Fields Institute Communications 62

The Fields Institute for Research in Mathematical Sciences

Chris Miller
Jean-Philippe Rolin
Patrick Speissegger
Editors

Lecture Notes on O-Minimal Structures and Real Analytic Geometry

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Fields Institute Communications

VOLUME 62

The Fields Institute for Research in Mathematical Sciences

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Lecture Notes on O-minimal Structures and Real Analytic Geometry

The Fields Institute for Research
FIELDS in the Mathematical Sciences

Springer

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ISSN 1069-5265 ISSN 2194-1564 (electronic)
ISBN 978-1-4614-4041-3 ISBN 978-1-4614-4042-0 (eBook)
DOI 10.1007/978-1-4614-4042-0
Springer New York Heidelberg Dordrecht London

Library of Congress Control Number: 2012939429

Mathematics Subject Classification (2010): 03C64, 14P15, 26A12, 26A93, 32C05, 32S65, 34C08, 34M40, 37S75, 58A17

Cover illustration: Drawing of J.C. Fields by Keith Yeomans

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface

The notes in this volume were produced in conjunction with the Thematic Program in O-minimal Structures and Real Analytic Geometry, held from January to June 2009 at the Fields Institute. Among the activities of our thematic program were three graduate courses, offered to participants and to graduate students from universities in the Greater Toronto Area. Each of these courses was, in turn, split into three modules, and most of these modules were taught by different instructors. Five of the six contributions to this volume arose from the modules taught by the authors: Felipe Cano on the resolution of singularities of vector fields; Chris Miller on o-minimality and Hardy fields; Jean-Philippe Rolin on the construction of o-minimal structures from quasianalytic classes; Fernando Sanzon non-oscillatory trajectories; and Patrick Speissegger on pfaffian sets. The sixth contribution, by Antongiuglio Fornasiero and Tamara Servi, is an adaptation of Wilkie's construction of o-minimal structures from total C^∞-functions to the nonstandard setting. Their adaptation was carried out concurrently with our program, and the resulting notes fit in naturally with the pfaffian portion of our lectures.

There are only a few dependencies between the contributions: Miller's is used in both Rolin's and Speissegger's, and Rolin's is used in Sanz's. In addition, familiarity with the basics is assumed for o-minimality (van den Dries [4] and Miller and van den Dries [5]) and semianalytic and subanalytic sets (Bierstone and Milman [2]). Further recommended reading are Marker [3] on model theory (basic aspects of which are used in Miller's notes) and Balser [1] on Borel-Laplace summation (used in Sanz's notes).

We thank the Fields Institute for the generous funding provided for our program, and we thank its very competent and helpful staff for making our stay there productive and very enjoyable. Participation of several US-based graduate students and junior postdoctoral researchers was partially funded by NSF Special Meetings Grant DMS-0753096.

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Blowings-Up of Vector Fields

Felipe Cano

Abstract A new proof of the reduction of singularities for planar vector fields is presented. The idea is to adapt Zariski's local uniformisation method to the vector field setting.

Mathematics Subject Classification (2010): Primary 32S65, Secondary 37F75

Introduction

These notes cover part of a course taught at the Fields Institute in January 2009, as part of the Thematic Program on O-minimal Structures and Real Analytic Geometry. I try to introduce the reader to a new proof of the reduction of singularities for vector fields in dimension two.

What is the reason for giving this new proof? Indeed, the original proof of 1968 given by Seidenberg [36] is complete and does not need much tweaking to be useful for most applications. Other proofs in dimension two were published, among them Giraud [20, 21], van den Essen [39], Dumortier [19] and one by myself [7], where I tried to recover Hironaka's way of reducing singularities.

In these notes, the idea is to recover the local uniformization method due to Zariski [42, 43], which dates back to 1940 (see also Vaquié [40] for a discussion of Zariski's method). The proof I present here can be generalized at least to dimension three, as done in joint work in progress with Roche and Spivakovsky [13, 14]. Also, as I explain later, the result in dimension three gives a global result as an application of Zariski's method.

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C. Miller et al. (eds.), *Lecture Notes on O-minimal Structures and Real Analytic Geometry*, Fields Institute Communications 62, DOI 10.1007/978-1-4614-4042-0_1,

For a more general elementary exposition of the theory of singular holomorphic foliations the reader may look at Camacho and Lins-Neto [5], Cano and Cerveau (Introduction aux feuilletages singuliers, Unpublished lecture notes available from the authors) and Brunella [3].

Historical note. Let us give a brief historical overview of the proof of reduction of singularities for vector fields in dimensions two and three. First of all, let us indicate that there are no known results in dimension greater than or equal to four, except for the specific case of absolutely isolated singularities (see Camacho et al. [6]).

The original proof of Seidenberg is based on the behavior of the multiplicity $i(C_1, C_2; p)$ of the intersection of two plane curves C_1 and C_2 at a point p under blowing-up. More precisely, Noether's formula states that

$$i(C_1, C_2; p) = m_p(C_1)m_p(C_2) + \sum_{p' \in E} i(C_1', C_2'; p'),$$

where E is the exceptional divisor of the blowing-up with center $\{p\}$, C_1', C_2' are the strict transforms of the curves C_1, C_2 and $m_p(C)$ denotes the multiplicity of the curve C at the point p. Van den Essen's, Dumortier's and Giraud's proofs follow this same idea; Dumortier's is specific to the real case and Giraud's to the framework of Algebraic Geometry in positive characteristic.

The use of the multiplicity of the intersection as a main invariant of control is based on the fact that the singularities considered are isolated, and hence the multiplicity of the intersection of the coefficients is finite. For vector fields this invariant is called *Milnor number* and generalizes, in the Hamiltonian case, the usual Milnor number of a function. If we can assure that the Milnor number remains finite under any blowing-up, then the method generalizes to higher dimension without obstruction. This is the case for absolutely isolated singularities in any dimension, as shown in our work with Camacho and Sad.

If one wants to look at the general case in dimension three, it is necessary to develop a method not based on control of the Milnor number. In [7], I gave a proof based on the ideas of Hironaka. This method can be interpreted as follows in dimension two: first, we need an invariant acting as the *Hilbert-Samuel function*; this invariant is the logarithmic multiplicity of the vector field, together with a description of a finite list of types. Second, we need maximal contact, which acts as a kind of reduction of the dimension from two to one. Finally, we consider a more specific invariant of control for the case of maximal contact, namely, the contact exponent associated to a Hironaka's characteristic polyhedron (in this case just a line).

More precisely, the first result in ambient dimension three was given by myself in [9, 15], in the form of a positive answer to Hironaka's game. This result is of a local nature, where we allow formal centers of blowings-up. In some sense, it is a strong local uniformization result, but it has the disadvantage that formal (non-convergent) centers of blowings-up are used. The statement is as follows: we start with the germ of a vector field at $(\mathbb{C}^3, 0)$, more precisely with the germ $\mathcal{L}$ of the foliation induced by the vector field. To this vector field, we associate a *logarithmic multiplicity* at a point p, the smallest multiplicity of the coefficients of the vector field expressed in a logarithmic way with respect to a normal crossings divisor (that is, we "force" the components of the divisor to be invariant). For instance, if p is the origin, the divisor is defined by $\prod_{i=1}^{e} x_i = 0$ and the vector field is given by

$$\xi = \sum_{i=1}^{e} a_i(x)x_i \frac{\partial}{\partial x_i} + \sum_{i=e+1}^{n} a_i(x)\frac{\partial}{\partial x_i},$$

then the corresponding **logarithmic** (or **adapted**) **multiplicity** is the minimum of the multiplicities of the coefficients $a_i(x)$ at the origin, for $i = 1, 2, \ldots, n$. We say that the point p is a **log-elementary** singularity of the vector field, if the logarithmic multiplicity at p is less than or equal to 1. Now we play Hironaka's desingularization game between two players A and B (where "A" is typically interpreted as "Abhyankar" in recognition of the latter's contribution to the understanding of singularities):

1. If p is log-elementary for the vector field, player A wins; otherwise, he chooses a formal center of blowing-up.
2. Player B chooses a point p' in the preimage of p under the blowing-up.
3. The game restarts with p' in place of p.

A **winning strategy** for player A is a decision method that makes sure the game stops in a finite number of steps, independently of the choices made by player B. In [15], I presented a winning strategy for player A. In [9], I extended this strategy to so-called **elementary singularities**, that is, singularities with non-nilpotent linear part.

At this point, the problems in dimension three are the following:

(a) To obtain a result where the centers of blowings-up are analytic; that is, the geometry of the ambient space is not destroyed by a blowing-up with a formal center.
(b) To obtain a global result. Instead of blowings-up with centers adapted to the point chosen by player B, try to obtain a global morphism such that all the points on the exceptional divisor are log-elementary or, even better, elementary.

A version of Hironaka's game can be played in the case of a non-oscillatory trajectory of the germ at the origin of real vector field ξ in $\mathbb{R}^3$ (see Sanz [35]). Let γ be a non-oscillatory trajectory of ξ that approaches to the origin, that is

$$\lim_{t\to\infty} \gamma(t) = 0.$$

We assume that γ is **non-oscillatory** (that is, γ crosses any analytic hypersurface at most finitely many times) and that γ is not contained in any analytic hypersurface. Then γ acts as player B in the following way: player A chooses a blowing-up with center the origin or a nonsingular analytic curve through the origin. The lifting of γ accumulates at only one point p' of the exceptional divisor: otherwise, we could produce an algebraic hypersurface that γ crosses infinitely many times, contradicting the non-oscillatory property of γ.

In joint work with Moussu and Rolin [12], we solved Hironaka's game in the case where player B is given by a non-oscillatory trajectory of the germ at the origin of a vector field in $\mathbb{R}^3$. Since we were working over the real field, we were interested in applications that are stable under ramifications, so we allowed ourselves to do ramifications; nevertheless, all our centers of blowings-up were analytic. In this way we obtained a local, non-birational reduction of singularities method over the real field that finishes in elementary (not just log-elementary) singularities.

The techniques used in [12] have a natural interpretation in terms of Zariski's method for the local uniformization. Indeed, a non-oscillatory trajectory γ of ξ induces an identification of the field of rational functions (even of meromorphic functions) in three variables with a Hardy field, via the substitution morphism

$$\frac{F(X,Y,Z)}{G(X,Y,Z)} \mapsto \frac{F(\gamma(t))}{G(\gamma(t))}.$$

This Hardy field has a natural valuation whose centers (in the sense of Zariski) are given by the accumulation points of γ under blowing-up. Thus, player B is in this case a valuation that chooses, at each step, the center of the valuation in the corresponding model of the field of rational functions. This is precisely Zariski's point of view for the local uniformization. The difference between his point of view and Hironaka's is that, in Zariski's case, we know the nature of player B (a valuation), and we can do arguments using this particular nature of player B.

The need for ramifications was evident in [12] for passing from a special nilpotent situation to an elementary case. More precisely, an example produced by F. Sanz and F. Sancho shows that the latter is not possible in general without using formal, but nonconvergent, blowings-up. Their example is the following:

$$\xi = x\left(x\frac{\partial}{\partial x} - \alpha y\frac{\partial}{\partial y} - \beta z\frac{\partial}{\partial x}\right) + xz\frac{\partial}{\partial y} + (y - \lambda x)\frac{\partial}{\partial z}.$$

This example is discussed in detail in the introduction of Panazzolo [30]. Let me just mention that, for this ξ, using a blowing-up with center a formal ξ-invariant curve transverse to $\{x = 0\}$, one obtains elementary singularities.

We are currently working, with Roche and Spivakovsky, on a local uniformization result, in the sense of Zariski, for a general situation of algebraic geometry in characteristic zero. We obtain, via a birational transformation along a given valuation, log-elementary singularities in ambient dimension three. Moreover, these log-elementary singularities satisfy a list of axioms given by Piltant [32] that allow us to globalize the local uniformization in an ambient space of dimension three. This result represents an axiomatic version of Zariski's gluing of local uniformizations in dimension three [42, 43]. As a consequence, we obtain a global and birational way of reducing singularities in dimension three, such that the final singularities are log-elementary.

In these notes, we present the two-dimensional version of this joint work, in order to introduce the reader to the key ideas of our method.

To finish this historical note, let us point out that log-elementary singularities are far from being elementary; for instance, nilpotent singularities are always log-elementary. In fact, Panazzolo's thesis [29] deals with transforming nilpotent to elementary in a global non-birational way, via real transformations of "quasi-homogeneous" type. This important work showcases just how far log-elementary signularities are from begin elementary.

The most complete result on reduction of singularities for vector fields in dimension three is Panazzolo's [30]. This is a global result, via non-birational transformations, that obtains elementary singularities in the real case. His techniques of control and globalization in [30] are close to Hironaka's; but he also uses weighted blowings-up, with weights associated to the Newton polyhedron of the vector field. These latter ideas are, arguably, the reason for the relative simplicity of his work.

More recently, as of May 2011, some new results on these matters have appeared: first, the valuation-theoretic arguments in dimension three in [14] can be generalized to any dimension in order to get maximal contact or resonance. Both these cases represent a reduction, in a certain sense, of the ambient dimension of the problem. Second, there is a preprint of McQuillan and Panazzolo in which they apply the techniques of [29] to obtain a three-dimensional reduction of singularities for vector fields in ambient dimension three, in the framework of stack theory.

Applications. A classical application of the reduction of singularities of vector fields is the theorem of Camacho and Sad [4], which proves the existence of an invariant holomorphic curve at a singularity of a holomorphic vector field in dimension two. This result was conjectured by R. Thom, based on the intuition that the invariant hypersurfaces should "organize" the dynamics. Their proof relies on reduction of singularities and the behavior of an index, now known as the *Camacho-Sad index*. A very short proof of this result may be found in [8].

In dimension two, the reduction of the singularities for vector fields has been a central result, providing an algebraic skeleton in the study of holonomy, formal and analytic classification, deformation, integrability, etc. Introductions to these topics can be found in [16, 17, 26–28].

In dimension three, fewer applications are known, due of course to the difficulties of the result itself. There is a counterexample to the existence of an invariant analytic curve, found by Gómez-Mont and Luengo [22], based on the behavior under blowing-up of elementary singularities. Besides the geometric study of oscillation presented in [35], I would like to mention a remark of Brunella [2] that shows that any real vector field in dimension three, with an isolated singularity at the origin, has at least one trajectory arriving at or exiting from the origin.

The reader may look at the references [10, 11, 18, 23, 31, 33, 34, 38, 41] as a small seletion of papers corresponding applications of reduction of singularities and some of the technics introduced in these notes.

1 Vector Fields and Blowings-Up

Germs of vector fields. The *ambient space* M is for us of one of the following types. We can have an ambient space which is a *real analytic variety*, that is M is described by a collection of real charts such that the compatibility conditions of the charts are real analytic applications. We can also consider the case that M is a *complex analytic variety*, with the same definition as before, except for the fact that the compatibility conditions of the charts are complex analytic (holomorphic) applications. We also consider the case that $M \subset \mathbb{P}^N_{\mathbb{C}}$ is an irreducible complex projective variety, where we can eventually have singular points. Most of the properties we are going to consider have *local nature* and thus they can be explained in terms of the local ring $\mathcal{O}_{M,p}$ of the germs of functions at a point p of M, whose maximal ideal $\mathcal{M}_{M,p}$ is given by the germs of functions $f \in \mathcal{O}_{M,p}$ such that $f(p) = 0$.

Since we work either over the real numbers or over the complex numbers, we denote $k = \mathbb{R}$ or $k = \mathbb{C}$, depending on the cases we are considering.

By definition, the *germs of vector field at* $p \in M$ are the k-derivations of the local ring $\mathcal{O}_{M,p}$. That is a germ of vector field is a map

$$\xi : \mathcal{O}_{M,p} \to \mathcal{O}_{M,p}$$

which is a homomorphism of k-vector spaces and satisfies to the Leibnitz rule

$$\xi(fg) = f\xi g + g\xi f.$$

We denote $\mathrm{Der}_k\mathcal{O}_{M,p}$ the set of germs of vector fields at p. It has a natural structure of k-vector space and moreover, it is a $\mathcal{O}_{M,p}$-module, where we have $(f\xi)g = f(\xi g)$.

The set of *tangent vectors* T_pM at p is the set of "centered derivations". That is, a tangent vector at p is a map

$$v : \mathcal{O}_{M,p} \to k$$

which is a homomorphism of k-vector spaces and satisfies to the "centered" Leibnitz rule

$$v(fg) = f(p)(vg) + g(p)(vf).$$

Obviously, any germ ξ of vector field at p induces a tangent vector

$$\xi|_p \in T_pM,$$

just by putting $\xi|_p f = (\xi f)(p)$. The *tangent space* T_pM has a natural structure of k-vector space.

Assume that p is a nonsingular point of M. This is always the case when M is a real or complex analytic variety. Then the maximal ideal $\mathcal{M}_{M,p}$ of $\mathcal{O}_{M,p}$ has a set of generators $x_1, x_2, \ldots, x_n$, where n is the dimension of M. Depending on the context, this set of generators is called *regular system of parameters* or *system of* *centered local coordinates*. There are particular germs of vector field that we denote $\partial/\partial x_i$, for $i = 1, 2, \ldots, n$ defined by the properties

$$\frac{\partial}{\partial x_i}(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

In fact, we obtain in this way a basis of the free $\mathcal{O}_{M,p}$-module $\mathrm{Der}_k \mathcal{O}_{M,p}$. So, any germ of vector field ξ has a unique expression as

$$\xi = a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + \cdots + a_n \frac{\partial}{\partial x_n},$$

where $a_1, a_2, \ldots, a_n \in \mathcal{O}_{M,p}$. Also, a k-basis of the tangent space $T_p M$ is given by $\partial/\partial x_i|_p$, for $i = 1, 2, \ldots, n$. In particular the map $\xi \mapsto \xi|_p$ is surjective.

Let us consider representatives X_i of the germs x_i for $i = 1, 2, \ldots, n$. There is an open neighborhood U of p satisfying the following property:

> For any point $q \in U$ there is a unique $\mathbf{q} = (q_1, q_2, \ldots, q_n) \in k^n$ such that the functions $X_1 - q_1, X_2 - q_2, \ldots, X_n - q_n$ define a regular system of parameters of $\mathcal{O}_{M,q}$.

In view of this property, we can consider *vector fields* defined in such neighborhoods U as expressions

$$\mathcal{V} = \sum_{i=1}^{n} A_i \frac{\partial}{\partial X_i}$$

where the $A_1, A_2, \ldots, A_n$ are functions defined in U. Obviously such a vector field $\mathcal{V}$ induces a germ of vector field $\mathcal{V}_q$ at each $q \in U$ in an evident way, as well as tangent vectors $\mathcal{V}(q) \in T_q M$.

Definition 1.1. A germ of vector field $\xi \in \mathrm{Der}_k \mathcal{O}_{M,p}$ is *non-singular* if p is a non-singular point of M and $\xi(\mathcal{M}_{M,p})$ is not contained in $\mathcal{M}_{M,p}$.

In terms of coordinates, this is equivalent to say that $\xi(x_i)(p) \neq 0$ for some of the parameters x_i. The next classical result justifies the interest of having a non-singular germ of vector field

Theorem 1.2 (Rectification). *Let $\xi \in \mathrm{Der}_k \mathcal{O}_{M,p}$ be a non-singular germ of vector field and let us assume that the ambient space M is a real or complex analytic variety. There is a choice of local coordinates $x_1, x_2, \ldots, x_n$ such that $\xi = \partial/\partial x_1$.*

Blowings-up of ambient space. Let $p \in M$ be a nonsingular point of the ambient space M. The blowing-up of M with center p is a morphism $\pi : M' \to M$ that we describe in this section.

Blowing-up of the projective space. Let us consider first the case where $M = \mathbb{P}^n_k$ is the n-dimensional projective space. Take a projective hyperplane $\Delta_\infty \subset \mathbb{P}^n_k$ such that $p \notin \Delta_\infty$. Now, we can choose homogeneous coordinates $[X_0, X_1, \ldots, X_n]$ in $\mathbb{P}^n_k$ such that $p = [1, 0, 0 \ldots, 0]$ and $\Delta_\infty = \{X_0 = 0\}$. Note that the points in Δ_∞

are of the form $[0, X_1, X_2, \ldots, X_n]$ and hence $[X_1, X_2, \ldots, X_n]$ can be considered as being homogeneous coordinates for Δ_∞. Let us denote by

$$\lambda : \mathbb{P}^n_k \setminus \{p\} \to \Delta_\infty$$

the linear projection defined by $\lambda(q) = (p+q) \cap \Delta_\infty$, where $p+q$ is the projective line through p and q. In terms of homogeneous coordinates, we have

$$\lambda([X_0, X_1, X_2, \ldots, X_n]) = [X_1, X_2, \ldots, X_n].$$

Let $G(\lambda)$ be the graph of λ and consider the topological closure

$$\overline{G(\lambda)} \subset \mathbb{P}^n_k \times \Delta_\infty.$$

The first projection $\pi : \overline{G(\lambda)} \to \mathbb{P}^n_k$ is by definition the *blowing-up* of $\mathbb{P}^n_k$ with center p. Let us note that the equations of $\overline{G(\lambda)}$ in homogeneous coordinates $[X_0, X_1, \ldots, X_n]$ for $\mathbb{P}^n_k$ and $[Y_1, Y_2, \ldots, Y_n]$ for Δ_∞ are

$$X_i Y_j = X_j Y_i; \text{ for } i, j = 1, 2, \ldots, n.$$

We see that $\overline{G(\lambda)} \setminus \pi^{-1}(p) = G(\lambda)$ and hence π defines an isomorphism

$$\pi : \overline{G(\lambda)} \setminus \pi^{-1}(p) \to \mathbb{P}^n_k \setminus \{p\}.$$

Moreover, there is an identification between $\pi^{-1}(p)$ and Δ_∞. We say that $\pi^{-1}(p)$ is the *exceptional divisor* of π and hence each of its points corresponds to a line through p.

The transformed space $\overline{G(\lambda)}$ is a nonsingular variety. To see a chart decomposition of it, we write

$$\overline{G(\lambda)} = G(\lambda) \cup \pi^{-1}(\mathbb{P}^n_k \setminus \Delta_\infty) = \pi^{-1}(\mathbb{P}^n_k \setminus \{p\}) \cup \pi^{-1}(\mathbb{P}^n_k \setminus \Delta_\infty).$$

Now, we already know that $\pi^{-1}(\mathbb{P}^n_k \setminus \{p\})$ is identified with the open set $\mathbb{P}^n_k \setminus \{p\}$ of the projective space $\mathbb{P}^n_k$. To describe $\pi^{-1}(\mathbb{P}^n_k \setminus \Delta_\infty)$, let us first recall that there is an identification

$$\mathbb{P}^n_k \setminus \Delta_\infty \leftrightarrow \mathbb{A}^n_k = k^n,$$

given in coordinates by $[1, x_1, x_2, \ldots, x_n] \leftrightarrow (x_1, x_2, \ldots, x_n)$. Now, we cover $\pi^{-1}(\mathbb{A}^n_k)$ by charts $\pi^{-1}(\mathbb{A}^n_k) = \bigcup_{j=1}^n U_j$ with

$$U_j = \pi^{-1}(\mathbb{A}^n_k) \cup \{Y_j \neq 0\}.$$

Each U_j has a coordinate mapping

$$\phi_j : U_j \to \mathbb{A}^n_k; \quad (\mathbf{x}, [\mathbf{Y}]) \mapsto (x_1^{(j)}, x_2^{(j)}, \ldots, x_n^{(j)}),$$

where $x_j^{(j)} = x_j$ and $x_i^{(j)} = Y_i/Y_j$ for $i \neq j$. In particular, the blowing-up π in the charts U_j has the equations

$$(x_1^{(j)}, x_2^{(j)}, \ldots, x_n^{(j)}) \mapsto (x_1, x_2, \ldots, x_n) \in \mathbb{A}_k^n,$$

where $x_j = x_j^{(j)}$ and $x_i = x_i^{(j)} x_j^{(j)}$, for $i \neq j$. Let us remark that the morphism π may be recovered starting with these equations.

Blowing-up of any variety. Let M be a variety, covered by charts $U \subset M$, that we identify as open sets $U \subset \mathbb{A}_k^n$. Take a point $p \in M$ and consider a chart U such that $p \in U$. We can shrink the other charts to assume that $p \notin U'$ for another chart U' different from U. Now, we can do the blowing-up of U with center p

$$\pi_U : \tilde{U} = \pi^{-1}(U) \to U \subset \mathbb{A}_k^n.$$

We glue the charts U' with $\tilde{U}$ by recalling the identification between $\tilde{U} \setminus \pi^{-1}(p)$ and $U \setminus \{p\}$. In this way we obtain the blow-up morphism

$$\pi : \tilde{M} \to M.$$

Blowing up along a subvariety. Let M be a variety and consider a closed subvariety $Y \subset M$. We can identify locally the pair (M, Y) with the pair $U \times V, \{\mathbf{0}\} \times V$, where U and V are open subsets $\mathbf{0} \in U \subset \mathbb{A}_k^{n-m}$ and $V \subset \mathbb{A}_k^m$. The blowing-up

$$\pi : \tilde{M} \to M,$$

of M with center π is obtained by gluing together the local blowings-up

$$\tilde{U} \times V \to U \times V,$$

where $\tilde{U} \to U$ is the blowing-up with center $\mathbf{0}$. Note that the *exceptional divisor* $\pi^{-1}(Y) \subset \tilde{M}$ is a hyper-surface covered by open sets of the form $\pi^{-1}(p) \times V$.

The universal property of the blowing-up. The above constructions seem to be highly non intrinsic. In particular one immediately sees a problem to justify the gluing procedures in the blowing-up along a subvariety. All this difficulties are solved by invoking the *universal property of the blowing-up*. In algebraic terms it can be stated as follows

> Let $\pi : \tilde{M} \to M$ be the blowing-up of M along a subvariety. Consider another proper morphism $h : M' \to M$ having the property that $h^{-1}(M \setminus Y)$ is isomorphic to $M \setminus Y$ and $h^{-1}(Y)$ is a hyper-surface (in the sense that the sheaf $\mathcal{J}_Y \mathcal{O}_{M'}$ is an inversible sheaf). Then there is a unique morphism $f : M' \to \tilde{M}$ such that $\pi \circ f = h$.

We will not insist in this property and the use of the blowing-up we will do is mainly through the equations and coordinates.

Transform of a vector field by blowings-up. Let ξ be a germ of vector field at $p \in M$. That is $\xi \in \mathrm{Der}_k \mathcal{O}_{M,p}$. Consider a blowing-up

$$\pi : \tilde{M} \to M,$$

along a subvariety $Y \subset M$ and fix a point $p' \in \pi^{-1}(p)$. We want to see if ξ defines in a natural way a germ of vector field at p'.

Remark 1.3. Let ω be a germ of differential 1-form. The standard pull-back of 1-forms by a morphism allows us to define $\pi^*\omega$ in a very natural way as a germ of differential 1-form at p'. The case of a germ of vector field is slightly more complicated.

The ring of germs of functions $\mathcal{O}'_{M,p'}$ is an extension of $\mathcal{O}_{M,p}$ through the blow-up morphism. More precisely, we can choose local coordinates $x_1, x_2, \ldots, x_n$ around $p \in M$ such that

1. The center Y of the blowing-up is locally given at p by

$$Y = \{x_1 = x_2 = \cdots = x_m = 0\},$$

   where m is the codimension of Y in M.
2. There are local coordinates $x'_1, x'_2, \ldots, x'_n$ at $p' \in M'$ such that

$$x'_j = x_j / x_m, \quad j = 1, 2, \ldots, m-1.$$
$$x'_j = x_j, \quad j = m, m+1, \ldots, n.$$

   (The equalities have to be interpreted locally at p' by identifying x_j with $x_j \circ \pi$).

Without doing the complete details, a necessary a sufficient condition to extend ξ to a derivation

$$\xi : \mathcal{O}_{M',p'} \to \mathcal{O}_{M',p'},$$

is that $\xi(x'_j) \in \mathcal{O}_{M',p'}$ for all $j = 1, 2, \ldots, n$. Of course, it is enough to verify that $\xi(x'_j) \in \mathcal{O}_{M',p'}$ for $1 \leq j \leq m-1$. Let us write

$$\xi = \sum_{i=1}^{n} a_i(x_1, x_2, \ldots, x_n) \frac{\partial}{\partial x_i}.$$

We have

$$\xi(x'_j) = \xi(x_j / x_m) = \frac{x_m a_j - x_j a_m}{x_m^2} = \frac{1}{x_m} \left(a_j - x'_j a_m \right).$$

That is, the condition we look for is: x'^2_m divides $x_m a_j - x_j a_m$ in the ring $\mathcal{O}_{M',p'}$, for all $1 \leq j \leq m-1$.

Proposition 1.4. *The following conditions are equivalent*

1. *ξ extends to a derivation $\xi : \mathcal{O}_{M',p'} \to \mathcal{O}_{m',p'}$.*
2. *x'^2_m divides $x_m a_j - x_j a_m$ in the ring $\mathcal{O}_{M',p'}$, for all $1 \leq j \leq m-1$.*
3. *$\xi(x_i)$ belongs to the ideal I of $\mathcal{O}_{M,p}$ generated by $x_1, x_2, \ldots, x_m$ (this is the ideal defining $Y \subset M$), for any $i = 1, 2, \ldots, m$.*

Proof. Obviously 3 implies 2. Conversely, the condition that x'^2_m divides $x_m a_j - x_j a_m$ in the ring $\mathcal{O}_{M',p'}$ is equivalent to say that $x_m a_j - x_j a_m$ is in $I^2\mathcal{O}_{M,p}$. Assume that $a_{j_0} \notin I$ for some $1 \leq j_0 \leq m-1$. Then

$$f = \frac{\partial(x_m a_{j_0} - x_{j_0} a_m)}{\partial x_m} = a_{j_0} + x_m \frac{\partial a_{j_0}}{\partial x_m} + x_{j_0} \frac{\partial a_m}{\partial x_m}$$

is not in I, contradiction, since $x_m a_j - x_j a_m$ is in $I^2\mathcal{O}_{M,p}$. If $a_m \notin I$, we do the same argument by taking the partial derivative with respect to x_j, for any $1 \leq j \leq m-1$. □

The third condition in the proposition means that Y is invariant for ξ. To be precise, we have the following definition:

Definition 1.5. Let $I \subset \mathcal{O}_{M,p}$ be a prime ideal, defining a germ of subspace $(Y, p) \subset (M, p)$. We say that (Y, p) is *invariant* for ξ if and only if $\xi(I) \subset I$.

Remark 1.6. The point $\{p\}$ is invariant for ξ if and only if ξ is singular at p (we also say that ξ has an *equilibrium point* at p). Consider the curve

$$Y = \{x_1 = x_2 = \cdots = x_{n-1} = 0\},$$

to say that Y is invariant means that the vector field is "vertical" along the curve, that is $a_i(0, 0, \ldots, 0, x_n) = 0$ for $i = 1, 2, \ldots, n-1$; in other words, the vector field is tangent to the curve at the points of Y and hence the trajectories of the integral curves of ξ starting at points in Y are contained in Y (this explains the word "invariant").

Foliations by lines. A foliation by lines $\mathcal{L}$ over M corresponds to the fact of considering locally a vector field "without velocity". The leaves will be the trajectories of the vector field, that is the images of the integral curves, where we do not consider the parametrization by the time.

To be precise, an *atlas* for a foliation $\mathcal{L}$ is a collection (U_i, ξ_i) of *foliated charts* such that the ξ_i are vector fields defined over the open sets U_i and

$$\xi|_{U_i \cap U_j} = h_{ij} \xi_j|_{U_i \cap U_j},$$

where the h_{ij} are invertible functions defined over $U_i \cap U_j$. As usual, we define the foliation by identifying it with a maximal atlas. The foliation is *reduced* if for any (nonsingular) point $p \in M$ we can write

$$\xi_i = \sum_{i=1}^{n} a_i(x) \frac{\partial}{\partial x_i}$$

where the coefficients $a_i(x) \in \mathcal{O}_{M,p}$ are without common factor. It is possible to pass from a foliation to a reduced one in a unique way just by taking the greatest common divisor of the coefficients. The singular locus Sing$\mathcal{L}$ of $\mathcal{L}$ is locally given by the singular locus of the vector fields ξ_i and it is of codimension greater or equal than two in the case of a reduced foliation.

We can also define *meromorphic foliations* as given by atlases of the form $\{(U_i, g_i^{-1}\xi_i)\}$ where $g_i \in \mathcal{O}_M(U_i)$ and the compatibility of the charts is defined as

$$g_j|_{U_i\cap U_j}\xi|_{U_i\cap U_j} = h_{ij}g_i|_{U_i\cap U_j}\xi_j|_{U_i\cap U_j}.$$

As before a meromorphic foliation gives in a unique way a reduced foliation.

Algebraic foliations. In the algebraic case we can define a meromorphic foliation in a particular way which is very convenient for the work in a bi-rational context. Let K be the field of rational functions of M, that we suppose to be an algebraic variety over a field k of characteristic zero (recall that we typically have $k = \mathbb{R}$ or $k = \mathbb{C}$). The K-vector field of derivations $\mathrm{Der}_k K$ has K-dimension $n = \dim M$. A *rational foliation by lines* is just a one dimensional K-vector subspace

$$\mathcal{L} \subset \mathrm{Der}_k K.$$

If induces a reduced foliation as follows. Let $p \in M$ be a nonsingular point. The regular local ring $\mathcal{O}_{M,p}$ has a regular system of parameters $x_1, x_2, \ldots, x_n$ (minimal set of generators of the maximal ideal) and

$$\mathrm{Der}_k\mathcal{O}_{M,p} = \sum_{i=1}^{n} \mathcal{O}_{M,p}\frac{\partial}{\partial x_i}.$$

Moreover, each germ of vector field $\xi \in \mathrm{Der}_k\mathcal{O}_{M,p}$ extends in a unique way to a derivation $\xi : K \to K$. Now $\mathcal{L} \cap \mathrm{Der}_k\mathcal{O}_{M,p}$ is a free $\mathcal{O}_{M,p}$-module of rank one generated by a germ of vector field without common factors in its coefficients. In this way we obtain a reduced foliation on M.

Blowing up foliations. We have seen that a vector field can only be blown up if the center of the blowing-up is invariant. Otherwise, we obtain a meromorphic vector field. This is not an obstruction for the blowing-up of a foliation. Hence any foliation can be transformed under a blowing-up with any center.

Dicritical vector fields. Let ξ be a germ of vector field in $p \in M$ and suppose that

$$\xi = \sum_{i=1}^{n} a_i(x_1, x_2, \ldots, x_n)\frac{\partial}{\partial x_i}$$

in local coordinates $x_1, x_2, \ldots, x_n$. Let us consider the blowing-up $\pi : M_1 \to M$ of M with center p. Assume that p is an equilibrium point of ξ and hence we have a transform $\tilde{\xi}$ of ξ by π. Let us denote by $E = \pi^{-1}(p)$ the exceptional divisor of the blowing-up π.

At each point $p_1 \in E$ we have that $\tilde{\xi} = h\xi'_1$, where $h \in \mathcal{O}_{M_1,p_1}$ and ξ'_1 has no common factors in its coefficients. We have the following properties

1. The exceptional divisor E is invariant for $\tilde{\xi}$. This is a consequence of the fact that p is an equilibrium point of ξ.
2. If ξ has no common factor in its coefficients, then $h = 0$ is contained in E. More precisely, we have that either h is a unit (that is $h = 0$ is empty) or $\{h = 0\} = E$.

Let us look in a more precise way this situation. Consider the example of the radial vector field

$$R = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}.$$

Take a point p_1 with local coordinates $\mathbf{x}'$ such that $x'_1 = x_1$ and $x'_i = x_i/x_1$ for $i \geq 2$. In this case $E = \{x'_1 = 0\}$ and

$$\tilde{R} = x'_1 \frac{\partial}{\partial x'_1}; \quad R'_1 = \frac{\partial}{\partial x'_1}.$$

Let us note that E is not invariant for R'_1.

Definition 1.7. In the above situation we say that ξ is *dicritical* at p or that π is a *dicritical blowing-up* for ξ if and only if E is not invariant for ξ'_1.

This definition works for the case of a foliation, just by considering the reduced foliation after blowing up.

Let us give a characterization of the dicritical vector fields at p. Let r be the *order* of ξ at p, that is the minimum of the orders of the zero p of each coefficient a_i. We can decompose each coefficient $a_i(x)$ as a sum of homogeneous polynomials

$$a_i(\mathbf{x}) = A_{i,r}(x_1, x_2, \ldots, x_n) + A_{i,r+1}(x_1, x_2, \ldots, x_n) + \cdots.$$

Now, the germ of vector field ξ is dicritical at $p \in M$ if and only if the vectors $(A_{1,r}, A_{2,r}, \ldots, A_{n,r})$ and $(x_1, x_2, \ldots, x_n)$ are proportional, that is

$$x_i A_{r,j} = x_j A_{r,i}; \quad \text{for all } i, j.$$

Let us remark that being dicritical is a very particular situation. A still unsolved problem is to show that under any infinite sequence of blowings-up centered at points the resulting foliation is dicritical only finitely many times. This is true in dimension two and three, but it is not known for higher dimensions.

Invariant curves. Let $\xi = \sum_{i=1}^{n} a_i(x)\partial/\partial x_i$ be a germ of vector field at $p \in M$. A germ of analytic parameterized curve at p is just a morphism $\gamma : t \mapsto \gamma(t)$, where $\gamma(0) = 0$. The curve γ is called an *integral curve* of ξ if and only if

$$\gamma'(t) = \xi(\gamma(t))$$

for all t, where $\gamma'(t)$ means the tangent vector of γ at t. We know that there is always a unique integral curve (in the analytic context) of ξ at p. In the case that $p \in M$ is an equilibrium point, the integral curve at p is just the constant curve $t \mapsto p$.

We can also consider the definition of invariant subvariety given in a previous section. Take a germ of curve $(Y, p) \subset (M, p)$ at p, defined by the ideal $I \subset \mathcal{O}_{M,p}$. Recall that (Y, p) is invariant for ξ if and only if $\xi(I) \subset I$. It is possible to show that this is equivalent to say that (Y, p) is union of leaves, that is of images of integral curves.

In a more algebraic frame, assume that we have a Puiseux parametrization

$$x_i = \phi_i(t); \quad i = 1, 2, \ldots, n$$

of the curve (Y, p). The necessary and sufficient condition to assure that (Y, p) is invariant for ξ is that

$$a_i(\phi(t))\phi'_j(t) = a_j(\phi(t))\phi'_i(t); \quad \text{for all } i, j.$$

This condition means that $\xi(q)$ is in the tangent space of Y at each point q of Y near p.

Formal invariant curves. A formal curve $(\hat{Y}, p)$ at $p \in M$ is by definition the kernel $\hat{I} \subset \widehat{\mathcal{O}}_{M,p}$ of a morphism of complete local rings

$$\hat{\phi} : \widehat{\mathcal{O}}_{M,p} = k[[x_1, x_2, \ldots, x_n]] \to k[[t]].$$

Here we can interpret $\hat{\phi}$ as a Puiseux parametrization of $(\hat{Y}, p)$. The derivation ξ extends to a derivation $\xi : \widehat{\mathcal{O}}_{M,p} \to \widehat{\mathcal{O}}_{M,p}$. As for the convergent case we have

Proposition 1.8. *In the above situation the following properties are equivalent*

1. $\xi(\hat{I}) \subset \hat{I}$.
2. $a_i(\hat{\phi}(t))\hat{\phi}'_j(t) = a_j(\hat{\phi}(t))\hat{\phi}'_i(t); \quad$ *for all* i, j.

If we have the equivalent properties of the above proposition, we say that $(\hat{Y}, p)$ is a *formal invariant curve* for ξ.

We shall see that there are formal invariant curves that are not convergent ones. This is one of the difficulties when doing reduction of singularities of vector fields, since the invariant objects are not necessarily convergent ones.

Definition 1.9. The formal curve $(\hat{Y}, p)$ is *non-singular* if and only if there is a Puiseux parametrization $\hat{\phi}(t)$ such that one of the $\hat{\phi}_i(t)$ has order 1.

This definition is equivalent to say that in formal coordinates, we have that $\hat{Y} = \{\hat{x}_2 = \hat{x}_3 = \cdots = \hat{x}_n = 0\}$. Moreover, if the curve is convergent, the rectification (in the analytic frame) may be done with convergent coordinates.

Behavior under blowing-up. Let $(\hat{Y}, p)$ be a formal curve at $p \in M$. Of course, a particular case is the case when $(\hat{Y}, p)$ is convergent. Consider the blowing-up

$$\pi : M_1 \to M$$

with center p. Up to do a linear change in the coordinates $x_1, x_2, \ldots, x_n$, we can assume that $\hat{Y}$ has a parametrization $\hat{\phi}(t)$ where $\hat{\phi}_1(t)$ has order d and $\hat{\phi}_i(t)$ has order $> d$ for all $i = 2, 3, \ldots, n$. Now, consider the point p_1 in the exceptional divisor E of π corresponding to the line

$$x_2 = x_3 = \cdots = x_n = 0.$$

At this point we have local coordinates $x'_1 = x_1, x'_i = x_i/x_1$, for $i = 2, 3, \ldots, n$. Now we have a Puiseux parametrization

$$x'_1 = \hat{\phi}_1(t),\ x'_i = \frac{\hat{\phi}_i(t)}{\hat{\phi}_1(t)}; \quad i = 2, 3, \ldots, n,$$

that defines a formal curve $\hat{Y}_1$ at p_1. We say that $(\hat{Y}_1, p_1)$ is the *strict transform* of $(\hat{Y}, p)$ by π and that p_1 is the *first tangent* or *first infinitesimal near point* of $(\hat{Y}, p)$.

Proposition 1.10. *Let ξ be a germ of vector field having an equilibrium point at $p \in M$ and let $(\hat{Y}, p)$ be a formal curve. Denote by $(\hat{Y}_1, p_1)$ the strict transform of (Y, p) by the blowing-up π of M with center p. We have*

1. *$(\hat{Y}_1, p_1)$ is convergent if and only if $(\hat{Y}, p)$ is convergent.*
2. *$(\hat{Y}_1, p_1)$ is invariant for ξ if and only if $(\hat{Y}, p)$ is invariant.*

Infinitely near points. Let $(\hat{Y}, p)$ be a formal curve in M. We can blow up successively $M = M_0$ to get an infinite sequence

$$\pi_{i+1} : M_{i+1} \to M_i$$

of blowings-up with centers $p_i \in M_i$, where $(\hat{Y}_{i+1}, p_{i+1})$ is the strict transform of $(\hat{Y}_i, p_i)$ and of course we put $(\hat{Y}_0, p_0) = (\hat{Y}, p)$. The points p_i are called the *iterated tangents of* $(\hat{Y}, p)$ or in another context the *infinitely near points* (although in [1] they consider only those points where the multiplicity does not drop).

Proposition 1.11 (Reduction of singularities of curves). *Given a formal curve $(\hat{Y}, p)$ in M, there is an index $N \geq 0$ such that $(\hat{Y}_i, p_i)$ is non singular for all $i \geq N$.*

Proof. Take coordinates $x_1, x_2, \ldots, x_n$ and a Puiseux expansion $\phi(t)$ such that $\phi_i(t) = t^{m_i} U_i(t)$, with $U_i(0) \neq 0$ and

$$m = m_1 < m_2 \leq m_3, \ldots, m_n$$

and moreover m does not divide m_2. Blowing up, we obtain $m'_1 = m_1$, $m'_i = m_i - m_1$, for $i \geq 2$ and the situation repeats if $m_1 < m'_2$. Note that $m_1 \neq m'_2$. After finitely many steps we get $m'_2 < m_1$ and we are done by induction on m. □

Take a (reduced) foliation by lines $\mathcal{L}$ in M locally generated at p by a vector field ξ. Let us denote by $\mathcal{L}_i$ the successive transformed foliations each one in M_i and let ξ_i be a local generator of $\mathcal{L}_i$ at p_i.

Proposition 1.12. *The following properties are equivalent:*

1. *$(\hat{Y}, p)$ is invariant for $\mathcal{L}$.*
2. *There is an index $N' \geq 0$ such that $p_i \in Sing\mathcal{L}_i$, for each $i \geq N'$.*

Proof. By reduction of singularities of the curve, we may assume that $\hat{Y}, p$ is given by $x_2 = x_3 = \cdots = x_n = 0$. Let us consider a logarithmic viewpoint relatively to $x_1 = 0$. To do this, we put $\eta_i = \xi_i$ if $x_1 = 0$ is invariant for ξ_i and $\eta_i = x_1\xi_i$ if $x_1 = 0$ is not invariant for ξ_i. We can write

$$\eta_i = b_{i1}(\mathbf{x}_i)x_{i1}\frac{\partial}{\partial x_{i1}} + \sum_{j=2}^{n} b_{ij}(\mathbf{x}_i)\frac{\partial}{\partial x_{ij}}$$

where the coefficients $b_{i1}, b_{i2}, \ldots, b_{in}$ have no common factor and the coordinates satisfy

$$x_{i1} = x_1; \; x_{ij} = x_j/x_1^i, \quad j \geq 2.$$

Let α_i be the minimum of the orders of $b_{i1}, b_{i2}, \ldots, b_{in}$ and put $\tau_i = \alpha_i$ if α_i is also the minimum of the orders of $b_{i2}, b_{i3}, \ldots, b_{in}$ and $\tau_i = \alpha_i + 1$ otherwise. We have

$$b_{i+1,1} = \frac{b_{i1}}{x_1^{\tau_i - 1}}; \quad b_{i+1,j} = \frac{b_{ij}}{x_1^{\tau_i}} - x_{i+1,j}b_{i+1,1}, \quad j \geq 2.$$

Let δ_{ij} be the order of $b_{ij}(x_1, 0, \ldots, 0) =$ and δ_i the minimum of the δ_{ij}, for $j \geq 2$. To say that (Y_i, p_i) is invariant is equivalent to say that $\delta_i = \infty$ and this implies that p_i is a singular point of ξ_i. Let us note that

$$\delta_{i+1} = \delta_i - \tau_i.$$

The only way to have a finite δ_i is that $\tau_i = 0$ for $i \geq N'$. But if $\tau_i = 0$ the point p_i is a nonsingular point of η_i and "a fortiori" of ξ_i. □

Elementary singularities. Let ξ be a germ of vector field at $p \in M$ and assume that p is an equilibrium point of ξ. That is $\xi(\mathcal{M}) \subset \mathcal{M}$, where $\mathcal{M}$ is the maximal ideal of the local ring $\mathcal{O}_{M,p}$ of M at p.

Let us recall that the quotient $\mathcal{M}/\mathcal{M}^2$ is a k-vector space (here k is the base field) of dimension n. More precisely, if $x_1, x_2, \ldots, x_n$ is a local system of coordinates at p, we have that

$$\overline{x}_j = x_j + \mathcal{M}^2; \; j = 1, 2, \ldots, n,$$