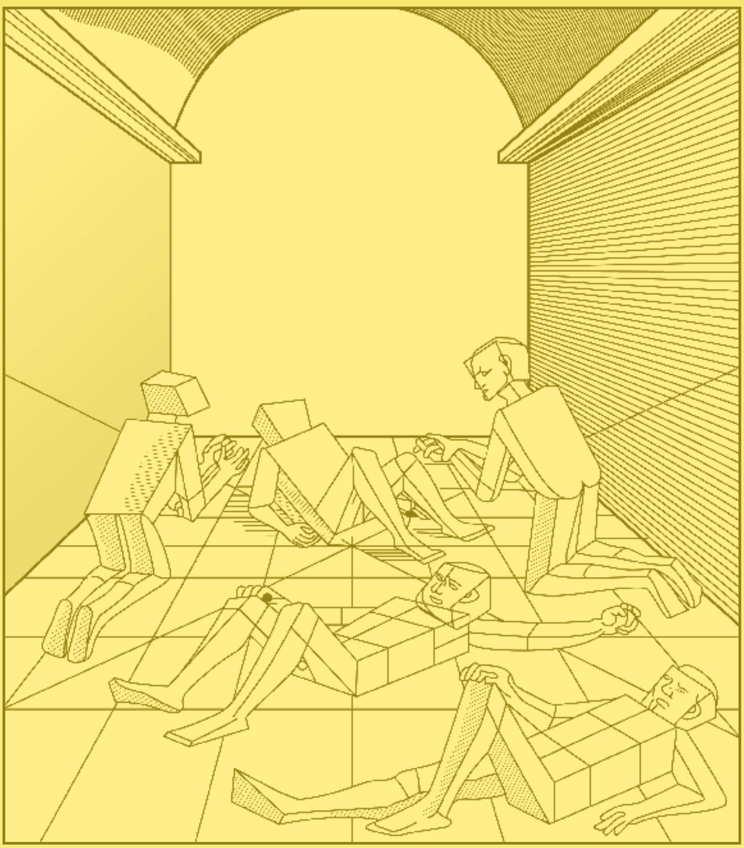
János Pach *Editor*

Thirty Essays on Geometric Graph Theory



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Thirty Essays on Geometric Graph Theory

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Introduction

János Pach

In the mathematical literature, the term "geometric graph theory" is often used in a somewhat vague sense: to cover any area of graph theory in which geometric methods seem to be relevant to the study of graphs defined by geometric means. In the present volume, by a *geometric graph* we mean a graph drawn in the plane so that its vertices are represented by distinct points and its edges by (possibly crossing) straight-line segments between these points such that no edge passes through a vertex different from its endpoints. *Topological graphs* are defined analogously, except that their edges can be represented by simple Jordan arcs [17].

In this sense, the theory of geometric and topological graphs starts with the study of *planar graphs*, initiated by Euler around 1750. For a long time it appeared that planar graphs, that is, graphs that can be drawn without edge crossings, do not have many interesting properties; they offer little excitement for mathematicians. *Kuratowski* and *Wagner* found simple characterizations of planar graphs in terms of forbidden subdivisions and minors, and it follows from *Steinitz's* work on convex polytopes that every planar graph can be drawn by noncrossing straight-line edges, as a geometric graph. In other words, every planar graph can be "stretched." This fact is usually referred to as *Fáry's theorem* [9]. One of the first really surprising results on planar graphs was the *Hanani–Tutte theorem* [7, 21], which states that if a graph can be drawn in such a way that any pair of its edges cross an even number of times, then it can also be drawn without edge crossing; that is, it must be a planar graph! It turns out that the reason why parity plays a role here lies in the *Jordan curve theorem*: Any curve connecting two points, both of which belong to the interior of a closed Jordan curve, must cross this curve an even number of times.

In the past quarter of a century, partially driven by the needs of computer graphics and other techniques of visualization, *graph drawing* has become a separate new area of research on the borderline of graph theory and computational geometry,

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with its annual international symposia and regular conference proceedings. The first such conference took place in June 1992, in Marino (near Rome). The subject has developed in close cooperation with industrial researchers developing software for visualization. Many interesting mathematical questions were asked, which were clearly motivated by potential applications. For instance, what is the size of the smallest integer grid such that every planar graph of n vertices admits a crossing-free straight-line drawing in which every vertex is mapped to a grid point [10]? Does there exist a positive-valued function $a(d)$ such that every planar graph of maximum degree d admits a crossing-free straight-line drawing, in which the angle between any two adjacent edges is at least $a(d)$ [16]? Many other more realistic measures of resolution were also considered. Geometric and topological graph theory has become one of the theoretical pillars of graph drawing.

Most graphs G are not planar. Nevertheless, we often have to represent them in the plane, and we may want to minimize the number of crossings in the resulting drawing. The smallest number of crossings that we can achieve is called the *crossing number* of G. *Turán's* famous "brick factory problem" asks for the crossing number of a complete bipartite graph with n vertices in each of its vertex classes [20]. In spite of many attempts to solve this problem, we still do not have even an asymptotically tight answer to this question. According to *Zarankiewicz's conjecture* [12], this quantity is equal to $\left(\frac{1}{16}+o(1)\right)n^4$. Many interesting related problems can be raised. For example, one can define the *pair-crossing number* of G as the minimum number of crossing pairs of edges over all possible drawings of G [8, 19]. Do the crossing number and the pair-crossing number coincide for every graph? We know that if we restrict our attention to straight-line drawings of G, we obtain a new parameter, different from the crossing number. The minimum number of crossings over all straight-line drawings of G is called the *linear crossing number* of G. It is known that there are graphs with crossing number 4 and with arbitrarily large linear crossing numbers [6]. According to a particularly useful inequality of Leighton [15] and, independently, of Ajtai et al. [3], the crossing number of any graph with n vertices and $e > 3n-6$ edges is at least a positive constant times e^3/n^2. Apart from the value of the constant, this bound is tight. It has found many interesting applications in combinatorial geometry and number theory.

A topological graph is called a *thrackle* if any pair of its edges meet precisely once, either at an endpoint or at a proper crossing [22]. According to *Conway's* celebrated *thrackle conjecture*, every thrackle has at most as many edges as vertices. The conjecture is known to be true for straight-line thrackles (geometric graphs) and for thrackles that can be drawn in such a way that that every vertical line meets every edge in at most one point [13]. However, the best-known general upper bound for the number of edges of an n-vertex thrackle is only $1.42n$ (see [11]). The fact that this simply formulated puzzle has been open for almost half a century indicates how little we know about crossing patterns of edges in a topological graph. A topological graph is called *simple* if any pair of its edges meet at most once, either at an endpoint or at a proper crossing. Conway's conjecture can now be rephrased as follows: Every simple topological graph with n vertices and more than n edges has two disjoint edges (that is, two edges that do not cross and do not share an endpoint).

In the same spirit, *Erdős, Hanani, Kupitz, Perles,* and others raised a number of exciting extremal problems [4, 5, 14]. What is the maximum number of edges that a simple topological graph of n vertices can have if it contains no k pairwise disjoint edges for a fixed $k > 2$ or some other fixed forbidden configuration? There has been a lot of activity in this area, yet most of the above questions are still open. For instance, it is conjectured that, for any fixed k, the maximum number of edges of a (simple) topological graph with n vertices and no k pairwise crossing edges is $O(n)$. The conjecture holds for $k \leq 4$ (see [1, 2]). It would be true for every k if the answer to the following question of *Erdős* were in the affirmative: Does there exist a constant $\chi(k)$ for every $k \geq 3$ such that any system of segments in the plane, no k of which are pairwise crossing, can be colored by $\chi(k)$ colors so that no two segments that cross each other receive the same color? However, it has been proved that the answer is no even for $k = 3$.

The first conference dedicated to geometric graph theory was held at Rutgers University, New Jersey, in the framework of the DIMACS Special Focus on Discrete and Computational Geometry, in the Fall of 2002 (see [18]). The second such meeting took place eight years later, as part of the Special Semester on Discrete and Computational Geometry organized by the Bernoulli Center at EPFL, Lausanne. The progress in this area made during this period is striking. The present volume is a careful selection of 30 invited and thoroughly refereed survey and research articles reporting on significant recent achievements in geometric graph theory.

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The Rectilinear Crossing Number of K_n: Closing in (or Are We?)

Bernardo M. Ábrego, Silvia Fernández-Merchant, and Gelasio Salazar

Abstract The calculation of the rectilinear crossing number of complete graphs is an important open problem in combinatorial geometry, with important and fruitful connections to other classical problems. Our aim in this chapter is to survey the body of knowledge around this parameter.

Mathematics Subject Classification (2010): 52C30, 52C10, 52C45, 05C62, 68R10, 60D05, 52A22

1 Introduction

In a *rectilinear* (or *geometric*) drawing of a graph G, the vertices of G are represented by points, and an edge joining two vertices is represented by the straight segment joining the corresponding two points. Edges are allowed to cross, but an edge cannot contain a vertex other than its endpoints. The *rectilinear* (or *geometric*) crossing number $\overline{\mathrm{cr}}(G)$ of a graph G is the minimum number of pairwise crossinsgs of edges in a rectilinear drawing of G in the plane.

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1.1 The Relevance of $\overline{\text{cr}}(K_n)$

As with every graph theory parameter, there is a natural interest in calculating the rectilinear crossing number of certain families of graphs, such as the complete bipartite graphs $K_{m,n}$ and the complete graphs K_n. The estimation of $\overline{\text{cr}}(K_n)$ is of particular interest, since $\overline{\text{cr}}(K_n)$ equals the minimum number $\square(n)$ of convex quadrilaterals defined by n points in the plane in general position; the problem of determining $\square(n)$ belongs to a collection of classical combinatorial geometry problems, the so-called Erdős–Szekeres problems. For a comprehensive survey on results and open questions on these and related problems, we refer the reader to the monograph by Brass et al. [16].

Another important motivation to study $\overline{\text{cr}}(K_n)$ is its close connection with the celebrated Sylvester four-point problem from geometric probability. Sylvester asked what the probability is that four points chosen at random in the plane form a convex quadrilateral [29]. After it became clear that this is an ill-posed question [30], Sylvester put forward a related conjecture. Let R be a bounded convex open set in the plane with finite area, and let $q(R)$ be the probability that four points chosen randomly from R define a convex quadrilateral. Then (Sylvester's conjecture [20]) $q(R)$ is minimized when R is a circle or an ellipse, and maximized when R is a triangle. This conjecture was proved by Blashke in 1917 [15]. Scheinerman and Wilf addressed in [27] the general problem when R is not required to be convex. It is easy to see that in this case $q(R)$ can be made arbitrarily close to 1 by choosing R to be a very thin annulus. The remaining problem is to determine the infimum $q_* := \inf q(R)$, taken over all open sets R with finite area. Scheinerman and Wilf established the striking connection

$$q_* = \lim_{n\to\infty} \frac{\overline{\text{cr}}(K_n)}{\binom{n}{4}}, \tag{1}$$

thus inextricably linking the estimation of *Sylvester's four-point constant* q_* to the (asymptotic) behavior of $\overline{\text{cr}}(K_n)$.

As we shall see below, recent developments have unveiled a close relationship between $\overline{\text{cr}}(K_n)$ and yet another classical combinatorial geometry parameter: the number of $(\leq k)$-edges in an n-point set.

1.2 Purpose and Timeliness of This Survey

Up until 2000, very little was known about $\overline{\text{cr}}(K_n)$. Since then, our knowledge of this problem has seen a tremendous growth. Surprising and useful connections to other classical problems have been unveiled. The current estimates for $\overline{\text{cr}}(K_n)$ have reached a point that would have seemed unlikely (to say the least) at the beginning of the previous decade.

For instance, before 2000 the ratio between the best lower and upper bounds for q_* was about 0.755; at the time of writing this survey, this ratio has been raised above 0.998. The implied success in our understanding of the problem cannot be understated—hence, the words "closing in" in the title of this chapter. Moreover, as we have already mentioned above and shall see below in more detail, the problem of estimating $\overline{\mathrm{cr}}(K_n)$ has turned out to be intimately related to other classical combinatorial geometry problems. Nowadays, anyone seriously interested in $(\leq k)$-edges or in halving lines has no alternative but to take a careful look at the literature on $\overline{\mathrm{cr}}(K_n)$ that has been produced in the last seven or eight years.

On the more cautious side, we must also note that the steady progress achieved on the estimation of $\overline{\mathrm{cr}}(K_n)$, both from the lower and the upper bounds' fronts, seems to have reached an impasse. To a researcher not too familiar with the field, the ratio 0.998 mentioned in the previous paragraph might signal an imminent closure on the determination of q_*. This is by no means the prevalent feeling among most (if not all) researchers actively working on this problem. Hardly any relevant new insights have been reported for some time. This humbling reality prompted us to include a word of caution ("or are we?") in the title of this chapter.

With this in mind, it makes sense to sit down and reflect on what has been done, to highlight the key developments, and to record the state of the art of the problem. We also see this as an opportunity to candidly (and, at times, informally) explain the obstacles that seem to prevent any further substantial progress with the current techniques, in the hopes that this will foster the development of refined or substantially novel techniques to attack this fundamental problem.

1.3 Structure of This Survey

The problem of estimating $\overline{\mathrm{cr}}(K_n)$ breaks into the two problems of establishing upper and lower bounds for this parameter, with the problem of finding exact values of $\overline{\mathrm{cr}}(K_n)$ lying, evidently, within both realms.

Before moving on to separate discussions on the problems of lower and upper bounding $\overline{\mathrm{cr}}(K_n)$, we shall review one of the main foundations behind our current knowledge of $\overline{\mathrm{cr}}(K_n)$. The Rectilinear Crossing Number (RCN) project, led by Aichholzer, has been a fruitful source of inspiration as well as an invaluable tool for establishing results and testing conjectures. In Sect. 2 we describe the nature and reach of the RCN project, which, as we will see, has both a claim and an impact on both the lower- and upper-bounding fronts.

In Sect. 3 we give an overview of the state of the art of the problem of lower bounding $\overline{\mathrm{cr}}(K_n)$ circa 2003.

Besides Aichholzer's RCN project, there seems to be a general consensus on the other main foundation behind our current knowledge of $\overline{\mathrm{cr}}(K_n)$. A major breakthrough was achieved around 2003, when two independent teams of researchers elucidated the close connection between $\overline{\mathrm{cr}}(K_n)$ and the number of $(\leq k)$-edges in an n-point set [4, 25]. A good estimate on the number of such $(\leq k)$-edges, also given in these papers, yielded an impressively improved lower bound on $\overline{\mathrm{cr}}(K_n)$. We devote Sect. 4 to a review of these cornerstone results.

In Sect. 5 we overview the subsequent efforts to refine the bounds for the number of $(\leq k)$-edges given in [4, 25], in the quest for improved lower bounds for $\overline{\mathrm{cr}}(K_n)$.

In Sect. 6 we discuss the different approaches to establishing upper bounds for $\overline{\mathrm{cr}}(K_n)$.

Section 7 contains a brief summary of the state of the art of the problem at the time of writing this survey. We present the current best estimates (lower and upper bounds) for q_*, as well as an annotated table with the values of n for which the exact value of $\overline{\mathrm{cr}}(K_n)$ is known.

We conclude this survey by reflecting on some possible future developments around this fundamental problem. We discuss the difficulties that lie behind our current impasse, and outline a somewhat promising approach that may pave the way toward future improvements.

2 The RCN Project

Around 2000, a team of researchers led by Aichholzer undertook the task of building databases with all the distinct (up to order type equivalence; see below) n-point configurations in general position, for $n \leq 10$ [8, 12, 24]. The raw knowledge of all possible n-point configurations put Aichholzer and his collaborators in a position to explore in depth several classical combinatorial geometry problems. In particular, it allowed for the exact calculation of $\overline{\mathrm{cr}}(K_n)$ for small values of n.

The criterion used by Aichholzer et al. to discriminate if two collections of points are nonisomorphic is based on the concept of order types. Consider an (ordered) n-point set $P = \{p_1, p_2, \ldots, p_n\}$ in general position. To each three integers i, j, k with $1 \leq i < j < k \leq n$, associate a *sign* (or *order type*) $\mathrm{sign}(ijk)$ according to the following rule. If, as we traverse the triangle defined by p_i, p_j, and p_k by following the edges $\overline{p_i p_j}$, $\overline{p_j p_k}$, and $\overline{p_k p_i}$ in the given order, the resulting closed curve has a clockwise orientation, then let $\mathrm{sign}(ijk) := +$. Otherwise, let $\mathrm{sign}(ijk) := -$. The collection of the order types of all the triples of points of P is the *order type* of P. Now let Q be another n-point set in general position. If the elements of Q can be ordered $\{q_1, q_2, \ldots, q_n\}$ so that the order types of P and Q are the same, then P and Q are *order type equivalent* (under the mapping $p_i \mapsto q_i$ for $i = 1, 2, \ldots, n$). We simply say that P and Q *have the same order type*.

Order type equivalence is a natural isomorphism criterion for point sets in general position. For crossing number purposes, it is certainly the relevant paradigm. Indeed, suppose that P and Q have the same order type. Then there is a bijection from the points of P to the points of Q so that four points in P span a convex quadrilateral if and only if the corresponding four points in Q span a convex quadrilateral. Conversely, if this last condition holds, then P and Q have the same order type.

Aichholzer et al. constructed the complete database of all distinct order types on n points, for all $n \leq 10$. As an application, they verified that $\overline{\mathrm{cr}}(K_{10}) = 62$ (this had been proved by Brodsky et al. in [17]).

Without building the complete database for $n = 11$, the information gathered by Aichholzer et al. for $n \leq 10$ allowed them to calculate $\overline{\mathrm{cr}}(K_{11})$ and $\overline{\mathrm{cr}}(K_{12})$.

To achieve this, taking their database for 10 points as a starting point, they analyzed (for $m = 10$, and then for $m = 11$) which m-point order types may possibly be extended to $(m+1)$-point sets that correspond to crossing-minimal drawings of K_{m+1}.

The determination of the rectilinear crossing numbers of K_{11} and K_{12} marks the beginning of the RCN project. As one of the major achievements of the RCN, Aichholzer developed some impressively accurate heuristics to generate geometric drawings of K_n with few crossings. Aichholzer set up a web page (http://www.ist.tugraz.at/staff/aichholzer/research/rp/triangulations/crossing/) to keep track of the best geometric drawings of K_n available, as well as of the number of distinct (up to order type equivalence) drawings achieving the current minimum.

The results reported by Aichholzer (http://www.ist.tugraz.at/staff/aichholzer/research/rp/triangulations/crossing/) have had a major lasting impact in the field. As new results and techniques to find improved lower bounds have become available (see Sects. 4 and 5), it has been possible to determine the exact value of $\overline{\mathrm{cr}}(K_n)$ for more values of n (see Sect. 7). The outstanding quality of the upper bounds obtained by Aichholzer is evidenced by the fact that the drawings reported in Aichholzer (http://www.ist.tugraz.at/staff/aichholzer/research/rp/triangulations/crossing/) turned out to be crossing-optimal for all $n \leq 27$ and for $n = 30$ (for $n = 28$ and 29 the exact value of $\overline{\mathrm{cr}}(K_n)$ is still unknown). At the time of writing this chapter, the best upper bounds available (see Sect. 6) are obtained from constructions that build upon "base" drawings of K_n for relatively small values of n. As further evidence of the influence of the RCN, we note that the base drawings used have been obtained by small modifications of drawings given in Aichholzer (http://www.ist.tugraz.at/staff/aichholzer/research/rp/triangulations/crossing/).

As a final note, we mention that Aichholzer and Krasser subsequently completed the database of all distinct order types of 11-point sets [13] (http://www.ist.tugraz.at/staff/aichholzer/research/rp/triangulations/ordertypes/). Using this database as a starting point, they were able to compute $\overline{\mathrm{cr}}(K_n)$ for all $n \leq 17$. Building the complete database of all the order type nonequivalent 12-point sets seems to be an unfeasible task; not only it is estimated that the storage of these 12-point sets would require several petabytes of memory, but there are also some important technical difficulties.[1]

3 Lower Bounds I: Before 2004

In a paper published in 1972, Guy [22] gave the exact value of $\overline{\mathrm{cr}}(K_n)$ for $n \leq 9$. Almost 30 years later, Brodsky et al. [17] pushed the existing techniques to their limit, and introduced some clever new arguments, to calculate the exact value of $\overline{\mathrm{cr}}(K_{10})$.

[1] Aichholzer, personal communication.

As one of the first results of the RCN project (see Sect. 2), Aichholzer et al. [9] gave computer-assisted proofs that $\overline{\text{cr}}(K_{11}) = 102$ and $\overline{\text{cr}}(K_{12}) = 153$.

Because each of the n subsets of size $n-1$ of an n-point set P has at most $\overline{\text{cr}}(K_{n-1})$ crossings, and each crossing of P appears in exactly $n-4$ such subsets, it follows that $(n-4)\overline{\text{cr}}(K_n) \geq n\overline{\text{cr}}(K_{n-1})$. This is equivalent to

$$1 \geq \frac{\overline{\text{cr}}(K_n)}{\binom{n}{4}} \geq \frac{\overline{\text{cr}}(K_{n-1})}{\binom{n-1}{4}},$$

which shows that Sylvester's four-point constant q_* defined in (1) actually exists. Starting from a lower bound for $\overline{\text{cr}}(K_m)$ for any fixed m, one can obtain a lower bound for $\overline{\text{cr}}(K_n)$ for every $n > m$ (and consequently a lower bound for q_*) by iterating $\overline{\text{cr}}(K_n) \geq \lceil \overline{\text{cr}}(K_{n-1})n/(n-4) \rceil$. This technique was used by Brodsky et al. [17] with $\overline{\text{cr}}(K_{10}) = 62$ to show that $q_* > 0.3001$. Adding to this argument the fact that $\overline{\text{cr}}(K_n)$ and $\binom{n}{4}$ have the same parity when n is odd (this easily follows from (2) but was proved for any nonnecessarily rectilinear drawing of K_n by Eggleton and Guy [21]), and using $\overline{\text{cr}}(K_{11}) = 102$, Aichholzer et al. [9] showed that $q_* > 0.3115$.

Building upon ideas from Welzl [34] and Wagner and Welzl [32], Wagner [31] used a completely novel approach to show that $q_* > 0.3288$. Wagner's work is particularly significant, since it deviates from the traditional approach of lower bounding q_* by using a particular lower bound and a counting argument. Indeed, the ideas in [31] are prescient of the revolutionary approach that will be reviewed in the next section.

4 Lower Bounds II: The Breakthrough

Our understanding of geometric drawings of K_n underwent a phase transition by unveiling a close relationship with k-edges. We recall that if P is an n-point set, and $0 \leq k \leq n/2-1$, a *k-edge* of P is a line through two points of P leaving exactly k points on one side. A $(\leq k)$-edge is a j-edge with $j \leq k$. The number of k- and $(\leq k)$-edges of P are denoted by $E_k(P)$ and $E_{\leq k}(P)$, respectively. Finally, let $E_{\leq k}(n)$ denote the minimum $E_{\leq k}(P)$, taken over all n-point sets P in general position.

For an n-point set P in the plane in general position, let $\overline{\text{cr}}(P)$ denote the number of crossings in the rectilinear drawing of K_n induced by P. The following was proved independently by Lovász et al. [25], and by Ábrego and Fernández-Merchant [4]:

$$\overline{\text{cr}}(P) = \sum_{k=0}^{\lfloor n/2 \rfloor - 2} (n-2k-3)E_{\leq k}(P) - \frac{3}{4}\binom{n}{3} + (1+(-1)^{n+1})\frac{1}{8}\binom{n}{2}. \tag{2}$$

The relevance of this connection between $\overline{\text{cr}}(P)$ and $E_{\leq k}(P)$ was made evident in both [4, 25] by proving that

$$E_{\leq k}(n) \geq 3\binom{k+2}{2}, \quad \text{for } 0 \leq k \leq n/2-1. \tag{3}$$

Substituting (3) into (2) yields

$$\overline{\mathrm{cr}}(K_n) \geq \frac{3}{8}\binom{n}{4} + \Theta(n^3), \tag{4}$$

thus implying the remarkably improved bound $q_* \geq 3/8 = 0.375$.

We recall that the *crossing number* $\mathrm{cr}(G)$ of a graph G is the minimum number of pairwise crossings of edges in a (nonnecessarily geometric) drawing of G in the plane. There are drawings of K_n with exactly $\lambda_n := (1/4)\lfloor n/2 \rfloor \lfloor (n-1)/2 \rfloor \lfloor (n-2)/2 \rfloor \lfloor (n-3)/2 \rfloor$ crossings, and it is widely believed that these drawings are crossing-minimal; that is, it is conjectured that $\mathrm{cr}(K_n) = \lambda_n$ for every positive integer n. This conjecture has been verified for $n \leq 12$ [22, 26]. Since $\mathrm{cr}(K_n) \leq \lambda_n$, it follows at once that $\lim_{n\to\infty} \mathrm{cr}(K_n)/\binom{n}{4} \leq 3/8$.

This last upper bound gives an additional significance to (4). With this motivation, Lovász et al. pushed a little further, invoking the following from [33]:

$$E_{\leq k}(n) \geq \binom{n}{2} - n\sqrt{n^2 - 2n - 4k^2 + 4k}. \tag{5}$$

This last bound is better than (3) for $k > 0.4956n$. Using (3) for $k \leq 0.4956n$, and (5) for $k > 0.4956n$, Lovász et al. derived the slightly improved bound $q_* > (3/8) + 10^{-5}$. Although numerically marginal, this improvement is significant because it shows that $\mathrm{cr}(K_n)$ and $\overline{\mathrm{cr}}(K_n)$ differ in the asymptotically relevant term.

5 Lower Bounds III: Further Improvements

Since the key connection (2) was proved in [4, 25], all subsequent efforts to lower bound $\overline{\mathrm{cr}}(K_n)$ have been focused on finding better estimates for $E_{\leq k}(n)$.

The first improvement was reported in [14], giving a lower bound for $E_{\leq k}(n)$ that is strictly better than (3) for $k > 0.4651n$. The bound given in [14] is in terms of a complicated expression. For our current surveying purposes, it suffices to mention that using this bound, Balogh and Salazar proved that $\overline{\mathrm{cr}}(K_n) > 0.37553\binom{n}{4} + \Theta(n^3)$.

Another significant improvement was achieved by Aichholzer et al. [10], who proved that

$$E_{\leq k}(n) \geq 3\binom{k+2}{2} + 3\binom{k+2-\lfloor n/3 \rfloor}{2} - \max\{0, (k+1-\lfloor n/3 \rfloor)(n - 3\lfloor n/3 \rfloor)\}. \tag{6}$$

A shorter proof of (6), given in the more general context of pseudolinear drawings, was given in [1].

Substituting (6) into (2), one obtains the improved estimate $q_* \geq 41/108 > 0.37962$. Moreover, it is possible to use the bound by Balogh and Salazar [14] in the range $k > 0.4864n$ to obtain the marginally better $q_* > 0.37968$.

The current best lower bound known for q_* is derived using a result recently reported by Ábrego et al. [3, 7]. They proved that for every k and n such that $\lceil(4n-11)/9\rceil - 1 \leq k \leq (n-2)/2$,

$$E_{\leq k}(n) \geq u_k(n) \geq \binom{n}{2} - \frac{1}{9}\sqrt{1-\frac{2k+2}{n}}(5n^2+19n-31). \tag{7}$$

The function u_k is asymptotic to the latter expression and it is better than all previous bounds [including (5) (6), and the bound in [14]] across its full range $\lceil(4n-11)/9\rceil \leq k \leq (n-2)/2$. In addition, Ábrego et al. [3] constructed point sets achieving equality on (6) for all $k < \lceil(4n-11)/9\rceil$. Using (6) for $k < \lceil(4n-11)/9\rceil$, and (7) for $\lceil(4n-11)/9\rceil \leq k \leq (n-2)/2$. It follows from (2) that $\overline{\text{cr}}(K_n) \geq (277/729)\binom{n}{4} + \Theta(n^3)$, thus implying that $q_* \geq 277/729 > 0.37997$.

6 Upper Bounds

The literature on crossing numbers of particular families of graphs is vastly dominated by papers that focus on establishing lower bounds. Most of the time, a natural drawing suggests itself with relatively little effort. When successive attempts to produce better drawings fail, this is seen as plausible evidence that the proposed drawing is indeed optimal. The efforts are then directed in the opposite, and usually remarkably harder, direction: proving nontrivial lower bounds for the crossing numbers of the graphs upon consideration.

The problem of upper bounding the rectilinear crossing number of K_n is a notable exception to this trend. The goal is to describe a way to draw K_n with as few crossings as possible, for arbitrarily large values of n, so as to have at least an educated guess at the asymptotic value $q_* = \lim_{n\to\infty} \overline{\text{cr}}(K_n)/\binom{n}{4}$. Over the years, several strategies to draw K_n with few crossings have been put forward. However, to this day there has not been a clear candidate for an optimal drawing. The only common characteristic is that almost all drawings with few crossings have (or are really close to have) threefold symmetry with respect to a point. That is, the underlying point set P of the drawing is partitioned into three sets (we call them *wings*) of size $n/3$ each, with the property that rotating each wing $2\pi/3$ and $4\pi/3$ around a suitable point generates the other two wings.

In the early 1970s, Jensen [23] was the first to propose a way to draw K_n for arbitrarily large values of n. His construction gave specific coordinates for $n/3$ points in a wing, and then he obtained the remaining two wings by rotating $2\pi/3$ and $4\pi/3$ around the origin. As a result, he obtained $q_* \leq 7/18 < 0.38889$.

At around the same time, Singer [28] started the trend of recursively constructing drawings of K_n. His idea was to start with a good drawing of $K_{n/3}$, apply an affine transformation to it to make the slope of each of its edges sufficiently close to zero, and then add the $2\pi/3$ and $4\pi/3$ rotations of the resulting drawing to obtain the other two wings (see Fig. 1a). This construction shows that

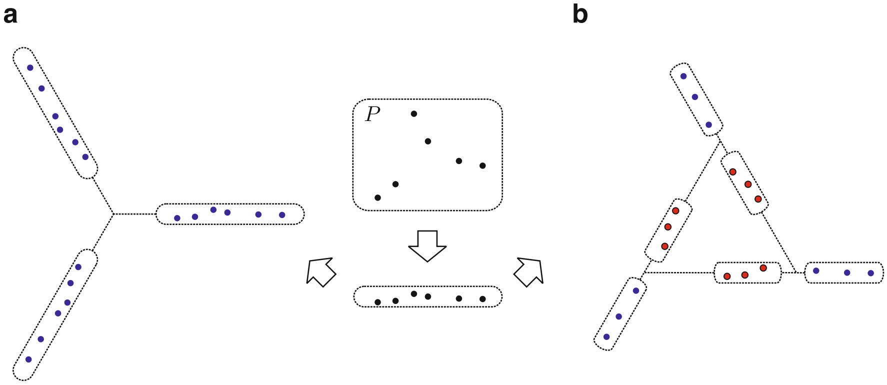

Fig. 1 (**a**) Recursive construction by Singer. (**b**) Recursive construction by Brodsky et al.

$$\overline{\mathrm{cr}}(K_n) \le 3\overline{\mathrm{cr}}(K_{n/3}) + 3\cdot\frac{n}{3}\binom{n/3}{3} + 3\binom{n/3}{2}^2.$$

Indeed, the first term consists of the crossings obtained from four points in the same wing, the next term counts the crossings from three points in one wing and the remaining in one of the other two wings, and the last term counts the crossings from two points in one wing and two points in another wing. Using $\overline{\mathrm{cr}}(K_3) = 0$ as a starting point, this inequality gives $q_* \le 5/13 < 0.38462$.

Brodsky et al. [18] modified Singer's construction by sliding three points in each wing toward the center of rotation as shown in Fig. 1b. Their construction gives $q_* \le 6{,}467/16{,}848 < 0.38385$.

Aichholzer et al. [9] devised a different replacement construction. They started with an underlying set P with an even number of points N. Instead of triplicating P, they replaced every point of P by a cluster of c points on a small arc of circle flat enough so that all lines among these c points leave $N/2$ points of P on one side and $N/2-1$ on the other side (see Fig. 2a). Letting $n = cN$, their construction gives

$$\overline{\mathrm{cr}}(K_n) \le \left(\frac{24\overline{\mathrm{cr}}(P) + 3N^3 - 7N^2 + 6N}{N^4}\right)\binom{n}{4} + \Theta(n^3).$$

Using a set P with $N = 36$ points and $\overline{\mathrm{cr}}(P) = 21\,191$ they obtained $q_* < 0.380858$. They explored further using different sizes for each of the clusters, which resulted in an improvement of the latter bound to $q_* < 0.380739$. This method of obtaining lower bounds allowed for improvements by using better initial sets P. Aichholzer and Krasser [13], as part of their computer-assisted search of the crossing numbers $\overline{\mathrm{cr}}(K_n)$ for small values of n, obtained a particular drawing of K_{54} that gives $q_* < 0.380601$.

Ábrego and Fernández-Merchant [5] started with an underlying set P with an even number of points N. They obtained a new set Q by replacing every point of P by a pair of points close to each other and spanning a line that divides the rest of Q

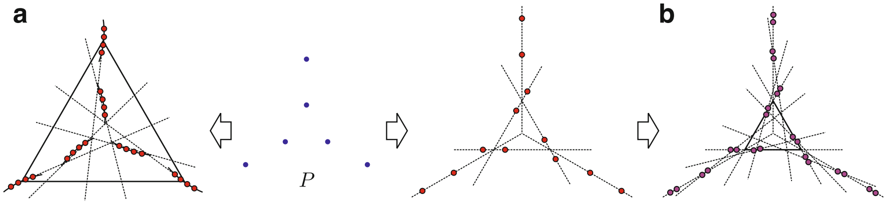

Fig. 2 (**a**) Replacement construction by Aichholzer et al. (**b**) Recursive construction by Ábrego and Fernández-Merchant

in half (see Fig. 2b). This property of having a *halving-line matching* is not satisfied by an arbitrary point set P, but fortunately it is satisfied by most of the small sets with optimal crossing number. Moreover, the resulting set Q inherits this property. Thus, if $n = 2^k N$, then iterating this construction k times gives

$$\overline{\mathrm{cr}}(K_n) \le \left(\frac{24\overline{\mathrm{cr}}(P) + 3N^3 - 7N^2 + (30/7)N}{N^4} \right) \binom{n}{4} + \Theta(n^3). \tag{8}$$

At the time, using the best-known drawing of K_{30} (now proved to be optimal) yielded $q_* < 0.380559$. To this date, (8) provides the currently best recursive construction. The restrictions on the base set P were subsequently weakened [2] in the sense that (8) also holds for arbitrary sets P with an odd number of points. Applying this inequality to a drawing of K_{315} with 152,210,640 crossings gives the currently best upper bound: $q_* < \frac{83{,}247{,}328}{218{,}791{,}125} < 0.380488$.

To support the belief that the crossing-minimal sets have nearly threefold symmetry, Ábrego et al. [2] constructed a threefold symmetric set of n points for each n multiple of 3, $n < 100$ (see Fig. 3). Moreover, threefold symmetry is inherited from the base set in all recursive constructions mentioned before. In fact, the drawing of K_{315} used as a base set to obtain the best current upper bound has threefold symmetry.

7 Summary

In this section we summarize, for quick reference, the state of the art on $\overline{\mathrm{cr}}(K_n)$ and q_* at the time of writing this chapter.

7.1 Sylvester's Four-Point Constant

$$0.379972 < \frac{277}{729} \le q_* \le \frac{83{,}247{,}328}{218{,}791{,}125} < 0.380488. \tag{9}$$

The lower and upper bounds in (9) are derived in [2,3] (see also [7]), respectively.

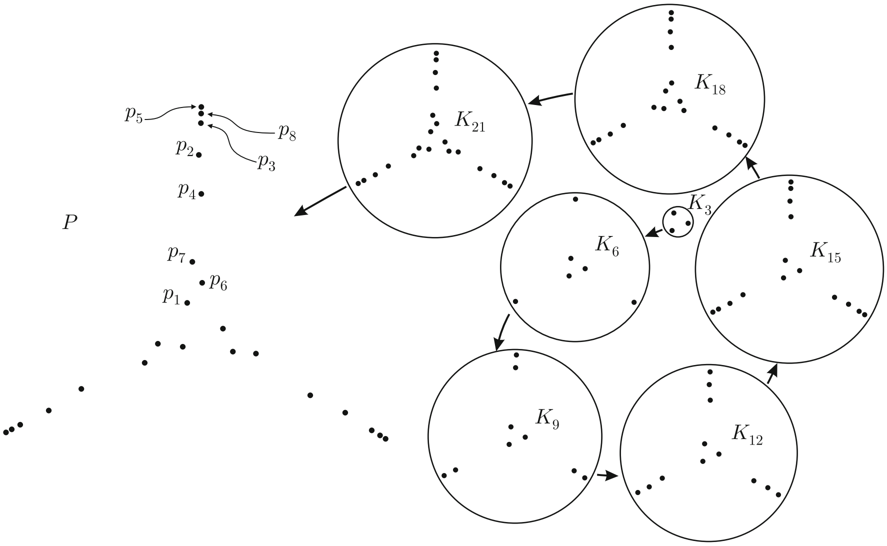

Fig. 3 The underlying vertex set of an optimal 3-symmetric geometric drawing of K_{24}. This point set contains optimal nested 3-symmetric drawings of $K_{21}, K_{18}, K_{15}, K_{12}, K_9, K_6$, and K_3

Table 1 Exact rectilinear crossing numbers known

n	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\overline{\text{cr}}(K_n)$	1	3	9	19	36	62	102	153	229	324	447	603	798	1,029

n	19	20	21	22	23	24	25	26	27	30
$\overline{\text{cr}}(K_n)$	1,318	1,657	2,055	2,528	3,077	3,699	4,430	5,250	6,180	9,726

7.2 Exact Values of $\overline{\text{cr}}(K_n)$

The exact value of $\overline{\text{cr}}(K_n)$ is known for $n \leq 27$ and for $n = 30$ (see Table 1).

For $n \leq 27$, the lower bound for $\overline{\text{cr}}(K_n)$ is derived in [3] (see also [6]). The bound $\overline{\text{cr}}(K_{30}) \geq 9726$ is proved in [19]. In all cases, the upper bounds were obtained by Aichholzer (http://www.ist.tugraz.at/staff/aichholzer/research/rp/triangulations/crossing/).

8 Further Thoughts and Future Research

Since the introduction of (2) in [4, 25], all the progress achieved on lower bounding q_* has been contingent on the derivation of improved bounds for $E_{\leq k}(n)$.

Although it may seem natural to expect the continuation of this trend, there is some evidence that suggests that this approach alone will not lead to the correct value of q_*. The reasons behind our caution lie in our own investigations of sets