Developments in Mathematics

Hershel M. Farkas
Robert C. Gunning
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B.A. Taylor *Editors*

From Fourier Analysis and Number Theory to Radon Transforms and Geometry

In Memory of Leon Ehrenpreis

Springer

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Leon Ehrenpreis (1930–2010)

Preface

This is a volume of papers dedicated to the memory of Leon Ehrenpreis. Although Leon was primarily an analyst, whose best known results deal with partial differential equations, he was also very interested in and made significant contributions to the fields of Riemann surfaces (both the algebraic and geometric theories), number theory (both analytic and combinatorial), and geometry in general.

The contributors to this volume are mathematicians who appreciated Leon's unique view of mathematics; most knew him well and admired his work, character, and unbounded energy. For the most part the papers are original contributions to areas of mathematics in which Leon worked; so this volume may convey a sense of the breadth of his interests.

The papers cover topics in number theory and modular forms, combinatorial number theory, representation theory, pure analysis, and topics in applied mathematics such as population biology and parallel refractors. Almost any mathematician will find articles of professional interest here.

Leon had interests that extended far beyond just mathematics. He was a student of Jewish Law and Talmud, a handball player, a pianist, a marathon runner, and above all a scholar and a gentleman. Since we would like the readers of this volume to have a better picture of the person to whom it is dedicated, we have included a biographical sketch of Leon Ehrenpreis, written by his daughter, a professional scientific journalist. We hope that all readers will find this chapter fascinating and inspirational.

Jerusalem, Israel — H.M. Farkas
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*Marvin Knopp (of blessed memory) passed away on December 24, 2011, after almost the entire volume was edited by the four of us. Without him, this volume would not have appeared.

Contents

A Biography of Leon Ehrenpreis

By: Yael Nachama (Ehrenpreis) Meyer

Dr. Leon Ehrenpreis (b. May 22, 1930; d. August 16, 2010), a leading mathematician of the twentieth century, proved the Fundamental Principle that became known as the Malgrange–Ehrenpreis theorem, a foundation of the modern theory of differential equations that became the basis for many subsequent theoretical and technological developments.

He was a native New Yorker who taught and lectured throughout the USA, as well as in academic institutions in France, Israel, and Japan. Ehrenpreis made significant and novel contributions to a number of other areas of modern mathematics including differential equations, Fourier analysis, Radon transforms, integral geometry, and number theory. He was known in the mathematical community for his commitment to religious principles and to his large family, as well as for his contributions to the essence of modern mathematics.

Leon Ehrenpreis published two major works: *Fourier Analysis in Several Complex Variables* (1970) and *The Universality of the Radon Transform* (2003), authored many papers, and mentored 12 Ph.D. students in New York, Yeshiva, and Temple Universities over the course of a mathematical career that spanned over half a century. What follows is his story.

Leon Ehrenpreis was born on May 22, 1930. His mother, Ethel, née Balk, was born in Lithuania; his father, William, a native of Austria, had changed his last name from that of his own father (Kalb) to that of his mother, in order to escape the Russian draft. And so "Ehrenpreis," the German word for "prize of honor," became the family surname.

Leon, whose parents also gave him the Hebrew name "Eliezer," was born just at the close of the era during which millions of Eastern European Jews had left behind the homes where their families had lived for generations and survived eras of persecution, in order to reach the land that promised to take in all of "your tired, your poor, your huddled masses yearning to breathe free…" and give their children the opportunity to become Americans. Coming ashore in New York City, many of these new immigrants settled in Manhattan's Lower East Side, in Brooklyn, and the

Bronx. Leon's family was no exception; over the course of his childhood, he lived in all three of these boroughs. Initially, Ethel and William, their baby Leon, and his older brother Seymour, settled in a home in the Marine Park neighborhood of Brooklyn.

When Leon was 10, the family moved to the Lower East Side, a neighborhood with a large Jewish community. Leon's home was one in which the kitchen was kosher, and the Sabbath recognized, and their Jewishness the defining personal, family, and communal identity, though without knowledge or emphasis on the subtle details of religious observance. So it was only there that Leon came into contact with boys of his own age whose families were strictly observant, an introduction to religious life that started Leon on his trajectory towards full-scale observance. He also attended a Jewish studies after-school program to prepare for his bar mitzvah, his entry into Jewish adulthood. Soon after his bar mitzvah, Leon stopped attending his after-school studies, though he continued to attend Sabbath services at the local synagogue as a result of his friends' influence.

The majority of New York's Jews at that time were aiming to raise their children to be successful, high-achieving Americans, with academic success and intellectual pursuits an important priority for many, including the Ehrenpreis family. So it was that soon after his bar mitzvah, Leon followed his brother into the prestigious Stuyvesant High School in Manhattan. Leon had skipped two grades in elementary school and then skipped his initial year of high school, beginning Stuyvesant in the tenth grade.

When Leon was 16, the family moved to the Bronx. Now more interested in learning about his Jewish heritage, Leon attended the Young Israel of Clay Avenue and joined Hashomer Hadati, a youth group that would be the forerunner of the religious Zionist Bnei Akiva movement. He now traveled downtown each day to Stuyvesant, where he continued to excel in his studies, though not in his class conduct! He recalled having the highest grades in French, but failing to be awarded the French medal because of his poor behavior. He also scored the highest on the chemistry medal qualifying exam (though a teacher's error meant that he never actually received it), and he was also awarded the mathematics medal—though most of these were won by his new best friend Donald Newman, whom Leon credited with influencing him to become a mathematician. His mother, he recalled, considered the choice of mathematics a "cop out" to avoid having to do the serious lab work that a physics major would require.

Leon initially met Donald Newman on his first day at Stuyvesant, where his classmate was seated just on the other side of the aisle in their first class of the day. Almost immediately, Donald handed a clipboard to Leon with the order to "solve this problem." The board read "Sierpnerhe"—Ehrenpreis backwards. Already in ninth grade, Leon said, Donald's reputation foreshadowed his greatness. The same was said about Leon from the tenth grade onward. "He was the great man of Stuyvesant—we already knew he would be a mathematical star." The two created lifelong nicknames for each other, and "Flotzo-Flip" (Donald) and "Glockenshpiel" (Leon) formed a friendship that would last forever. "I felt like a real mathematician when Flotz and I discussed mathematics together," Leon recalled. The two friends,

who were considered the best mathematicians in the class, would ultimately follow much the same path throughout their mathematical lives, remaining close personal friends throughout.

At the age of 20, Leon joined the National Guard, which involved training for 2 hours each week and 2 weeks in the summer. His youthful military duty provided him with a lifelong repertoire of "war stories." He was fond of recalling for his children how, to maintain a kosher diet, he subsisted on thrice-a-day meals of ice cream, and how his commanding officer, who initially refused his request for time off on Saturdays, finally told him to "disappear on Friday night—and don't come back until Sunday." Leon also liked to describe how his superiors eventually worked out what his strong points were—and weren't—and so assigned him to calculate the trajectory of the shots being fired instead of actually firing them. His speedy calculation ability made him popular among his fellow reservists as well, as he would finish all the work assigned to his group within the first hour of the morning—and then the entire troop would go to sleep for the rest of the day.

By this time, Leon had nearly completed his university studies, having been in college since the age of sixteen-and-a-half. Leon was enrolled at City College, the "Jewish Harvard," as it was known during those years when the Ivy League still maintained a quota of Jewish students, leaving many of the best and brightest to attend New York City's public university. In addition to his old friend Donald Newman, the class included Robert "Johnny" Aumann, Lee Rubel, Jack Schwartz, Allen Shields, Leo Flatto, Martin Davis, and David Finkelstein, a group of individuals who would go on to change the face of mathematics, computer science, and the sciences for decades to come. This high-powered group of students formed a math club and had their own table in the cafeteria—the "mathematics table"—where, Aumann recalled, the group would sit together, eating ice cream, discussing the topology of bagels, and enjoying "a lot of chess playing, a lot of math talk… that was a very intense experience."

In addition to his university studies (where handball and weightlifting competed with his mathematics major for his attention; he was the handball champion of New York City during his early college years), and his military activities, Leon expanded his Jewish education by enrolling in an evening Jewish studies program, where Bible and Talmud, as well as Hebrew language and literature, were taught entirely in Hebrew. This represented Leon's first formal attendance in an academic Jewish studies program. "It was the first time I ever studied a page of Talmud!" Leon recalled.

While attending City, Leon audited a series of lectures on probability theory given by Professor Harold Shapiro at NYU's Courant Institute. He identified an event that occurred during the course of these lectures as a "turning point" in his development into a "true" mathematician. Professor Shapiro wrote a statement on the board that he thought was obvious. Then he began writing out the proof—until he came to a step of the proof that he couldn't carry out. "You've learned more from my not knowing how to do it than by my presenting a proof," Shapiro told them. So Leon became determined to correct the proof himself. "I ate, breathed and slept correcting that step and…

Sunday: nothing.

Monday, late afternoon: Eureka! I can't fix that step in the proof because the theorem itself is wrong! So I corrected the theorem itself. Then I returned to Shapiro to inform him—it's wrong! Erdös and Chung had stated the theorem incorrectly. Although I was only eighteen, I was convinced that I was right. I showed Shapiro a counterexample to demonstrate without question that I had created the correct proof. I beat Erdös and Chung! I'm a mathematician!! No doubt anymore—I am the real thing."

That same year, Leon registered in joint mathematics–physics graduate programs at both Columbia and NYU simultaneously. He actually had not yet completed his Bachelor's degree at CUNY, during the course of which he had also "illegally" taken several advanced classes before completing the relevant prerequisites, and so for years to come would have "nightmares" that the university powers-that-be would suddenly discover his crimes and come to take away his B.S.—and his Ph.D.

Between 1952 and 1953, he worked on his doctorate with Claude Chevalley (whom Leon termed "the best in the world") as his thesis advisor. He completed his thesis, entitled, "Theory of Distributions in Locally Compact Spaces," in 1953, earning a Ph.D. from Columbia University at the age of 23.

Nearly 20 years later, Alan Taylor would ask Leon how, as a student of Chevalley, he had come to work on problems that led to what would ultimately be called the "fundamental principle." Leon explained to him that Chevalley had suggested that Leon write to Laurent Schwartz for thesis-problem suggestions. Schwartz, in turn, had responded with a list of questions about partial differential operators, along with the details he knew about them at the time, including the fundamental questions. The answers given by Leon and others, in the 1950s, would form the basis of the modern theory of linear constant-coefficient partial differential operators.

After Leon earned his doctorate, Chevalley arranged a first teaching position at Johns Hopkins University in Baltimore, Maryland, for him. It was there that Leon met Shlomo Sternberg, later a mathematics professor at Harvard, then a Johns Hopkins student, who reminisced:

> "Thinking back through the years, I can't recall a single time, no matter how trying the circumstances may have been, whether casual or serious, that his voice, his eyes, his whole demeanor conveyed less than deep warmth, profound generosity, an optimism, a hopefulness that was pure Leon. When we were young, 'pure Leon' might include a dash of madcap charm, a directness, a boyish whimsy, a ruefulness, that belied his distinguished mathematical achievements. His style was not professorial. He was not into style or image—then or ever. Leon retained and presented an honesty, a disarming forthrightness, a genuineness, a profound generosity and sheer vitality that he carried with him all of his life…"

Soon afterwards, Leon went on to the first of what would be four sabbaticals at the Institute for Advanced Study, where he remained for 3 years (1954–1957) as an assistant to Arne Beurling, a permanent professor at the Institute and the man who took over Einstein's office. Leon also renewed a friendship from his City College days with Robert Aumann, who was also doing a postdoctorate at Princeton at that time.

It was during his first 2 years at Princeton, from 1954 to 1955, that Leon proved the fundamental theorem that would forever bear his name and later that of Malgrange as well, after French mathematician Bernard Malgrange independently proved the same theorem in 1955-1956. The Malgrange–Ehrenpreis theorem, which states that every nonzero differential operator with constant coefficients has a Green's function, was a foundation of the modern theory of differential equations that would serve as the basis for a range of theoretical and technological advances in the years to come.

Leon's presence in Princeton during these years proved to be crucial for the career of a younger friend, Hillel Furstenberg, who was a graduate student then, and some years later, took a position at the Hebrew University. At that time the graduate math department was a bulwark for the prevailing mathematical currents, with a clear inclination for the fashionable. Someone not entirely attuned to this would be less than comfortable pursuing his own line of research. Furstenberg describes his experience: "I was then experimenting with certain ideas which were later to prove fundamental for my work, but these deviated from the main thrust of activity in Fine Hall. Like every other mathematician, I needed someone to bounce ideas off, and Leon turned out to be the ideal partner—someone open to everything, willing to think deeply about just about anything, and having the ability to contribute with intelligence and insight to other people's problems. I think of Leon as my mathematical 'big brother.'"

In 1957, Leon went on to a 2-year teaching stint at Brandeis, followed in 1959 by his joining the teaching staff at Yeshiva University for 2 years. Then it was back to the Institute for another year (1961–1962), followed by his appointment in 1962 to full professor of mathematics to the Courant Institute at New York University. During his tenure at Courant, Leon lived on the NYU campus in Washington Square Village.

His NYU colleague, Sylvain Cappell, a raconteur of "Leon stories," recalled one particular moment during Leon's time at Courant Institute, when Institute administrator Jay Blaire, who had heard about the brilliance of this member of the mathematics faculty, headed over to meet him. He knocked on Leon's office and when a voice said, "please come in," Jay opened the door to behold a nearly empty office in which all the furniture was piled on itself in a corner. He later learned that this was because Leon had converted his new office into a handball court—driving Professor Donsker in the next office nuts with the ping! In the midst of this otherwise empty office was Leon standing on his head, a position he maintained during their entire meeting. At its end, Leon extended an upside-down arm to shake hands and asked Jay to kindly let himself out of the office and please close the door behind himself.

The year 1970 saw the publication of what Leon considered his "best work," his first major volume, *Fourier Analysis in Several Complex Variables*, in which he developed comparison theorems to establish the fundamentals of Fourier analysis and to illustrate their applications to partial differential equations. Leon began the volume by establishing the quotient structure theorem or fundamental principle of

Fourier analysis, then focused on applications to partial differential equations, and in the final section, explored functions and their role in Fourier representation.

Alan Taylor in his memorial essay, "Remembrances of Leon Ehrenpreis," recalled following Leon's suggestion to attend Courant for a postdoctoral year, which he did in 1968. That year, which Taylor described as "the most interesting and fun year of my professional life," Leon's student Carlos Berenstein was completing his doctoral thesis at Courant while helping Leon with the final editing of his Fourier Analysis volume. Meanwhile, Leon had moved to Yeshiva University, where he was giving a course on the book, so each Thursday,

> "Carlos and I would take the A train uptown to spend the day with Leon, attending his class and talking about mathematics. I really saw Leon's style of doing mathematics in that class. He was always interested in the fundamental reasons that theorems were true and in illustrative examples, but less interested in the details. It seemed to me that he could look at almost any problem in analysis from the point of view of Fourier analysis. Indeed, his book on Fourier analysis, in addition to presenting the proof of his most important contribution, the fundamental principle, contains chapters on general boundary problems, lacunary series, and quasianalytic functions… Leon was doing mathematics 100% of the time I spent around him and I think it was true always, especially when riding the train and in his jogging…."

While his appointment at Courant had been intended as a lifetime position, Leon received a "summons" from Dr. Belkin, president of Yeshiva University, to educate the "next generation" of Jewish academics. So, in 1968, 6 years after joining the NYU faculty, Leon returned to YU, where he would remain a member of the Belfer Graduate School faculty for 18 years—riding his bicycle through the dignified halls of academia, reuniting with his old friend "Flotz," after Leon encouraged Newman to join him on the YU mathematics faculty, and impacting upon hundreds of students—until the doors of the university's graduate school of arts & sciences were shut in 1984.

As a Jewish institution, Yeshiva also provided a fertile environment for Leon's synthesis of Talmudic and mathematical concepts. He taught a class entitled, "Modern Scientific and Mathematical Concepts in the Babylonian Talmud," and also introduced his calculus class with a page from the Talmud discussing the area of a circle as it relates to the size of a *sukkah,* a temporary booth built annually for the Jewish holiday of *Sukkot.* One of the students for his "Mathematics and the Talmud" class was undergraduate student Hershel Farkas. Hershel and his wife Sara, who would host both Leon and Ahava's first date and their wedding, would become among their dearest friends, the "family" waiting to welcome them home when their oldest child was born, to celebrate their greatest joys and share their major life moments. Indeed, over 40 years later, it was Hershel, just off the plane from Israel, whom Leon would plan to meet on August 15, 2010—a final mathematical conversation that never took place.

Yeshiva's new faculty member was also known for his rather laid-back attitude to the course schedule: One of Leon's students recalled his professor informing his class on the first day that while the course was scheduled for Tuesdays and Thursdays, he couldn't make it on Thursdays—and actually Tuesdays didn't work

for him either. They settled on Sunday afternoons for their weekly study of differential equations, complex analysis, and number theory.

There was the time that Leon informed his class that he would be running the New York City marathon that coming Sunday, so he might be a little late for class. True to his word, he completed the race, took a taxi uptown, showered in his nephew's dorm room and came to lecture. Leon also used to tell the story of his stint as a teacher of an undergraduate math class at Stern College, YU's women's college. This "favorite Leon story," which Peter Kuchment, of Texas A&M University, likes to relate often to his students, describes him teaching a calculus class to this new group of students. "As any good teacher would do," Kuchment tells, "he tried to lead his students, whenever possible, to the discovery of new things. So, he once said: 'Let us think, how could we try to define the slope of a curve?' 'What is there to think about?' was the reply from one smart student, 'it says on page 52 of our textbook that this is the derivative.' 'Well,' replied Leon, 'I haven't read till page 52 yet.' The result was that the class complained to the administration that they were given an unqualified teacher. So much for inspiring teaching; it can backfire!"

Meanwhile, in 1954, Leon's brother Seymour had gotten married, Leon himself had headed back to the Institute, and their parents had moved again, this time to the Brighton Beach section of Brooklyn. Leon described himself as "always in search of new vistas of knowledge," so now, at the age of 24, he took advantage of the opportunities in his family's new neighborhood to expand his Jewish textual knowledge. He bought himself a copy of the English translation of the Talmud. Leon used to read the English side of the page—and viewed himself as the very personification of a Torah scholar because he could quote from the Talmud—in English!—with ease. But he was still searching for a more intensive learning experience.

It was his mother who found the way. She asked the local kosher butcher who could teach her son and received the response that if he wanted to "study seriously" he should go to Brighton resident Rabbi Yehudah Davis. Leon headed off to Rabbi Davis, and upon seeing the long-bearded rabbi, assumed he would speak only Yiddish. But in fact, the American-born rabbi spoke perfect English, and upon hearing Leon's background, addressed him with a simple question: "Why does a negative times a negative equal a positive?" "Here I was," Leon would later tell Dr. Yitzchak Levine, a member of the Department of Mathematical Sciences at New Jersey's Stevens Institute of Technology, "a mathematician at the Institute for Advanced Study at Princeton and I could not answer his question. I still do not know why conceptually a minus times a minus is a plus—and this was not the only question about mathematics he asked which I could not answer!"

Teacher and student, renaissance men both, began to study together regularly, taking long walks on the boardwalk to discuss Jewish philosophy and the lessons to be learned from the lives of the great men of Jewish history. Three years later, when Leon took a position in Brandeis, Rabbi Davis had just been appointed as dean of a yeshiva in Boston. Leon lived in the yeshiva, and the two continued to study together. Then, when Leon returned to the Institute for the 1961–1962 academic year, he invited his Jewish studies mentor for a visit to the Faculty Tea Room,

introducing him to some of the greatest mathematicians and scientists of the day, including André Weil, with whom the rabbi conversed at great length. Later, Leon hosted a group of the yeshiva's students at the Institute as well.

Leon would later credit Rabbi Davis for having had "a great influence on me and my life," establishing the foundation to his approach to Torah learning. Certain concepts in Rabbi Davis's philosophy of analysis became well-known facets of Leon's own way of viewing Biblical texts, including the idea that no two Biblical terms are synonymous; rather, each apparently similar term actually carries with it an entirely unique connotation.

During the 1960s, after Leon had begun teaching at Courant, a friend suggested that for Jewish studies on the highest intellectual level, he should attend a class by Rabbi Moshe Feinstein. Rabbi Feinstein, considered the leading rabbinic authority of the twentieth century, had established his yeshiva, Mesivta Tiferes Jerusalem (MTJ), where the highest level of intellectual study took place in the least pretentious of environments, in the nearby Lower East Side neighborhood of Manhattan. It was the perfect study environment for Leon, who was described by many as having infinite patience for academic achievement but zero patience for bureaucratic convention.

Leon attended these classes with supreme dedication, even driving to New York from Princeton when he returned for additional semesters at the Institute. Within a few years—legend has it as a mere 5 years later—Leon had received his rabbinic ordination from Rabbi Feinstein, and remained his de facto advisor on scientific and technology issues until the famed authority on Jewish law passed away in 1986.

During his first marriage, to Ruth née Bers, daughter of the renowned mathematician and human rights activist Lipman Bers, Leon became the father of Ann (b. 1962) and Naomi (b. 1965, in Boston, during her father's sabbatical at Harvard; Naomi was the only one of Leon's children not born in New York City). Leon and Ruth had been introduced by Bers at a Jerusalem mathematics conference in 1960. They were married in June 1961, spending their first year of marriage at the Institute for Advanced Study. At Princeton, Leon developed close friendships with colleagues Bernie Dwork and Eli Stein, during a period Ruth later described as one in which "we all spoke freely about our families, laughed at ourselves and shared our concerns about the conditions of the world." They subsequently returned to NYU, with an intervening 1964–1965 sabbatical at Harvard, a year, Ruth recalled, that "was exciting. Leon was delighted to be surrounded by the mathematicians at Harvard and MIT whose families welcomed us warmly and shared their love of music and good food." The marriage ended in divorce in 1968.

In January 1972 Leon met Ahava Sperka, a native of Detroit, Michigan, the daughter of the Polish-born Rabbi Joshua and Canadian-born Yetta Sperka. Both Leon and Ahava were fond of recalling the immediate "kinship" of their first meeting: Leon picked up his date, and the two headed out to the event to which they had been invited, a "math party" at the Brooklyn home of Hershel and Sara Farkas. Leon commented, "you know, I've never actually been on time to any party before," to which Ahava replied, "neither have I." So they got out of the car to pass the time drinking tea until they could head to the event, comfortably late.

Thus began a "mathematical courtship," one that consisted primarily of evenings at "math parties," at which, Ahava would often reminisce, Leon would "wander off to 'talk math,' leaving me to fend for myself." It was a good preparation for a marriage in which "vacation" would come to mean "trip to another city, state, or country, where Leon would head off to his seminar, lecture, or conference, and leave me to entertain our growing family in yet another new place." Happily, Leon had found his soulmate, a kindred spirit who shared his dedication to principle, his love of adventure, and his yearning to explore new horizons.

As their mathematical social life continued, Ahava came to know many of the academics who played a role in Leon's life, including Lewis Coburn, graduate mathematics departmental chairman at YU, where Leon had begun teaching, and his wife, Charlene—who then discovered that their wedding had been officiated by Ahava's father. Then one day, as a change from the math party scene, Leon invited Ahava to New York's Metropolitan Museum of Art—and did she mind if they were joined by his coauthor on a new paper who had come in from Paris to work with him? Ahava's new "date" turned out to be, as she recalled, "a charming gentleman by the name of Paul Malliavin, a fifth-generation French aristocrat."

Leon, having done his Ph.D. at Columbia with French mathematician Chevalley, continued to be highly involved with the French school of mathematics. He spoke a fluent French and for many years spent several weeks each spring lecturing at the University of Paris, as well as at the Institut des Hautes Études Scientifiques (IHÉS) in Bures-sur-Yvette—invariably with a visit to the Malliavin home on the exclusive Isle de Paris, where one locked cabinet, nicknamed "Leon's kosher kitchen," would be opened upon their arrival.

It was Paul Malliavin as well who accompanied Ahava on a date with Leon in November of 1972, to the third-ever New York City marathon. This competition had begun as Central Park's "Earth Day Marathon" in 1970, a small race around Central Park that attracted few participants and even less media attention. After that first marathon, Leon recalled, "I said to a fellow runner 'I'll never do this again!' I had a mathematics conference at Princeton University the next day, and I was in such excruciating pain that I had to crawl out of bed to soak in the bathtub before I could get down the steps…"

The marathon grew substantially each year to become a global phenomenon that now attracts over 35,000 runners and two million spectators and turns all of the city's five boroughs into parts of the race track. Meanwhile, the group that gathered to watch Leon run would expand to include ever more of his growing family, as his wife and children—and in later years, his sons- and daughters-in-law, and grandchildren—would stand behind the barricade at their designated stop near the end of the marathon in Manhattan's Central Park, cheering wildly for "Aba" (in the spirit of the day, nearby spectators would eagerly join them in the call), as they peered at the thousands of runners passing by, eyes seeking that one familiar figure… who would suddenly appear, wave, pause long enough to be photographed, and then continue on to the finish line.

He would train throughout the year—"I find that I can train without wasting time," Leon once explained, "because I think about mathematics while I'm

running"—and in 37 years, he never missed a single marathon, despite a broken arm one year, a baby's due date another (he ran with a beeper that year, promising his wife that if summoned he would 'meet her at NYU's emergency room—after all it's right on the marathon route!'), and the commitment—which he kept—to officiate at the wedding of a fellow mathematician that same evening, completing his final 26-mile, 385-yard run at the age of 77.

A year after they met, Leon offered a romantic proposal to the woman he hoped would become his wife: "I'd really like to marry you," he explained, "but I just don't want the fuss and bother of preparing for a wedding." "So then let's just get married," his now-fiancée replied. And 10 days later, on January 25, 1973, in the Farkas home that had been the venue for their first date, they did.

The romantic times continued, as they headed off a few months later for a several-month-long honeymoon in the city of Kyoto, a distant setting in which the new couple, while eschewing the nonkosher Japanese cuisine (they subsisted there on bananas, rice, and peanut butter), thoroughly enjoyed Japanese cultural, botanical, and mathematical offerings. It was also the city in which a local physician informed Leon, in his best English, that "Mrs. Ehrenpreis would 'not, not' be giving birth in February of the following year" to the couple's first child.

It would be a number of years before they would return. Indeed it was only when that eldest child turned 15 that Leon and Ahava would take four of their children, five suitcases of clothing, and six boxes of matzah, granola, tuna fish, pasta, and other staples sufficient to feed six kosher-only individuals for 4 weeks, and fly across the horizon to the Land of the Rising Sun.

Takahiro Kawai, professor emeritus at Kyoto University, later described that first meeting between him and the man whose "fundamental principal" had been "a guiding principle for many young analysts, including me… When I first met Leon… I got the impression that he was a kind man of sincerity. The impression has continued until now.… I cannot forget the warm atmosphere full of intellectual curiousity, which led to our [joint] paper. Another incident… is that I once happened to notice that he had not taken anything [to eat] for two days and that the reason was that he was dubious about the date of the [Jewish] fast day in Kyoto due to the effect of the International Date Line.…"

One of the hallmarks of Leon's uniqueness was the fact that while he remained dedicated to every detail of his religious observance, he never saw that as an obstacle to being open to all; indeed his friends, colleagues, even those who met him only briefly, would reflect on the broad spectrum of his interests—from classical music to the great works of Western literature to Aramaic grammar—and of his openness to new ideas, new people, and new experiences within the consistent framework of his steadfast principles. As his close friend Hershel Farkas would later write about him, "Ehrenpreis's diversity extended way beyond mathematics. He was a pianist, a marathon runner, a talmudic scholar, and above all a fine and gentle soul."

Over the first two decades of their marriage, Leon, the man who had told Alan Taylor that he wished to have "as many children as a baseball team," would, in partnership with his wife, Ahava, raise their three girls and three boys: Nachama Yael (b. 1974), Raphael David (b. 1975), Akiva Shammai (b. 1977), Bracha Yehosheva

(Beth, b. 1983), Saadya (b. 1984), Yocheved Yetta (b. 1986). Their gang of eight—these six children along with their two older sisters—believed that summer and semester itineraries were built around sabbaticals, university schedules, and AMS conferences in Berkeley, Bowdoin, Jerusalem, and Japan, and were bewildered to discover that their elementary-school peers did not categorize their playdate options as "commutative," count to a "google" (that was before the Internet!), or dismiss errors as "trivial" in casual conversation. Each one went through the third-grade experience of informing his or her science teacher that her explanation of Copernican heliocentrism failed to take into account new perspectives on the solar system achieved by Einstein's Theory of Relativity.

The closing of Yeshiva University's graduate arts and science school in 1984 left Leon in a quandary: What would become his new mathematical "home"? His old friend Donald Newman rose to the occasion, encouraging Leon to join him at Temple University in Philadelphia, whose faculty Newman had joined in 1976. Leon formally accepted the offer of a position at Temple, where he remained for what would be his longest period in a single university: 26 years, until his death in 2010. So Flotz & Glock were back together again, each with his own style in their shared commitment to mathematics. Jane Friedman later recalled that

> "Dr. Newman paid more attention to the little details, so Dr. Ehrenpreis might be lecturing and say something like "the answer is this, or maybe this plus or minus one" and Dr. Newman would be in the back of the room yelling at him to get it right. You don't often think of professors yelling at each other over the heads of the students, but those two did it, with affection."

The two friends, who had been born just two months apart in 1930, would ultimately travel along the same path through high school, university, and their academic positions, their lives remaining intertwined until Newman's death on March 28, 2007, a loss greatly mourned by "Glockenspiel," who eulogized his lifelong friend at Newman's memorial service.

One of the Leon's Ph.D. students at Temple, Tong Banh, recalled the details of his mentor's years at Temple, depicting him best as "a person who preferred 'soft' solutions to human problems… I remember one day when we were approached by a beggar in the street. Leon immediately drew out a handful of quarters and handed them to him…" On the other hand, Banh emphasized, Leon was not at all "soft" when it came to mathematics, reviewing papers for potential publication and writing student letters of recommendation with a characteristic intellectual integrity and perfectionism that demanded the highest standards of academic achievement.

"But at Temple University," Banh described, "people mostly saw only the 'soft' part of his personality. He was extremely flexible in trying to accommodate everybody who ever needed anything from him."

Sylvain Cappell once asked Leon whether by usually taking the local train from NY to Philadelphia he didn't risk arriving late. Leon replied that, "in all the years I've been teaching at Temple, I've never arrived late." Sylvain couldn't help but wonder how it was that Leon, not known for his impeccable promptness, had achieved such a stellar punctuality record. Replied Leon: "Because class starts when I arrive."

In 1987, Leon and Bob Gunning of Princeton University directed the American Mathematical Society Theta Functions conference, which was held at Maine's Bowdoin College. Gunning later recalled their work together in a eulogy he wrote for Leon:

> The opportunity I had to work most closely with him was in organizing and managing the Theta Functions conference at Bowdoin College in the summer of 1987. I had experienced Leon's energy and enthusiasm before, and was not too surprised, although a bit overwhelmed, by the intensity with which he threw himself into organizing the conference schedules and the participants, as well as the AMS and NSF and who knows what foreign organizations for the participants coming from abroad; but it was an exhausting effort even to keep track of what we were doing. What did surprise me, although really it should not have, was the remarkable breadth of Leon's interests, and the depth with which he really understood what was going on in so many areas that the conference covered. I could not have found a better colleague to join in running a conference on that topic; and I am sure that I learned much more from Leon about so many aspects of theta functions than he did from me. Like so many other friends and colleagues, I shall miss his wild, but surprisingly often successful, ideas about how to approach problems, and his eagerness to talk about, and think about, a wide range of mathematics.

Two years later, in June 1989, Leon's student, Carlos Berenstein, and "grand-student," Daniele Struppa, organized a 60th birthday conference for him in the southern Italian coastal town of Cetraro. At the conference, entitled, "Geometrical and Algebraical Aspects in Several Complex Variables," Leon gave the keynote speech and a beautiful presentation on extension of solutions of partial differential equations, a topic that he had investigated for many years, and to which he made lasting contributions.

Struppa, who is today chancellor of California's Chapman College, recalled how Leon used to dine in his room, since the picturesque Calabria region did not have an available source for kosher food. So, the night that the participants wished to surprise the "birthday boy" with a formal dedication of the conference to him, they had to lure him down on a pretext to the dining room where the celebration awaited. He also remembered the conference as the time Leon asked Struppa's mother for help in having a uniquely designed candelabra, with ten branches (one for each member of the family), crafted as a gift for Ahava. The result, a one-of-a-kind—immensely heavy—Italian silver showpiece, was carried by Leon from Milan back to Brooklyn, to take a place of pride as his wife's Sabbath candelabra.

That same year, Leon attended the integral geometry and tomography conference in Arcata, California, where for the first time he met Peter Kuchment, who later recalled:

> It was my very first trip outside the former USSR, and it felt like being in a dream… Another shock during my first visit and my emigration soon afterwards, was that names like Leon Ehrenpreis… which obviously existed only on book covers, or at least referred to semi-gods somewhere well above this Earth, corresponded to mere mortals… Meeting Leon in Arcata was my first experience of this kind… Just like everyone else, I loved Leon from the first encounter. His unfailing cheerful disposition and his abundant eagerness to discuss any kind of mathematics at any time made every occasion we met feel like a holiday… Leon always liked to crack or to hear a good joke. He was smiling most of the time that I saw him. It was a joy to discuss with him not only mathematics, but also religion,

> music, or anything else. What made this even more enjoyable, was that in my experience he never imposed his opinions, beliefs or personal problems (and he unfortunately had quite a few) on others. It was relaxing to talk to him. He must have been a wonderful *Rebbe* [teacher/spiritual guide]...

He was indeed "a wonderful Rebbe," asserted Temple colleague and dear friend Marvin Knopp, recalling how, "when Leon arrived at our department, he walked into each person's office and asked what work he or she was doing. If he didn't find it interesting, he never returned—but if he did, he kept coming back over and over again." Throughout his years at the university, Leon—always with a mug of tea in his hand—could constantly be found encouraging, inspiring, talking, and teaching, playing a formative role in the development of his own department and the mathematical community of his time. "He was our mentor," Knopp described, "giving us projects to do and problems to solve, spreading enthusiasm and ideas every day, and inspiring our research. That's the kind of effect of he had—and not too many people have that kind of impact.

"Leon had a quality of walking in halfway through a lecture—and rapidly understanding the material far better than the lecturer himself. This happened to me once: he came into my talk after I had already covered the board with figures, saying something about the train being late—and within two minutes he was asking me questions I couldn't answer!"

Jane Friedman, one of the Leon's Ph.D. students at Temple, later eulogized her advisor, writing:

> "I have the career and the life that I do, only because of his help, his kindness and his support. And I am truly grateful to him. As we all know, he was a brilliant mathematician. I feel tremendously privileged to have studied with him and to have had the benefit of his deep insights. Dr. Ehrenpreis was not only an inspiration to me as a mathematician, he was inspirational as a person."

Later on, Jane described Leon as someone who

> "had an amazing gift for seeing the big picture, how concepts fit together in a deep way. He was able to understand mathematics in a way which could be transformative. This was a gift he gave his students—a vision of what it was to understand deeply, to see the forest and not the trees. I was inspired by him to always try to understand deeply, not superficially, and to get beyond the details. I was also inspired by him as a person, by his evident love for his wife and children, by his commitment to his community and by his joy in his family... Nonmathematicians and beginning students have a superficial view of mathematics; they have mostly experienced math as computation and symbol manipulation. Professor Ehrenpreis helped me grow beyond this beginner's view of math. I will never understand as much and as deeply as he does, but because of him I understand more and more deeply than I would otherwise."

Jane told his daughter that "your father got all the important things right and many of the nonimportant things wrong. He always knew which was which."

In March of 1992, Leon officiated at the wedding of his eldest daughter, the first of three daughters at whose weddings he would officiate. Immediately afterwards, he was confronted with what would become the long-term illness of his son Akiva,

a medical situation that would represent a major challenge to Leon and his family for years to come—although Leon, with his consummate optimism, never gave up hoping for his son's full recovery.

Peter Kuchment recalled Leon's frequent visits:

> It is well known that he was an avid runner and had run the NY marathon every year since its inception in 1970 till 2007. He also liked to run during his visits, so when he visited me in Wichita, Kansas, I would sometimes pick a room for him in a hotel seven miles away from the campus, with a sufficiently attractive route to run between the two. So, after his lecture, or just a working day, he would give me his things to take back to the hotel, while he would run. Every time I would meet him after the run, he would have some new ideas (and he had so many great ideas!) about the problem we were working on at the time. Once, when he came back and I was waiting for him in the hotel's lobby, the receptionist at the front desk asked him: "Did you really run all the way from the campus?" Leon's reply was: "What else could I do? He refused to give me a ride"—and he pointed at me. I think I lost all the receptionist's respect at that time...

On April 6 and 7, 1998, "Analysis, Geometry and Number Theory: A Conference Celebrating the Mathematics of Leon Ehrenpreis" took place in Philadelphia, under the auspices of Temple University and the National Science Foundation. The 2-day event culminated in an honorary banquet with Leon's entire family in attendance. The proceedings of the conference were published by the American Mathematical Society 2 years later.

During the decade from 1993 until its publication in 2003, Leon devoted himself to the writing of his second major work, *The Universality of the Radon Transform*. The title, his choice after deep consideration, was one he felt reflected his profound belief that "mathematics is poetry," as were the words he composed to his wife for the book's dedication:

> *Many are the*
> *Inspirations of the heart*
> ***But that borne by love***
> *Surpasses all the rest*

In this volume, he expanded upon the concept of the Radon transform, an area with wide-ranging applications to X-ray technology, partial differential equations, nuclear magnetic resonance scanning, and tomography. In covering such a range of topics, Leon focused on recent research to highlight the strong relationship between the pure mathematical elements and their applications to such fields as medical imaging.

Eric Todd Quinto, a friend and collaborator, referred to the book, to which he and Peter Kuchment wrote an appendix, as reflecting Leon's "emphasis on unifying principles." Quinto explained that in the book, Leon "developed several overarching ideas and used them to understand properties of the transforms, such as range theorems and inversion methods... The book draws connections between several fields, including complex variables, PDE, harmonic analysis, number theory, and distribution—all of which benefitted from his contributions over the years."

Leon was diagnosed with prostate cancer in 2003. However, he chose to reveal this information to no one outside his immediate family, because, he stated firmly, "I don't want people to view me as a sick person." Indeed, over the next few years, he maintained his regular routine: He continued to commute on the train to Temple, 3 hours each way; he traveled to conferences and seminars; he enjoyed the births of his grandchildren. He continued running the marathon until 2007, completing this 26-mile, 385-yard race for the last time at the age of 77.

In the summer of 2008, he was invited to attend the conference in honor of Jan Bowman's birthday in Stockholm. At the age of 78 he was an honored guest who was surrounded throughout the week by young scientists eager to hear his ideas. Two months later, he took what would be his last overseas trip, to Israel, where he found opportunities for mathematical tête-à-têtes while celebrating the birth of a granddaughter. So it was that Leon continued to live life to the fullest, reflecting, as Shlomo Sternberg would later describe, that "vitality that perhaps for us best describes Leon. The years passed; life transpired with its joys and sorrows. For Leon and his family, the sorrows were of such immensity that would otherwise crush anyone. But Leon bore his with unimaginable courage and responsibility. Courage that, we dare say, none of us could have possibly comprehended, let alone mustered. But despite it all, and no matter what transpired, Leon retained every bit of the vitality of our earlier years. His mathematical work continued. His, along with Ahava's, loving care and unstinting dedication to his family continued. His kindness and loyalty to us, his friends, continued. It was who he was."

Leon spent the Fall 2008 semester on sabbatical at Rutgers University, where he had, in the words of faculty member Steve Miller, "a big fan base," with students and faculty alike affording him a deep respect. Both before and after his sabbatical term, he spent quite a bit of time at Rutgers, working primarily with both Miller and Abbas Bahri. He was active in the nonlinear analysis and PDE seminars as well as in the number theory seminar that Miller ran.

Two years later, on Tuesday, April 20, 2010, at 1:40 pm, Leon was presenting what would ultimately be his final lecture, in Rutgers mathematics department room 705, giving a continuation of earlier talks in that seminar on analytically continuing complex functions in a strip in the complex plane. A few minutes into the talk, Leon collapsed: he had suffered a stroke. Bahri and Miller rushed him to the hospital, where, as Steve Miller recalled, "many of us waited hours even without a chance to see him, just to be near this great man."

Subsequently, with his usual optimism and force of character, Leon devoted himself to restoring his health, all with his characteristic good humor. Even then he continued to "talk math" and to challenge the idea of giving up teaching, determined to "never retire." Indeed he had not yet formally retired from his position as professor of mathematics at Temple University when, on August 16, 2010, having suffered heart failure, he passed away, at Sloan-Kettering Memorial Hospital in New York City.

Six years before Leon's death, his son, Akiva, whose lifetime of ill-health, beginning with the discovery of a brainstem tumor at the age of 14, remained a

relentless challenge which Leon consistently faced with the greatest optimism, had suffered a catastrophic choking episode that left him in a long-term coma. Leon, along with the entire family, had remained devoted to Akiva throughout this painful period. One year and two months after Leon's death, on October 23, 2011, Akiva too passed away.

One month after Leon's passing, Abbas Bahri of Rutgers knocked on the door of the home in Brooklyn that had been his primary residence for 30 years. In his hand was a copy of the newest issue of the journal *Advanced Nonlinear Studies* with the entry for an article, entitled "Microglobal Analysis," by Dr. Leon Ehrenpreis. To his very last day he had continued to think, to create, to develop new ideas, and to write and transmit those ideas for future generations; he would truly be, as Hershel Farkas later wrote, "sorely missed by the mathematical community as both a scholar and a gentleman."

Several months later, paying tribute to Leon at the Memorial Conference at Temple University held during the year after Leon's death, Bahri wrote:

> "There are several good mathematicians, as well as there are several important mathematicians. But the fundamental ones are few. Leon is one of them... Leon has passed away; but the influence of his mathematical work is just at its beginning. Leon, I felt, was different because he clearly has longed to be a deeper person, a person with a soul and with a quest for another world, for a better and different world.... As Leon Ehrenpreis starts to find his final place in history, these are the two fundamental facts that make him stand out among us: the importance and depth of his work in mathematics and, beyond this work, the constant search for another, a better and more moral world."

"Leon Ehrenpreis: A Mathematical Conference in Memoriam" took place at Temple University on November 15 and 16, 2010. The panel of speakers throughout the 2-day event included Charles Epstein, University of Pennsylvania; Erik Fornaess; University of Michigan; Rutgers faculty Xiaojun Huang, Henryk Iwaniec, and Francois Treves; Joseph Kohn and Eli Stein of Princeton; Temple professors Igor Rivin and Cristian Gutierrez, and Peter Sarnak of the Institute for Advanced Study. Perhaps the most powerful testament to all that he had been, as mathematician and as mentor, was expressed by one Ph.D. student: "The joy of solving a problem is gone," Tong Banh mourned, "because I cannot share the solution with Professor Ehrenpreis."

There is much more that could said about Dr. Leon Ehrenpreis, more elements to portray, more anecdotes to relate, more tales to tell. This man, who touched so many lives and shaped so much of modern mathematics, lived a personal and professional life that continues to impact, to inform, and to inspire. He truly was—and remains—the "stuff of stories," for the reason that, as Sylvain Cappell described:

> Part of what makes "Leon Stories" so memorable – and why mathematicians delight in them—is that Leon juggled two quite opposite approaches to rule and structures. To the common, nuisance strictures and structures of quotidian life, Leon paid singularly little attention. But he accorded unbounded respect and love for the structures of mathematics and Judaism, and combined these with unbounded human insight and responsiveness. We will treasure our "Leon Stories" and tell them to our students, but they can hardly convey the unbounded joy he'd shared with us.

Differences of Partition Functions: The Anti-telescoping Method

George E. Andrews

Dedicated to the memory of the great Leon Ehrenpreis.

Abstract The late Leon Ehrenpreis originally posed the problem of showing that the difference of the two Rogers–Ramanujan products had positive coefficients without invoking the Rogers–Ramanujan identities. We first solve the problem generalized to the partial products and subsequently solve several related problems. The object is to introduce the anti-telescoping method which is capable of wide generalization.

1 Introduction

At the 1987 A.M.S. Institute on Theta Functions, Leon Ehrenpreis asked if one could prove that

$$\prod_{j=1}^{\infty} \frac{1}{(1-q^{5j-4})(1-q^{5j-1})} - \prod_{j=1}^{\infty} \frac{1}{(1-q^{5j-3})(1-q^{5j-2})}$$

has nonnegative coefficients in its power series expansion without resorting to the Rogers–Ramanujan identities.

In [4], Rodney Baxter and I answered this question "sort of." Actually, the point of our paper was to show that if one begins trying to solve Ehrenpreis's problem, then there is a natural path to the solution which has the Rogers–Ramanujan identities as a corollary. Indeed, as we say there [4, p. 408]: "It may well be objected that we presented a somewhat stilted motivation. Indeed if [the Rogers–Ramanujan

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identities] were not in the back of our minds, we would never have thought to construct [the path to the solution of Ehrenpreis's problem]." Subsequently in 1999, Kadell [9] constructed an injection of the partitions of n whose parts are $\equiv \pm 2 \pmod 5$ into partitions of n whose parts are $\equiv \pm 1 \pmod 5$. Finally in 2005, Berkovich and Garvan [6, Sect. 5] improved upon Kadell's work by providing ingenious, injective proofs for an infinite family of partition function inequalities related to finite products (including Theorem 1 below).

In this chapter, we introduce a new method which mixes analytic and injective arguments. We illustrate the method on the most famous problem, Theorem 1. We note that Theorem 2 is also a direct corollary of [6, Sect. 5].

Theorem 1 (The Finite Ehrenpreis Problem, cf. [6]). *For $n \geq 1$, the power series expansion of*

$$\prod_{j=1}^{n} \frac{1}{(1-q^{5j-4})(1-q^{5j-1})} - \prod_{j=1}^{n} \frac{1}{(1-q^{5j-3})(1-q^{5j-2})}$$

has nonnegative coefficients.

We should note that the original question can be answered trivially if one invokes the Rogers–Ramanujan identities [5, p. 82] because

$$\begin{aligned}
&\prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-4})(1-q^{5n-1})} - \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-3})(1-q^{5n-2})} \\
&\quad = \left(1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}\right) \\
&\qquad \times \left(1 + \sum_{n=1}^{\infty} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)}\right) \\
&\quad = q + \sum_{n=2}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^{n-1})}, \qquad (1.1)
\end{aligned}$$

which clearly has nonnegative coefficients.

However, there is no possibility of proving Theorem 1 in this manner because there are no known refinements of the Rogers–Ramanujan identities fitting these finite products. A new method is required.

Our method of proof might be called "anti-telescoping." Namely, we want to write the first line of (1.1) as

$$\sum_{j=1}^{n} (P_j - P_{j-1}) \qquad (1.2)$$

where each P_i is a finite product with

$$P_0 = \prod_{j=1}^{n} \frac{1}{(1-q^{5j-3})(1-q^{5j-2})}$$

and

$$P_n = \prod_{j=1}^{n} \frac{1}{(1-q^{5j-4})(1-q^{5j-1})}.$$

We construct the P_i so that they gradually change from P_1 to P_n. The proof then follows from an intricate, term-by-term analysis of (1.2).

In Sect. 2, we construct (1.2) and provide some analysis of the terms. In Sect. 3, we provide an injective map of partitions to show that each term of the constructed (1.2) has at most one negative coefficient. From there the proof of Theorem 1 is given quickly in Sect. 4.

We wish to emphasize that anti-telescoping is applicable to many problems of this nature. To make this point, we provide three further examples.

Theorem 2 (Finite Göllnitz-Gordon). *For $n \geq 1$, the power series expansion of*

$$\prod_{j=1}^{n} \frac{1}{(1-q^{8j-7})(1-q^{8j-4})(1-q^{8j-1})} - \prod_{j=1}^{n} \frac{1}{(1-q^{8j-5})(1-q^{8j-4})(1-q^{8j-3})}$$

has nonnegative coefficients.

This theorem falls to the anti-telescoping method much more easily than the finite Ehrenpreis problem (Theorem 1).

Theorem 3 (Finite little Göllnitz). *For $n \geq 1$, the power series expansion of*

$$\prod_{j=1}^{n} \frac{1}{(1-q^{8j-7})(1-q^{8j-3})(1-q^{8j-2})} - \prod_{j=1}^{n} \frac{1}{(1-q^{8j-6})(1-q^{8j-5})(1-q^{8j-1})}$$

has nonnegative coefficients.

This theorem requires a rather intricate application of anti-telescoping. We have chosen it to illustrate the breadth of this method.

We note that the partial products in Theorem 2 are from the Göllnitz-Gordon identities [2, (1.7) and (1.8) pp. 945–946] and the partial products in Theorem 3 are from identities termed by Alladi, The Little Göllnitz identities, [7, Sätze 2.3 and 2.4, pp. 166–167] (cf. [3, pp. 449–452]).

We conclude our applications of anti-telescoping by proving a finite version of differences between partition functions from the Rogers–Ramanujan–Gordon theorem ([8], cf. [1]). Again, the proof goes without difficulty; however, a few cases must be excluded including the result in Theorem 1.

Theorem 4 (Finite Rogers–Ramanujan-Gordon). *For* $\frac{k}{2} > s > r \geq 1$ *and* $n \geq 1$, *the power series expansion of*

$$\prod_{\substack{j=1 \\ j \not\equiv 0, \pm s \pmod k}}^{kn} \frac{1}{1-q^j} \quad - \quad \prod_{\substack{j=1 \\ j \not\equiv 0, \pm r \pmod k}}^{kn} \frac{1}{1-q^j}$$

has nonnegative coefficients except possibly in the case s prime and $s = r + 1$ *and* $k = 3r + 2$.

The final section of this chapter provides a number of open problems.

2 Anti-telescoping

In this short section, we construct the telescoping sum (1.2). Namely,

$$P_j = \frac{1}{(q, q^4; q^5)_j (q^{5j+2}, q^{5j+3}; q^5)_{n-j}} \tag{2.1}$$

where

$$(a; q)_s = (1-a)(1-aq)\cdots(1-aq^{s-1}),$$

and

$$(a_1, a_2, \ldots, a_r; q)_s = \prod_{i=1}^{r} (a_i; q)_s.$$

Clearly,

$$P_n = \frac{1}{(q, q^4; q^5)_n}$$

and

$$P_0 = \frac{1}{(q^2, q^3; q^5)_n}.$$

So,

$$\frac{1}{(q, q^4; q^5)_n} - \frac{1}{(q^2, q^3; q^5)_n} = \sum_{j=1}^{n} (P_j - P_{j-1}). \tag{2.2}$$

We let

$$T(n, j) := P_j - P_{j-1}.$$

So for $1 \leq j \leq n$,

$$\begin{aligned} T(n,j) &= \frac{(1-q^{5j-2})(1-q^{5j-3})-(1-q^{5j-4})(1-q^{5j-1})}{(q,q^4;q^5)_j\,(q^{5j-3},q^{5j-2};q^5)_{n+1-j}} \\ &= \frac{q^{5j-4}(1-q)(1-q^2)}{(q,q^4;q^5)_j\,(q^{5j-3},q^{5j-2};q^5)_{n+1-j}} \\ &= \frac{q^{5j-4}(1-q^2)}{(q^6;q^5)_{j-1}(q^4;q^5)_j\,(q^{5j-3},q^{5j-2};q^5)_{n+1-j}} \end{aligned} \tag{2.3}$$

and for $2 \leq j \leq n$

$$\begin{aligned} T(n,j) = q^{5j-8}&\left(\frac{q^4}{1-q^4}-\frac{q^6}{1-q^6}\right) \\ &\times\frac{1}{(q^{11};q^5)_{j-2}(q^9;q^5)_{j-1}(q^{5j-3},q^{5j-2};q^5)_{n+1-j}}. \end{aligned} \tag{2.4}$$

So for $n \geq 1$

$$T(n,1) = \frac{q}{(1-q^3)(1-q^4)(q^7,q^8;q^5)_{n-1}}, \tag{2.5}$$

for $n \geq 2$

$$T(n,1)+T(n,2) = \frac{q+q^4+q^5+q^6+q^9}{(1-q^6)(1-q^7)(1-q^8)(1-q^9)(q^{12},q^{13};q^5)_{n-2}}, \tag{2.6}$$

for $n \geq 3$,

$$\begin{aligned} &T(n,1)+T(n,2)+T(n,3) \\ &\quad= \frac{q+q^{11}+q^{21}}{(1-q^8)(1-q^9)(1-q^{11})(1-q^{14})(q^{12},q^{13};q^5)_{n-2}} \\ &\qquad+\frac{q^4+q^{11}}{(1-q)(1-q^9)(1-q^{11})(1-q^{14})(q^{12},q^{13};q^5)_{n-2}}, \end{aligned} \tag{2.7}$$

and for $n \geq 4$

$$\begin{aligned} &T(n,1)+T(n,2)+T(n,3)+T(n,4) \\ &\quad= \frac{q+q^{12}}{(1-q^3)(q^{12};q)_3(q^{16};q)_4(q^{22},q^{23};q^5)_{n-4}} \\ &\qquad+\frac{(2q^{11}+q^{21})(1+q^3+q^6+q^9+q^{12}+q^{15}+q^{18})}{(q^{11};q)_4(q^{16};q)_4(q^{22},q^{23};q^5)_{n-4}} \end{aligned}$$