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M.A. (Ken) Clements
Alan J. Bishop
Christine Keitel
Jeremy Kilpatrick
Frederick K.S. Leung *Editors*

Third International Handbook of Mathematics Education

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M. A. (Ken) Clements • Alan J. Bishop
Christine Keitel • Jeremy Kilpatrick
Frederick K. S. Leung
Editors

Third International Handbook of Mathematics Education

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Editors
M. A. (Ken) Clements
Illinois State University
Department of Mathematics
Normal, IL
USA

Christine Keitel
Freie University
Berlin
Germany

Frederick K. S. Leung
The University of Hong Kong
Pokfulam, Hong Kong
People's Republic of China

Alan J. Bishop
Monash University
Clayton, Victoria
Australia

Jeremy Kilpatrick
University of Georgia
Athens, Georgia
USA

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Past, Present and Future Dimensions of Mathematics Education: Introduction to the Third International Handbook of Mathematics Education

M. A. (Ken) Clements

Abstract The four major sections in this *Third International Handbook* are concerned with: (a) social, political and cultural dimensions in mathematics education; (b) mathematics education as a field of study; (c) technology in the mathematics curriculum; and (d) international perspectives on mathematics education. These themes are taken up by 84 internationally-recognized scholars, based in 26 different nations. Each of the *Handbook*'*s* four sections is structured on the basis of past, present and future aspects. The first chapter in a section provides historical perspectives ("How did we get to where we are now?"); the middle chapters in a section analyze present-day key issues and themes ("Where are we now, and what recent events have been especially significant?"); and the final chapter in a section reflects on policy matters ("Where are we going, and what should we do?"). An overview of the major common recurring themes and issues in the *Handbook* is presented. It is argued that mathematics education research has a vitally important role to play in improving mathematics curricula and the teaching and learning of mathematics. As a result of the expertise, wisdom, and internationalism of both authors and section editors, this *Handbook* provides an invaluable, state-of-the-art compendium of the most recent, and promising, developments in the field.

Keywords Globalization and mathematics education • History of mathematics education • *International Handbook of Mathematics Education* • Mathematics education research • Mathematics education policy • Social turn • No Child Left Behind • Technology in mathematics education

There are a number of facts about this *Third International Handbook* that should be made clear at the outset. These are:

- All 31 chapters were specifically written for this *Handbook.* There is no chapter in this *Handbook* which appeared in either the first *International Handbook of*

M. A. (Ken) Clements
Illinois State University, Department of Mathematics, Normal, IL, USA

Mathematics Education (Bishop, Clements, Keitel, Kilpatrick & Laborde, 1996) or the *Second International Handbook of Mathematics* (Bishop, Clements, Keitel, Kilpatrick & Leung, 2003).

- Although authors were expected to pay special attention to developments in scholarship, and in practice, that have occurred since the publication, in 2003, of the *Second International Handbook,* this *Third International Handbook* should not be seen merely as an update of the earlier handbooks. From the beginning, the editors aimed for a state-of-the-art compendium that identified and examined four major dimensions of contemporary mathematics education.
- The contents of this *Third International Handbook* are consistent with the inclusion of the word "International" in the title. Altogether, there are 84 authors who contributed to the 31 chapters, and at the time the chapters were written (between September 2010 and December 2011), the authors were working in a total of 26 nations: Australia, Austria, Brazil, Canada, China, Colombia, Czech Republic, Denmark, France, Germany, Hong Kong, Iran, Israel, Italy, Japan, Malaysia, Mexico, New Zealand, Portugal, Singapore, Spain, Sweden, The Netherlands, UK, USA, and Venezuela. Although we would have liked the *Third Handbook* to have been even more international than it is in its outlook, we recognize that given that there were to be only 31 chapters, it would not have been realistic, or prudent, to have attempted to have more nations represented among the authors.
- In July 2010 the editorial team met for a week to discuss the structure, likely chapter emphases, and authors for the *Third Handbook.* The first decision made, at that time, was that there would be the following four sections:
  - *Section A:* Social, Political and Cultural Dimensions in Mathematics Education;
  - *Section B:* Mathematics Education as a Field of Study;
  - *Section C:* Technology in the Mathematics Curriculum; and
  - *Section D:* International Perspectives on Mathematics Education.
  - We recognized that these sections did not cover all of the important areas of mathematics education—but we chose these major themes after reflection on what we thought offered the best follow-up potential to the *Second Handbook*, in terms of developments between 2003 and the present.
- It was also agreed that each section would be structured on the basis of past, present and future aspects. Thus, the first chapter in each of the four sections is concerned with analyses of historical antecedents ("How did we get to where we are now?"); the "middle" chapters provide analyses of present-day key issues and themes ("Where are we now, and what events since 2003 have been especially significant?"); and the final chapter in each section reflects on policy matters ("Where are we going, and what should we do?"). As far as we know, this *Handbook* is the first major mathematics education publication to adopt, consciously, this past–present–future organizational structure.
- Each author was selected, jointly by the editors, on the basis of her or his recognized excellence and experience in relation to the theme that needed to be addressed in a chapter.

Major International Developments in Mathematics Education Since 2003

I have read each chapter in this *Handbook* several times. One cannot read the chapters carefully without beginning to recognize the pervasiveness of certain influences on the field of mathematics education. It is not my intention here to comment on each chapter in the *Handbook*—the section editors will have the opportunity to do that in their own introductions, placed at the beginning of the sections. Rather, I wish to draw attention to several major developments, and sometimes associated tensions, over the last decade.

The "Social Turn" Versus Control Groups, Random Assignment, and Randomized Trials

The first major development has been in relation to what Lerman (2000) called the *social turn* in mathematics education research. Many of the authors (especially of chapters in Sections *A* and *B* of this *Handbook*) draw attention to the increasing use of socio-cultural theories in the field. Some see the selection, use and refinement of such theories as the main way by which mathematics education is developing into a discipline in its own right. This emphasis on the social, cultural and political aspects of mathematics education has resulted in many of the traditional assumptions in mathematics education, about *who* should study *what* mathematics, and *why*, being problematized. In relation to issues associated with the call for "mathematics for all," traditional concepts of "disadvantage" have been questioned and re-defined, and traditional classroom discourse patterns have been subjected to scrutiny, especially from vantage points offered by different theorists. Issues associated with the role of assessment have never been far away, and the matter of what should constitute the most appropriate forms of assessment in a given context is widely discussed. There has also been much discussion and research on the concept of teachers as researchers, and on what collaboration might mean in different areas of mathematics teaching and mathematics education research. Globalization tendencies have been, and continue to be, scrutinized from various theoretical perspectives.

An interesting feature of the last decade has been the roles and status of mathematics education researchers in the USA, where the 2001 Federal Education initiative *No Child Left Behind* (NCLB) Act heralded a series of significant reforms which sought to improve student, teacher, school, and system performance in mathematics through test-based school accountability (Learning Point Associates, 2007). The NCLB Act called for education policy to rely on a foundation of *scientifically-based research* which employed rigorous methodological designs and techniques, including control groups, random assignment, and randomized trials. National Science Foundation (NSF) grant applicants were strongly advised to strive for randomized designs, and the Department of Education's 2002–2007 strategic plan (see

Shavelson & Towne, 2002) stated that, by 2004, 75% of new research and evaluation projects funded by the Department which address causal questions should use randomized experimental designs. Mathematicians, as well as mathematics educators, were expected to be included in mathematics education research teams. The tension between those requirements and the spirit of Lerman's (2000) social turn is discussed in several chapters in this *Handbook*.

Technology

Some of the authors in this *Third International Handbook* make it very clear that these days the world of mathematics education is changing very rapidly, and that technology is a major factor influencing the directions of change. Writers in Sections *C* and *D* of this *Handbook* emphasize that recent technological developments are challenging traditional views on curriculum, teaching, learning, and assessment. What forms of curriculum, teaching, learning, and assessment are the most appropriate given the rapid technological developments? How can teachers keep up with developments and, simultaneously, cope with their often-too-heavy teaching loads? Given recent developments, what should algebra, geometry and calculus curricula look like in the future? What should proof in school mathematics look like? What technological aids should students be allowed to use in examinations, and what are the implications of that question for those responsible for developing policies with respect to assessment and evaluation?

Given the rapidity of ongoing technological developments, and the increasing reach of new technologies into even remote areas of the world, one cannot help wondering whether in 20 years time, say, there will be an agreed international mathematics curriculum. Many writers committed to the need to link curricula and teaching to social and cultural factors view such a possibility as extremely unwelcome. Issues associated with online and other distance forms of mathematics education are frequently discussed, and there is a concern that despite the socializing potential of new technology, an international mathematics curriculum would result in mathematics education becoming even more separated from local aspects of culture than it is now.

Globalization and Internationalization of Mathematics Education

When the *Handbook* editors initially met to work out the *Handbook* structure, chapter titles, authors, etc., it was agreed that it would be wise to try to avoid unnecessary repetition. In particular, it was agreed that we should try to restrict, to just a few chapters, discussion of the influence of the International Association for the Evaluation of Educational Achievement's (IEA's) "Trends in Mathematics and Science Study" (TIMSS), and the Organisation for Economic Co-operation and Development's (OECD's) "Programme for International Student Assessment" (PISA). Despite the best efforts of our editors, we failed in this regard, largely because many authors recognized the huge impact that TIMSS and PISA (and other international studies such as the Learner's Perspective Study—LPS) have had during the past decade.

There is a concern that TIMSS, PISA, and other international testing programs will have a standardizing effect on school mathematics that will cramp promising developments arising from the "social turn" in research. But some authors have argued that despite this potential danger, these international studies have drawn attention to well-performing nations like Japan, Singapore, Hong Kong, and Finland and have more or less forced researchers and policy makers to face the question: "Why have the students in such nations performed so well—and why have students in some extremely well-resourced nations performed considerably less well?" This has given rise to additional questions like: "How can we make mathematics education research more responsive to national needs, as those needs are perceived by politicians and education policy makers?" The possibility that influential policy makers do not regard the results of much mathematics education research as useful has been raised.

Who Should Read This *Handbook*?

As I read the draft chapters of this *Third International Handbook* I often found myself thinking that all mathematics educators, including mathematics teachers at all levels, should read some or all of the chapters. Then, when teaching graduate classes, I often thought that all of my graduate students would benefit from reading some of the chapters. I certainly intend to use this *Handbook* as a text for my future graduate students and, of course, I hope that other persons teaching graduate mathematics education students will do the same.

Various *Handbook* authors have drawn attention to the tendency for much mathematics education research to be carried out in teams that include school teachers and mathematics educators normally based outside of schools. Every person involved in collaborative studies of this type would likely benefit from becoming aware of what authors in this *Handbook* have said.

Chapters in this *Handbook* can provide important insights into how teachers and researchers around the world are working towards providing answers to issues that can no longer be ignored. For example, we need to answer questions such as: "What can a school do if it wants to engage all of its students actively and productively in relevant mathematics learning?" And, "What about those outside of the normal school and college system (many adults, for example) who want to learn mathematics?—What should we be doing, for them, to facilitate top-quality, and satisfying mathematics learning?"

This *Handbook* is the most-internationalized of all mathematics education handbooks that have been prepared thus far. Its chapters provide up-to-the-minute, state-of-the-art reviews on major themes; invariably, there has been an attempt to make readers aware of the international spread of opinion, methodologies, research and practice. The *Handbook* provides much insight, not only from researchers in the traditional European and North American nations but also from researchers in many other parts of the world. Throughout, any suggestion that the best mathematics education research wisdom has largely emanated from "the West" has been problematized, and basic questions—such as: "Why have Confucian-based cultures generated such productive forms of school mathematics?"—have been carefully considered. Authors charged with the responsibility of presenting historical perspectives (and authors of some of the other chapters, too) have deliberately argued from international, global, vantage points rather than from distinctly Western vantage points.

Whither Mathematics Education?

I have been privileged to work on the three Springer/Kluwer international handbooks on mathematics education. For almost all of my professional career I have worked in the field of mathematics education, and it has been a matter of principle for me to be able to say why I believe, strongly, that mathematics education is a crucially important field of endeavour.

Mathematics is one of the few areas in an individual's life in which she or he is required to spend between three and five hours per week (and, in addition, more hours on homework or with a tutor), for between 10 and 12 years (at least) studying a curriculum defined by others. What a waste of everyone's time, energy, and money, if students do not learn school mathematics as well as they possibly can, so that they develop an interest in the subject and an appreciation of its power to help them deal efficiently with important everyday problems. Furthermore, I believe that success with the subject is likely to be associated with greater satisfaction in later life (because successful students are more likely to take up vocations of their choice, or gain entry to a wider range of courses in higher education institutions). From a national perspective, the benefit of having a mathematically-competent citizenry is, it is often asserted, likely to result in strong economic performance (or, at least, stronger than would be the case if most citizens were not mathematically competent). Thus, it is important that research be conducted which will take into account students' attitudes towards mathematics, as well as their mathematical problem-solving and problem-posing performances.

But if mathematics education research is important, then how well are we doing in fostering the highest possible quality of mathematics learning as a result of our mathematics education research? Let us not put our heads in the sand on this matter. There is certainly a lot of room for improvement! The nation which has the most qualified mathematics education researchers is probably the USA—yet, many indicators (including results on international comparative studies) suggest that many US students fail to learn mathematics well. How could that be the case, considering the amount of research that has been conducted, and published within the USA, over so many years?

It is well known that many students, in most nations (perhaps all nations), experience difficulty in understanding fractions, the four operations with integers, and elementary algebra. We need to face the reality that many learners experience much difficulty in mathematizing situations for which mathematical approaches to problem solving would be informative and efficient. Why has there not been a marked improvement, given the large amount of mathematics education research conducted around the world, and over a very long period of time, with respect to such fundamentally important curriculum matters? Should our standard curricula and teaching approaches be problematized and reconceptualized? Various chapters in this *Third Handbook* consider issues such as these.

I could say much more—but perhaps, now I have succeeded in stimulating your interest and arousing an argumentative spirit within you. I should leave the core of what is said in this *Third International Handbook* to our team of very competent authors. As you read each chapter, I urge you to reflect on the basic question: Whither mathematics education?

Acknowledgments As many readers might imagine, it has been an honour and privilege to work with a team of such outstanding editors. It hardly needs to be said that Alan Bishop, Christine Keitel, Jeremy Kilpatrick and Frederick Leung are world-class mathematics educators and researchers. And I can say, sincerely, that each has expended a huge amount of effort, and time, on this project.

I would also like to thank the authors, who have worked hard to meet demanding deadlines. Together, twhe editors and authors for this Third International Handbook have provided enormous experience, expertise and wisdom on key mathematics education issues. Has any other book in the field of mathematics education ever had such a wide-ranging and authoritative international set of authors?

I would also like to thank the large number of professional reviewers of chapters. Every chapter has been read by at least four people—the appropriate section editor, at least two especially-appointed external reviewers, and myself. None of the Handbook's 31 chapters appears in the form in which it was first submitted. Always, authors were willing to respond professionally to suggestions for improvement.

During the period in which the Third Handbook was being prepared I was based at Illinois State University's Department of Mathematics. I would especially like to thank Dr George Seelinger, Head of the Department of Mathematics, for his encouragement and support.

Throughout the planning and manuscript preparation process, Harmen van Paradijs, Editorial Director, Human Sciences, for Springer, was supportive, wise and understanding. We also profited from the knowledge and cooperation of Melissa James and Natalie Rieborn, Springer's Publishing Editor, and Senior Editorial Assistant, respectively. We could not have asked for a better relationship with our publisher.

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Part I
Introduction to Section A: Social, Political and Cultural Dimensions in Mathematics Education

Christine Keitel

Abstract There are eight chapters in this first main section of the *Third International Handbook of Mathematics Education* and, altogether, there are 22 contributing authors, from 13 different nations. The first chapter, prepared by the five editors of the *Third Handbook*, provides historical perspectives on how far we have progressed towards the goals of mathematics for all—and also on different interpretations of that goal—over the past 200 years. The authors of the other chapters present various theoretical positions that are informing mathematics education researchers as they strive to achieve more equitable and effective environments in which the teaching and learning of mathematics occurs. Cultural, social, linguistic and political factors that not only affect views on the nature of mathematics, but also the structuring of curricula and education environments, are emphasized.

Keywords Deficit models in mathematics education • Disadvantage in mathematics education • Equity in mathematics education • Language and mathematics learning • Numeracy • Social justice in mathematics education • Social turn in mathematics education • Sociocultural directions in mathematics education • Transition between contexts

In Chapter 1, the editors argue that historically the acceptance of reckoning or mathematics as something to be taught in classrooms came rather late. Although, immediately after the invention of printing, reckoning books for independent learning appeared in Europe, and early in the 16th century private reckoning schools for bourgeois pupils were operating in central Europe, all of this happened rather slowly. At first, printed arithmetics were written in Latin, but then followed vernacular texts—like the famous arithmetic book written, in a German language, by Adam Ries. But it was mainly the children of wealthy bourgeois families in cities—and almost always boys—who attended such schools and usually the emphasis was on

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C. Keitel
Freie University, Berlin, Germany

mechanistic, rule-based calculations. Although famous mathematicians like, for example, Descartes and Leibniz, were advocating that their own revolutionary mathematical discoveries (e.g., Descartes' "Cartesian Geometry" and Leibniz's "Calculus") be taught in schools and universities, those who ultimately went on to study any form of higher mathematics were few in number.

High-level schools and universities were rare and expensive, and in any case, within such institutions mathematics was rarely regarded as a subject of educational value. Not only was the number of persons capable of teaching mathematics beyond elementary arithmetic small, but also general parental attitudes to schooling, the economic circumstances of most families, and social and psychological presuppositions and prejudices about mathematical ability or giftedness, combined to condemn forms of mathematics education into a precarious position. Mathematics teaching was the domain of the private tutor or the barely mathematically literate teacher in private schools. Chapter 1 identifies a historical progression underlying the evolution of the current expectation that relevant and applicable mathematics education should be available to all people: the sequence begins with schooling for all, and proceeds to arithmetic for all, to mathematics for all, and to quantitative or mathematical literacy for all.

In mathematics education research and practice today there is a noticeable change in approaches to researching the diverse social, political and cultural dimensions of mathematics education. In Chapter 2, Eva Jablonka, Margaret Walshaw and David Wagner provide an overview of a growing number of theories that are allowing us to widen our perspectives on these dimensions. Jablonka et al. identify and discuss theoretical trends and provide critical discussion not only of the theories themselves but also of the ways they are being used to discuss and critique research and practices in mathematics education. The authors successfully summarize, compare and exploit theories and their applications from research presented at the annual meetings of the International Group for the Psychology of Mathematics Education (PME).

Past research has largely characterized disadvantage as an individual or social condition that somehow impedes mathematics learning. That approach resulted in the marginalization of individuals whose physical, racial, ethnic, linguistic and social identities were different from normative identities constructed by dominant social groups. Recent studies have consciously avoided equating difference with deficiency and instead have sought to understand mathematics learning from the perspective of those whose identities are not consistent with norms constructed by dominant social groups. With this way of thinking, traditional concepts of "disadvantage" can be interpreted as having not only been socially constructed but also as having perpetuated disadvantage among certain types of individuals. Overcoming disadvantage can be achieved by analyzing how learning scenarios and teaching practices can be more finely tuned to the needs of particular groups of learners, empowering them to demonstrate abilities beyond the limits generally set and expected within dominant discourses.

In Chapter 3, Lulu Healy and Arthur Powell consider—under the heading "Understanding and Overcoming Disadvantages in Mathematics"—theoretical and methodological perspectives associated with the search for a more inclusive mathematics education, one which generally perceives and conceptualizes the role of the teachers as active participants in the process of researching and interpreting students' learning. Drawing from examples from a diverse range of learners including linguistic, racial and ethnic minorities, as well as deaf and blind students, the authors argue that by carefully studying and trying to get a much deeper understanding of the learning processes of such students we may not only be able to design pedagogical means to allow children to learn better, but also to better understand mathematics learning in general.

Cristina Frade, Nadja Acioly-Régnier and Li Jun described the aim of Chapter 4—titled "Beyond Deficit Models of Learning Mathematics: Sociocultural Directions for Change and Research"—as providing a theoretical exposé of the inherent weaknesses of deficit models. The identification of those weaknesses only came to be recognized following major paradigmatic changes in mathematics education research which drew attention to new perspectives on learning. Whereas, previously, deficit models were foregrounded in research designs, they have now been replaced by a wide variety of theoretical directions for studying diverse approaches to learning mathematics. This has resulted in an acceptance of the need for richness and variety in research practices, so that approaches can be studied, compared, and mutually applied and improved. Psychological and quantitative approaches and methods are now increasingly complemented, or even replaced, by new directions that rely on social and anthropological theories and methods. Rather than reviving ideas about deficit research in mathematics education, Frade et al. present sociocultural perspectives of learning mathematics, and show how these perspectives demand answers to important questions that were not even considered when deficit models of learning framed research. Having placed the main traditional markers of discrimination in school mathematics—gender, social class and ethnicity—within a perspective of social justice, the chapter concludes with a reflection on equality in terms of the democratic principle of meritocracy in mathematics education.

The recognition by recent researchers that learning mathematics is a culturally-influenced activity has become increasingly more apparent as research aims, technological advances, and methodological techniques have diversified, enabling more detailed analyses of learners and what they learn. Increased opportunities for studying learners in different cultural, social and political settings have also been provided by online access to results of international benchmark testing programs. The availability of data sets from large-scale quantitative studies—like, for example, "Trends in International Mathematics and Science Study" (TIMSS) and the "Program for International Student Assessment" (PISA)—and from comprehensive qualitative studies—like the international "Learners' Perspective Study" (LPS)—have facilitated careful investigation of research questions about learners and the contexts in which they learn. In Chapter 5, "Studying Learners in Intercultural Contexts," Yoshinori Shimizu and Gaye Williams point to how results and methods from large-scale quantitative studies have stimulated questions that demand qualitative research designs for their exploration. The increasing adoption of qualitative research has raised awareness with respect to the importance of historical, social and cultural perspectives when considering the dimensions of learning. This raises questions about the roles of "local" theories in investigations involving intercultural analyses.

In Chapter 6, "Learners in Transition Between Contexts," Tamsin Meaney and Troels Lange explore conceptions of learners in transition between contexts, and evaluate pedagogical practices that have been advocated for such learners. They point out that learning occurs as learners reflect on their transition between contexts, particularly when there are differences in what content knowledge is valued, the relationships between participants, and how activities are undertaken. From this perspective, productive pedagogical practices for learners in transition are those that build and sustain relationships between learners and mathematics and between learners and others, including especially those that lead outside the mathematics classroom. Meaney and Lange look, for their inspiration, specifically at examples of pedagogical practices that draw on principles associated with ethnomathematics and critical mathematics education.

Chapter 7 provides a focussed discussion of the goals and achievements of a movement that is concerned with adults' mathematics education (AME) as a field of study and practice. Jeff Evans, Tine Wedege and Keiko Yasukawa draw attention to a broad range of settings for teaching and learning, as well as for research. AME, whose activities have developed in a dynamic context of globalization, competition, and social insecurity, has faced the same struggle for its justification, in terms of humanistic and human capital goals of education, that adult education and lifelong education have been facing over the last half-century. This struggle is well reflected in current AME practices, research and policy. Evans et al. formulate critical perspectives for examining AME in the three connected dimensions of practice, research, and policy, always with the intention of clarifying assumptions, concepts, and actions with respect to crucial areas. Thus, for example, they examine multiple and contested meanings of key terms like "numeracy," and point out that definitions vary depending on whether they seek to foreground the needs of individual learners or whether they are more concerned with particular economic imperatives (such as "needs" of the labour market). Evans et al. illuminate how variations in such definitions can affect the experiences of AME learners and practitioners. They problematize ideas associated with "the transfer of learning" of mathematics from school to work, and from formal to non-formal or informal learning situations. They argue that because a new international survey of adults' skills—the OECD-sponsored Program for International Assessment of Adult Competencies (PIAAC) is now being conducted—it is timely to question what such surveys can tell us about the development of AME as a field, and to consider which questions need to be pursued independently.

In the last chapter of Section A, Chapter 8, on "Politics of Equity and Access in Teaching and Learning Mathematics," Neil A. Pateman and Lim Chap Sam, besides clarifying definitions of equity and access, briefly contrast two philosophical positions on the nature of mathematics and speculate about the consequences of these different positions for equity and access. They raise the question "whose mathematics?" and provide a developing viewpoint on how mathematics learning depends on equity and access for students. After considering the roles of mathematics teachers and how these are related to equity and access for students, they broaden their discussion to consider political influences on both teachers and learners of mathematics. Their observations relate to the role that politics plays at different levels in influencing access and equity for teaching and learning mathematics. Pateman and Lim illustrate their position through a discussion of particular examples, some from history, and others documenting more recent events. Finally they offer a brief discussion of several international cases which, they believe, demonstrate how a form of colonization is occurring in relation to contexts in which authorities insist on an “English-first” policy whereby the language of instruction in school mathematics must be English despite the fact that English is not the pupils’ first language.

Chapter 1
From the Few to the Many: Historical Perspectives on Who Should Learn Mathematics

M. A. (Ken) Clements, Christine Keitel, Alan J. Bishop, Jeremy Kilpatrick, and Frederick K. S. Leung

Abstract Today we take for granted that everybody should be offered the opportunity to learn mathematics. However, it was not until well into the 20th century that "mathematics for all" became an achievable goal. Before then, the geographical location of schools in relation to children's homes, the availability (or non-availability) of teachers capable of teaching mathematics, parental attitudes to schooling, economic circumstances of families, and social and psychological presuppositions and prejudices about mathematical ability or giftedness, all influenced greatly whether a child might have the opportunity to learn mathematics. Moreover, in many cultures the perceived difference between two social functions of mathematics—its utilitarian function and its capability to sharpen the mind and induce logical thinking—generated mathematics curricula and forms of teaching in local schools which did not meet the needs of some learners. This chapter identifies a historical progression towards the achievement of mathematics for all: from schooling for all, to arithmetic for all, to basic mathematics for all; to secondary mathematics for all; to mathematical modelling for all; and to quantitative literacy for all.

M. A. (Ken) Clements (✉)
Illinois State University, Department of Mathematics, Normal, IL, USA
e-mail: clements@ilstu.edu

C. Keitel
Freie University, Berlin, Germany

A. J. Bishop
Monash University, Clayton, Victoria, Australia

J. Kilpatrick
University of Georgia, Athens, GA, USA

F. K. S. Leung
The University of Hong Kong, Pokfulam, Hong Kong, People's Republic of China

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"Mathematics for all" is the kind of goal that anticipates a world in which all people have the opportunity to learn, and benefit from learning, mathematics. This chapter offers historical perspectives, not only on who has had the opportunity to learn mathematics, but also on forms of mathematics that have been embraced by the expression "mathematics for all." We take that expression to mean a situation in which all living people, in all nations, at any particular time, will have formally studied, or are studying, or will be expected to study at least some form of mathematics. There is also an implied additional assumption that studying mathematics will bring associated benefits—personal, social, and political—for all.

Our decision to interpret "mathematics for all" in that way means that we shall not be focussing on higher-order mathematics as found in universities. We shall be more concerned with providing a historical analysis of how, gradually, during the 19th and 20th centuries, more and more people gained the opportunity to study mathematics. Our decision implies that part of our analysis needs to be concerned with the concept of "schooling for all" because progress towards mathematics for all, as we are interpreting it, presupposes schooling for all.

Towards Mathematics for All

Perhaps the best-known set of statements on "mathematics for all" is a collection of 22 papers (Damerow, Dunkley, Nebres, & Werry, 1984), published in 1984 by the United Nations Educational, Scientific, and Cultural Organization (UNESCO). Since that publication, there have been many calls for "mathematics for all" (e.g., Gates & Vistro-Yu, 2003; Krygowska, 1984), or variations or extensions of that theme, such as "algebra for all" (e.g., Viadero, 2009) and "numeracy for all" (e.g., Robinson, 1996).

One of the most stimulating papers in the UNESCO collection was jointly prepared by Peter Damerow, of the Max Planck Institute for Human Development and Education in Germany, and Ian Westbury, of the University of Illinois. Damerow and Westbury (1984) addressed the problem of designing a mathematics curriculum which meets the mathematical needs of all students in a nation. They asserted that history had shown that in such efforts the politics of the situation inevitably led to a curriculum which met the needs of only a small group of students. They maintained that that was precisely what had happened, in many nations, during the "modern mathematics" era from the mid-1950s to the mid-1970s.

In the following passage, Damerow et al. (1984) identified a major stumbling block in efforts to achieve "mathematics for all":

> Mathematics curricula were developed for an élite group of students who were expected to specialize in the subject, and to study mathematics subsequently at higher levels in a tertiary institution. As education has become increasingly universal, however, students of lesser ability and with more modest vocational aspirations and daily life requirements have entered the school system in greater numbers. A major problem results when these students are exposed to a curriculum designed for potential specialists. This same type of traditional curriculum

> has frequently been transferred to developing and third world countries where, because of different cultural and social traditions, the inappropriateness for general mathematics education has only been compounded. (p. 4)

Notice how Damerow et al. assumed that students have different “abilities” with respect to mathematics, and that a curriculum for “élite” students would not be suited to the needs of other students (Kamens & Benavot, 1991).

In another paper in the 1984 collection, Ben Nebres, of the Philippines, introduced the twin concepts of vertical and horizontal curriculum relationships. He argued that education authorities in developing countries typically kept their eyes on vertical curriculum requirements in developed countries, because they not only wanted their élite students to be qualified to study in developed countries, but they also wanted their own graduates to be professionally accepted for registration purposes in those nations. Nebres (1984) pointed out that this resulted in local curricula in developing countries failing to meet the needs of the majority and, indeed, failing to provide courses that were of interest, or suitable, for, most local students.

Historically, the numbers of students permitted to study mathematics, formally, have varied from community to community, from nation to nation, and from era to era (Gates & Vistro-Yu, 2003; Li & Ginsburg, 2006; Wu & Zhang, 2006). Even within the same community, or the same school, at a particular time, there may not be agreement on which students should be allowed to study the different forms of mathematics that are offered.

If everyone is to study mathematics then should there be a “core” mathematics curriculum, and if so, what should that core mathematics curriculum look like? And, to what extent should the mathematics-for-all expectation take into account cultural factors and individual differences? Should “mathematics for all” mean that students in schools in Paris, France, be taught the same mathematics as students in schools in the remote and mountainous regions of Vietnam? If one answers no, then immediately should follow the uncomfortable but important question, why not? H. R. W. Benjamin’s (1939) classic *Saber-Tooth Curriculum* helped us recognize that there are important areas of life—like, for example, sports—in which it makes little sense for everyone to be asked to learn and practise the same skills. It may not be reasonable to require all people everywhere to learn the same mathematics.

In all societies, most adults use what Bishop (1988) called “small-m” mathematics, on a daily basis. They count, reason, and use concepts like “more,” “less,” “the same,” and so forth, to perform actions in appropriate sequences. We all estimate and measure context-relevant quantities involving money, distances, times, capacities, areas, and other quantities. In this chapter we take such ethnomathematical practices for granted and focus more on the “big-M” forms of Mathematics (Bishop, 1988) that are offered in formal education institutions.

The perspectives we provide in this chapter will mainly take account of developments over the past 200 years. The coverage provides a broad sweep, and it has not been possible to take account of changing circumstances in all nations.

From our perspective, any scholarly discussion of mathematics-for-all phenomena ought to address the following questions:

1. Should all school children be expected to study mathematics, and if so what mathematics and for how long?
2. Should different students in different cultural settings study the same mathematics?
3. Should different students in the same nation, and even at the same school or college, study the same forms of mathematics?

Although these questions appear to be straightforward, they can be interpreted in different ways.

In this chapter we identify progress towards "mathematics for all" by providing commentary on the history of the development of the concepts of "schooling for all," "arithmetic for all," "basic mathematics for all," "secondary school mathematics for all," "mathematical modelling for all," and "quantitative literacy for all" (or "numeracy for all").

Schooling for All

Table 1.1, which is adapted from the United States' Commissioner of Education's reports for 1905 and 1907, shows proportions of people, in 37 nations and states on 6 continents, who were enrolled in schools around 1900. Entries are suggestive of the acceptance, or otherwise, of formal schooling in the various countries and states that are listed.

Around 1900, many school-age children, in many parts of the world, were not enrolled in a school. In some nations—for example, in regions now known as Bhutan and Brunei Darussalam—there were no formal schools, although there were small local temple- or mosque-related arrangements in which mainly religious knowledge was taught (Horwood & Clements, 2000).

School enrolment was one thing and attendance another. Although Table 1.1 indicates, for example, that percentages of children enrolled in schools in the USA were relatively high, many boys in the north-eastern and mid-western states worked on their parents' farms for most of the year and attended local one-room schoolhouses during winter months only (Cubberley, 1920; Zimmerman, 2009). Furthermore, in the USA in 1900, "only five percent of one-room school graduates proceeded to urban high schools" (Grove, 2000, p. 75).

More generally, at the beginning of the 20th century, school mathematics beyond the most elementary forms of arithmetic was not something that most people, in most countries, had experienced, or would experience (West, Greene, & Brownell, 1930). In many nations, children did not attend school regularly, and often they received no formal instruction in mathematics at all. Thus, it would have made little sense at that time to try to create an international policy on "mathematics for all," even if someone had thought of trying to achieve that goal. Often there was no

Table 1.1
Percentage of Populations Enrolled in Schools, in Various Nations, Around 1900

State or Country	Approx. % of Population Enrolled in Schools	State or Country	Approx. % of Population Enrolled in Schools	State or Country	Approx. % of Population Enrolled in Schools
Ontario (Canada)	21	Sweden	14	Costa Rica	6
USA	21	Belgium	12	Roumania	6
Switzerland	20	Québec (Canada)	12	Mexico	5
Prince Edward Island (Canada)	20	Japan	11	Honduras	5
Victoria (Australia)	20	Cuba	10	Nicaragua	4
England and Wales	18	Cape of Good Hope	10	Portugal	4
Scotland	17	Argentina	9	Servia	4
Ireland	17	Bulgaria	9	Bombay (India)	3
German Empire	17	Italy	8	Russia	3
Norway	15	Greece	7	Egypt	2
The Netherlands	14	Puerto Rico	7	Burma	1
Austria-Hungary	14	Spain	7		
France	14	Uruguay	6		

Note. Data are taken from reports by the U.S. Commissioner of Education (1905, 1907).

school within walking distance of a child's home; often, schools were available but parents did not want their children to attend them; sometimes, teachers capable of teaching forms of mathematics beyond the four operations and simple measurement were not available (Kamens & Benavot, 1991).

Achmad Arifin (1984), an Indonesian mathematician, emphasized the need for mathematics programs to be available in all schools in developing nations like Indonesia. He added, however, that such programs needed to be related to societal needs and cultural expectations: Although mathematical correctness in school textbooks and instruction was important, and something not to be taken for granted, unless there were frequent and positive interactions between schools, mathematics educators, and mathematicians, an acceptable mathematics-for-all agenda would be difficult to develop and implement. But, Arifin argued, if well-organized school mathematics programs could be worked out, then this could have beyond-school local benefits because mathematical solutions might then be applied to social problems.

Was the ideal of schooling for all achieved during the course of the 20th century? The short answer is no. A longer answer would elaborate on the fact that although, during the 20th century, schooling for all became a reality in most nations, in many Asian, African, and Central and South American nations it has never been achieved (Freire, 1996). Nevertheless, in many nations, there was significant progress towards schooling for all.

Take, for example, the nation of Brunei Darussalam, where it was not until 1914 that the first government-supported primary school was opened, and for many years even after that, most Bruneian children—and especially girls—never attended school (Upex, 2000). It was only in the 1950s that the first government secondary school was opened. However, in Brunei Darussalam today, almost all children attend primary and secondary schools, and mathematics is a mandatory part of the curriculum that they study. Likewise, in the Malaysian states of Sarawak and Sabah, which share their borders with Brunei Darussalam, it was not until the late 19th century that government-supported schools were first established, and the value and utility of such schools were not accepted by the majority of the local populations until well into the 20th century (Abu Zahari, 1977).

Progress Towards Schooling for All

The principle of schooling for all was declared, confirmed and reconfirmed by powerful organizations at various times during the 20th century. In 1948, for example, part of the Universal Declaration of Human Rights adopted by the United Nations asserted that "everyone has a right to education." In 1990, a World Conference on "Education for All," held in Jomtien (Thailand), and sponsored by UNESCO (1990), laid down that every person—child, youth or adult—should be able to benefit from educational opportunities designed to meet his or her basic learning needs.

The Jomtien delegates set the goal that by the year 2000 every child in every country should have the chance to complete at least a primary education. However, the goal was not reached, for in 2000 UNESCO estimated that 16% of the world's children did not attend school (Skovsmose, 2006). Of those who attended school, about 20% failed to complete a primary school education (Bruns, Mingat, & Rakotomalala, 2003). A World Education Forum in Dakar, Senegal, in 2000, reaffirmed the Jomtien commitment to schooling for all and added a note about the quality of education that should be expected in schools. The following Dakar goal for universal education (UNESCO, 2000) specifically mentioned numeracy:

> Improve all aspects of the quality of education and ensure excellence of all so that recognized and measurable learning outcomes are achieved by all, especially in literacy, numeracy, and essential life skills. (Quoted in Bruns et al., 2003, p. 2)

Dakar delegates decided that strategies should be devised that would enable all children to receive instruction in elementary numeracy, and that this goal should be achieved early in the 21st century.

Although the Jomtien and Dakar meetings presented an optimistic face, at the beginning of the 21st century universal primary education was far from having been achieved. UNESCO's (1998a) *World Education Report* revealed that in some Southeast Asian nations (e.g., Cambodia, Indonesia and Laos) millions of children never attended primary school. Of those who did, many did not remain at school after Grade 5 (UNESCO, 1998a, 1998b). Towards the end of the century, UNESCO (1998b) estimated that between 100 and 140 million of the world's primary-school-aged children had never attended school.

Around 2005 there were about 860 million illiterate adults in the world, of whom about 60% lived in India, China, Pakistan or Bangladesh. Whereas middle-class families in large cities valued the processes and products of primary school arithmetic, that was not always the case with poor families—especially those in remote regions or in slum areas in large cities. Often parents of poor families found it difficult to comprehend why their children should be required to spend many years in schools being drilled on "useless" facts when the children were needed at home or in the fields (Horwood & Clements, 2000).

Harding (1995) reported that in the 1990s well over 100 million adults aged between 15 and 35 were illiterate, and of these, 62% were women. Immediately before the floods which devastated the island nation of Haiti in 2010, about 65% of school-age children in that nation had never attended school, and the country's adult literacy rate was less than 50%. In Afghanistan, the primary-school completion rate dropped from 22% in 1990 to an estimated 8% in 1999 (Bruns et al., 2003). During the 1990s, Zambia, the Republic of Congo, Albania, Cameroon, Kenya, Madagascar, Qatar, Iraq, the United Arab Emirates, Bahrain, and Venezuela, made little, if any, progress on primary-school completion rates (Bruns et al., 2003; Delors, 1996).

Even today, many children do not get the chance to complete a primary-school education because they never go to school. According to data presented at a United Nations Summit in 2010, about 30 million school-age children in sub-Saharan African nations had never attended school (UNESCO, 2010). In war-ravaged Mekong Basin nations—Cambodia and Laos, for example—many children attend school only spasmodically, if at all. Harding (1995) cited UNESCO data indicating that between 19 and 24 million children aged between 6 and 14 years in India in 1995 had never attended school, and 60% of these were girls. According to Harding (1995), almost half of the children who entered Grade 1 in India dropped out before they reached Grade 5, with the highest drop-out rate occurring immediately after Grade 1 (see also UNESCO, 1998b).

Those who have learned to value formal education can find it difficult to understand why some parents avoid sending their children to school. The educated élite tend to think that schools provide a bridge to a better world. However, those who think that way have something to learn from the following comments by Ben Nebres (2006) on education in the Philippines:

> The first impression of a visiting mathematics educator from countries with a stronger mathematics education tradition in discussions with counterparts from the Philippines might be that of similarities in situations. As solutions begin to be discussed, however, he might begin to realize that beneath these similarities are greater differences. The dominant reality in a country like the Philippines is the scarcity of resources, both human and material. Five or six students have to share a textbook. Many schools lack classrooms, so classrooms meant for 40 children are crammed with 80 students. Or schools have double sessions, in some cases triple sessions, a day. Teachers are poorly trained and have to teach in very difficult environments. (p. 278)

In school education, conditions and contexts matter.

In the first *International Handbook of Mathematics Education*, Stephen Arnold, Christine Shiu and Nerida Ellerton (1996) emphasized the potential of distance education for improving access to mathematics learning, especially, but certainly not only, in geographically remote areas. For several decades, in China, India, Indonesia and Thailand, for example, there have been large enrolments in distance courses in mathematics, especially from adults seeking to qualify for professional appointments (such as teaching). Although this movement has been accelerated by developments in information and communication technologies (hereafter "ICT"), particularly in relation to online education, too often these developments have not given sufficient credence to local cultural and societal factors (Clements & Ellerton, 1996).

Arithmetic for All

The *Abbaco* Tradition in Arithmetic

Modern scholarship has revealed that many aspects of current school mathematics curricula have descended from what has been called the *abbaco* tradition in arithmetic (Ellerton & Clements, 2012; Franci, 1992; Høyrup, 2005; Long, McGee, & Stahl, 2009; Swetz, 1987, 1992; Van Egmond, 1980). It is likely that this tradition emerged from practices associated with so-called *trattati* or *libri d'abbaco*, vernacular Italian pedagogic manuals of commercial mathematics, accounting, and geometry widely used in Italian reckoning schools from the 13th century (Long et al., 2009; Van Egmond, 1980). Sharp increases in international trade and banking in Renaissance Europe prompted city republics to form vernacular schools in which commercial mathematics, accounting and writing were taught to sons of merchants or to apprentices with important responsibilities.

In Western Europe it became common for merchant-class parents to send their sons for two-year courses at these reckoning schools, where they learned commercially-oriented *abbaco* mathematics (see, e.g., Swetz, 1987, for details of an *abbaco* text, the *Treviso Arithmetic*, an Italian arithmetic first printed in 1478). Thus, for instance, in 1522, a book by Adam Ries, the noted German *rechenmeister* (reckoning master), showed how the use of Hindu-Arabic numbers could simplify calculations. The language of the text was German, not Latin, and although the book was probably aimed at male students, Ries thought that all students should learn to use written methods for calculation. According to Karpinski (1925), 40 editions of Ries's arithmetics were published in the vernacular in the 16th century alone, and many more appeared after that.

According to the *abbaco* tradition, children were not expected to begin to study in the reckoning schools until they were about 10 years of age. Then, for several years, boys would prepare cyphering books in which they neatly made entries on a standard sequence of topics (Van Egmond, 1980). They recorded rules, cases, examples and exercises concerned with Hindu-Arabic numeration, the four operations on numbers