cup \{\infty\}$. We consider the stereographic projection

$$\varphi : H^3_+ \cup H^3_- \ni (x_0, x_1, x_2, x_3) \longmapsto \frac{(x_1, x_2, x_3)}{1 - x_0} \in S^3_H, \tag{22}$$

which is an isometric embedding, and S^3_H can be considered as a compactification of $H^3_+ \cup H^3_-$. Thus, the unit normal vector field as in Eq. (21) induces a smooth map:

$$\nu : M \longrightarrow S^3_H.$$

We now give the following definition:

Definition 1.18. A C^∞-map $\nu : M \to S^3_H$ is called an *HMC*-1 *face* (i.e. harmonic-mean curvature 1 face) if it is a unit normal vector field of a CMC-1 face $f : M \to S^3_1$.

As pointed out in [11], such a ν actually has the property that the harmonic mean

$$HM := \left(\frac{(\lambda_1)^{-1} + (\lambda_2)^{-1}}{2} \right)^{-1} = \frac{2\lambda_1\lambda_2}{\lambda_1 + \lambda_2}$$

of the two principal curvatures λ_1, λ_2 of ν is identically equal to 1. In Sect. 4, we classify S^3_1-catenoids, and then we will also comment on the associated HMC-1 faces that are their unit normal vector fields (see Fig. 7 in Sect. 4).

2 Extended Hyperbolic Metrics on Riemann Surfaces

Let M be a Riemann surface.

Definition 2.1. A C^∞-metric $d\sigma^2$ defined on an open dense subset W of M is called an *extended hyperbolic metric* on M (or sometimes just called a *hyperbolic metric* for the sake of brevity) if it satisfies the following two properties:

1. $d\sigma^2$ is a Hermitian metric of constant curvature -1 on W.
2. There exists a discrete subset S of M such that for each local complex coordinate (U,z) of $M \setminus S$, $e^{-\omega}$ gives a smooth function on $U \setminus S$, where we use a local expression
$$d\sigma^2 = e^{\omega}|dz|^2 \qquad (|dz|^2 := dz\,d\bar{z})$$
on $U \cap W$. Moreover,
$$h(z) := \omega_{zz} - \frac{(\omega_z)^2}{2} \tag{23}$$
can be extended to a holomorphic function on $U \setminus S$.

The word "extended" expresses that the hyperbolic metric might have not only isolated singularities, but also singularities consisting of curves. We choose this terminology, since there are already a number of notions of "generalized hyperbolic metrics".

The first condition is independent of the second condition. In fact, if $e^{\omega}|dz|^2$ is positive definite, the holomorphicity of $h(z)$ implies that $e^{\omega}|dz|^2$ has constant curvature, but one cannot specify that constant. The holomorphic 2-differential $h(z)\,dz^2$ defined on each local complex coordinate induces a projective connection (cf. Appendix A) on $U \setminus S$. The smoothness of $e^{-\omega}$ is required since $U \setminus W$ may be disconnected and the extended hyperbolic metric associated to a given projective connection may not be uniquely determined in general, because of the ambiguity of Möbius transformations of the developing map [cf. Eq. (24) and Appendix A].

Let p be a point in the discrete subset S as above. Then by our definition of extended hyperbolic metric, p is an isolated singularity of the function $h(z)$ [cf. Eq. (23)] defined on a local complex coordinate (U,z) around p. If p is not a removable singular point of $h(z)$, then p is called a *proper singular point*. Moreover, if $h(z)$ has at most a pole of order 2 at $z = p$, then p is called a *regular singular point* of $d\sigma^2$. An extended hyperbolic metric is *proper singularity free* (or *PS-free*) if it has no proper singular points (i.e. $h(z)$ is holomorphic).

Definition 2.2. Two extended hyperbolic metrics on a Riemann surface M are *isometric* if one is obtained as the pull-back of the other by a holomorphic or anti-holomorphic automorphism on M.

It is well-known that there is a unique Hermitian metric of constant curvature -1 defined on an arbitrary closed Riemann surface M of genus greater than one, which can then be considered as a PS-free extended hyperbolic metric. General

PS-free extended hyperbolic metrics on closed Riemann surfaces are discussed in Goldman [10].

We now consider a metric on $S^2 = \mathbf{C} \cup \{\infty\}$, called a *spherical Poincaré metric*, defined by

$$d\sigma_P^2 := \frac{4|dz|^2}{(1-|z|^2)^2},$$

which is a Hermitian metric of constant Gaussian curvature -1 defined on $\{z \in \mathbf{C} \cup \{\infty\}; |z| \neq 1\}$. The metric $d\sigma_P^2$ is a PS-free extended hyperbolic metric defined on $\mathbf{C} \cup \{\infty\}$. In fact, if we set

$$\omega = \log\left(\frac{4}{(1-|z|^2)^2}\right),$$

then $\omega_{zz} - (\omega_z)^2/2$ vanishes identically. We call the pair

$$S_H^2 := (\mathbf{C} \cup \{\infty\}, d\sigma_P^2)$$

the *hyperbolic sphere*, which was introduced in [11], but has already appeared in Goldman [10, Sect. 2] and [20]. The hyperbolic sphere S_H^2 can be considered as an attachment of two hyperbolic planes at their ideal boundaries, as discussed in the Introduction. The orientation-preserving isometry group $\mathrm{Isom}_+(S_H^2)$ of $d\sigma_P^2$ is generated by $\mathrm{PSU}(1,1)$ and the matrices $\pm\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$.

We fix an extended hyperbolic metric $d\sigma^2$ on a connected Riemann surface M. Let $\{(U_\lambda, z_\lambda)\}_{\lambda \in \Lambda}$ be a covering of M consisting of local complex coordinates such that $d\sigma^2 = \exp(\omega^\lambda)|dz_\lambda|^2$, where $\omega^\lambda\,(\lambda \in \Lambda)$ is a C^∞-function on U_λ. We set

$$P_\lambda := \left(\omega^\lambda_{z_\lambda z_\lambda} - \frac{(\omega^\lambda_{z_\lambda})^2}{2}\right)(dz_\lambda)^2 \qquad (\lambda \in \Lambda), \tag{24}$$

which gives a projective connection (cf. Appendix A) defined on $M \setminus S$, where S is the set of proper singular points of $d\sigma^2$. We call $P = \{P_\lambda\}_{\lambda \in \Lambda}$ the *projective connection* induced by $d\sigma^2$, and denote it by

$$S(d\sigma^2) := \{P_\lambda\}_{\lambda \in \Lambda}.$$

In fact, the projective connection P can be considered as a Schwarzian derivative of $d\sigma^2$ because of the identity

$$S_\lambda(g)(dz_\lambda)^2 = P_\lambda \qquad (\lambda \in \Lambda),$$

where g is a developing map of $d\sigma^2$ (see Theorem 2.3).

Theorem 2.3. *Let $d\sigma^2$ be an extended hyperbolic metric on a connected Riemann surface M whose proper singular set is S. Then there exists a meromorphic function g defined on the universal covering space $\pi : \widetilde{M \setminus S} \to M \setminus S$ such that*

$$\pi^* d\sigma^2 = g^* d\sigma_P^2.$$

(Such a meromorphic function g is called a developing map of $d\sigma^2$.)

Conversely, let S be a discrete subset of M and g a meromorphic function on $\widetilde{M \setminus S}$ such that

$$d\sigma^2 := \frac{4\, dg\, d\bar{g}}{(1-|g|^2)^2}$$

gives a positive definite metric defined on an open dense subset W of M. Then $d\sigma^2$ is an extended hyperbolic metric on M, whose developing map is g.

To prove the assertion, we need the following:

Lemma 2.4. *Let g_1 and g_2 be two nonconstant meromorphic functions on a Riemann surface M such that there exists a matrix $a \in \mathrm{SL}(2,\mathbf{C})$ satisfying $g_2 = a \star g_1$. If there exists a neighborhood U of a point $p \in M$ such that*

$$\{q \in U\,;\, |g_1(q)| = 1\} = \{q \in U\,;\, |g_2(q)| = 1\},$$

and if p is not a branch point of g_1 and $|g_1(p)| = 1$, then $\hat{\pi}(a) \in \mathrm{Isom}_+(S^2_H)$.

Proof. Since p is not a branch point of g_1, we may assume that $z = g_1$ gives a local complex coordinate on U. Then we have that

$$\{z \in U\,;\, |z| = 1\} = \{z \in U\,;\, |a \star z| = 1\},$$

which implies that $\hat{\pi}(a) \in \mathrm{Isom}_+(S^2_H)$. □

Proof of Theorem 2.3. We take the maximal open dense subset W of $M \setminus S$ so that $d\sigma^2$ is a Hermitian metric of constant curvature -1 on W. Let $\{W_\lambda\}_{\lambda \in \Lambda}$ be the connected components of W. We take a reference point $p_\lambda \in W_\lambda$, and fix a simply connected local complex coordinate (U, z) centered at p_λ such that $U \subset W_\lambda$. Then there exists a holomorphic function

$$g_\lambda : U \longrightarrow \mathcal{D} = \{w \in \mathbf{C}\,;\, |w| < 1\}$$

such that $g_\lambda^* d\sigma_P^2$ coincides with $d\sigma^2$ on U. Then it can be easily checked that

$$S_z(g_\lambda) = \omega_{zz} - \frac{(\omega_z)^2}{2}$$

holds, where we set $d\sigma^2 = e^{\omega} |dz|^2$ on U.

On the other hand, let

$$g_0 : \widetilde{M \setminus S} \to \mathbf{C} \cup \{\infty\}$$

be a developing map of the associated projective connection of $d\sigma^2$ (cf. Fact A.3 of Appendix A). By definition, it holds that

$$S_z(g_0) = \omega_{zz} - \frac{(\omega_z)^2}{2} \left(= S_z(g_\lambda)\right).$$

Then there exists a matrix $a_\lambda \in \mathrm{SL}(2, \mathbf{C})$ such that $g_\lambda = a_\lambda \star g_0$ on U. So if we set

$$\tilde{g}_\lambda := a_\lambda \star g_0,$$

then $\tilde{g}_\lambda$ is a meromorphic function defined on $\widetilde{M \setminus S}$. Regarding (U, z) as a local coordinate of $\widetilde{M \setminus S}$, $\tilde{g}_\lambda$ is a meromorphic extension of $g_\lambda \circ \pi$, where $\pi : \widetilde{M \setminus S} \to M \setminus S$ is the covering projection. Since $1/d\sigma^2$ is smooth on $M \setminus S$ as a differential of type $(-1, -1)$ (cf. the property 2 in Definition 2.1 of extended hyperbolic metrics), the real analyticity of $d\sigma^2$ on W_λ implies that

$$1/d\sigma^2 = 1/\left(\tilde{g}_\lambda^* d\sigma_P^2\right) \tag{25}$$

holds on $\overline{W_\lambda}$ as a differential of type $(-1, -1)$. If $W_\lambda = W$, then $\tilde{g}_\lambda$ is the desired developing map of $d\sigma^2$. So we consider the case that W has at least two connected components. In this case, ∂W_λ is not discrete, and we can find a point $p \in \partial W_\lambda \setminus S$. Since the branch point of $\tilde{g}_\lambda$ is discrete, we may also assume that p is not a branch point of $\tilde{g}_\lambda$. If we take a sufficiently small neighborhood U of p, then $U \cap \partial W_\lambda$ is the set of points satisfying $|\tilde{g}_\lambda| = 1$. If $p \in \overline{W}_\lambda \cap \overline{W}_\mu$, then $|\tilde{g}_\lambda| = |\tilde{g}_\mu| = 1$ holds on $U \cap \partial W_\lambda$. By Lemma 2.4,

$$1/(\tilde{g}_\lambda^* d\sigma_P^2) = 1/(\tilde{g}_\mu^* d\sigma_P^2)$$

holds on $\overline{W_\lambda \cup W_\mu}$. Thus Eq. (25) holds on all of $M \setminus S$, and each $\tilde{g}_\lambda$ $(\lambda \in \Lambda)$ gives the desired developing map of $d\sigma^2$. The second statement of the theorem can be proved easily. □

Let

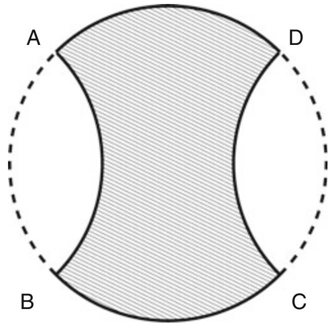
$$g : \widetilde{M \setminus S} \longrightarrow \mathbf{C} \cup \{\infty\}$$

be a developing map of an extended hyperbolic metric $d\sigma^2$ defined on a Riemann surface M. Since $\pi^* d\sigma^2 = g^* d\sigma_P^2$, g induces a group homomorphism

$$\rho_g : \pi_1(M \setminus S) \longrightarrow \mathrm{Isom}_+\left(S_H^2\right) (\subset \mathrm{PSL}(2, \mathbf{C}))$$

such that $g \circ T^{-1} = \rho_g(T) \star g$ holds for each $T \in \pi_1(M \setminus S)$. We call the group homomorphism ρ_g a *monodromy representation* of the extended hyperbolic metric

Fig. 1 Example 2.5



$d\sigma^2$. Since the developing map g is not uniquely determined, the representation ρ_g has an ambiguity of the conjugate actions in $\mathrm{Isom}_+(S^2_H)$. On the other hand, the subset

$$\partial^\infty(d\sigma^2) := \left\{\pi(p) \in M \setminus S\, ;\, |g(p)| = 1 \ \left(p \in \widetilde{M \setminus S}\right)\right\} \tag{26}$$

does not depend on the choice of g, which is called the *ideal boundary set* of the metric $d\sigma^2$. Each point in $\partial^\infty(d\sigma^2)$ is called an *ideal boundary point*, or ∂^∞-point. By definition, $d\sigma^2$ is not defined at ∂^∞-points, and also not at proper singular points. The representation ρ_g canonically induces a new representation

$$\hat{\rho}_g : \pi_1(M \setminus S) \longrightarrow \mathbf{Z}_2 := \mathrm{Isom}_+(S^2_H)/\mathrm{PSU}(1,1).$$

If the induced representation $\hat{\rho}_g$ is trivial, that is, if

$$\rho_g\bigl(\pi_1(M \setminus S)\bigr) \subset \mathrm{PSU}(1,1)$$

holds, then the extended hyperbolic metric $d\sigma^2$ is called *co-orientable*. An extended hyperbolic metric which is not co-orientable is said to be *non-co-orientable*. By taking a double covering, a non-co-orientable extended hyperbolic metric becomes co-orientable (see Remark 2.12).

Example 2.5. Consider a 4-gon ABCD in the closed unit disk $\overline{\mathcal{D}} \subset \mathbf{C}$ as in Fig. 1 whose two edges AB and CD are complete geodesics in $\mathcal{D}$ as the Poincaré disk, and BC and DA lie in the ideal boundary of $\mathcal{D}$. Gluing each pair of edges, we get a PS-free extended hyperbolic metric $d\sigma_1^2$ on a torus. This metric $d\sigma_1^2$ is non-co-orientable, since the ideal boundary set is connected (see Proposition 2.18). Similarly, considering a $4k$-gon in $\overline{\mathcal{D}}$, we get a non-co-orientable extended hyperbolic metric defined on a closed Riemann surface of genus $k > 1$ whose ideal boundary set is connected. Moreover, taking its double covering, we also get a co-orientable PS-free hyperbolic metric defined on a closed Riemann surface of positive genus.