Static & Dynamic Game Theory:
Foundations & Applications

Pierre Bernhard
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The Interval Market Model in Mathematical Finance

Game-Theoretic Methods

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The Interval Market Model in Mathematical Finance: Game-Theoretic Methods

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Preface

Mathematical finance was probably founded by Louis Bachelier in 1900 [19]. In his thesis and subsequent contributions, he constructed a stochastic model of stock price processes, essentially inventing the random walk or Brownian motion. But this was five years before Einstein investigated Brownian motion and long before Kolmogorov refounded probability on sound mathematical grounds; some basic probabilistic tools were missing. Bachelier's contribution was considered nonrigorous and, consequently, not recognized for its true pioneering value.

In contrast, few works in mathematical finance have enjoyed the fame and had the impact of Fischer Black and Myron Scholes' seminal paper [46]. In a bold move, it took the subjective concept of risk aversion out of the rationale for pricing financial derivatives, grounding such pricing on purely objective considerations.

"Objective," though, does not mean that no arbitrariness remains. In line with Bachelier, the Black–Scholes theory is based on an arbitrary choice of mathematical, stochastic model for underlying stock prices, which we will call "Samuelson's model," although some authors trace it back to earlier works.

Samuelson's model is called a "geometric diffusion," or "lognormal distribution." In that model, the price process $S(t)$ is assumed to obey the following Itô stochastic equation:

$$\frac{\mathrm{d}S}{S} = \mu \mathrm{d}t + \sigma \mathrm{d}B , \tag{0.1}$$

where μ and σ are known, deterministic parameters, or time functions, called "drift" and "volatility," respectively, and $B(\cdot)$ is a standard Brownian motion (or Wiener process).

Following these prestigious forerunners, most of the literature in mathematical finance relies on Samuelson's model, although notable exceptions have existed ever since, for example, [56, 57, 77, 87, 109, 117, 122, 124, 133].

The aim of this volume is to report several accomplishments using another class of models that we call, after [132], interval models. In these models, if n stocks are considered, it is assumed that a compact convex set of $\mathbb{R}^n$ is known that always contains the vector of relative stock price velocities (in a continuous-time setting) or one-step relative price changes (in a discrete-time setting). In the scalar case, corresponding to the classic Black–Scholes problem, and in discrete time, this means that we know two constants $\mathbf{d} < 1$ and $\mathbf{u} > 1$ – the notations used here are in reference to [57] – such that for a given $\delta t > 0$ and for all possible price trajectories

$$S(t+\delta t) \in [\mathbf{d}S(t), \mathbf{u}S(t)],$$

a line segment. In contrast, Cox et al. [57] assume that

$$S(t+\delta t) \in \{\mathbf{d}S(t), \mathbf{u}S(t)\},$$

the end points of a line segment, of course, a huge difference in terms of realism, and also of mathematics, even if in some cases we recover some of their results. More generally, in higher-dimensional problems, whether discrete or continuous time, this results in a tube of possible trajectories, or a so-called trajectory tube model.

These *interval models* were introduced independently, and almost simultaneously, by the authors of this volume. We only cite here some earlier papers as a matter of historical record. A common feature of these works is that, far remote from the mainstream finance literature, they suffered long delays between the date when they were written and their eventual publication, usually not in finance journals. Beyond Roorda et al. [132] already cited, whose preprint dates back to 2000, we mention here a 1998 paper by Vassili Kolokoltsov [95] and a paper from 2003 that only appeared in 2007 [86], a thesis supervised by Jean-Pierre Aubin defended in 2000 [128] – but a published version [17] had to wait till 2005 – and a conference paper by Pierre Bernhard, also in 2000 [37], an earlier form of which [35] did not appear in print until 2003.

If probabilities are the *lingua franca* of classic mathematical finance, it could be said that, although probabilities are certainly not ruled out, the most pervasive tool of the theories developed in this volume is some form of dynamic game theory. Most developments to be reported here belong to the realm of robust control, i.e., minimax approaches to decision making in the presence of uncertainty. These take several forms: the discrete Isaacs equation, Isaacs and Breakwell's geometric analysis of extremal fields, Aubin's viability approach, Crandall and Lions' viscosity solutions as extended to differential games by Evans and Souganidis, Bardi, and others, Frankowska's nonsmooth analysis approach to viscosity solutions, and geometric properties of risk-neutral probability laws and positively complete sets.

As a consequence, we will not attempt to give here a general introduction to dynamic game theory, as different parts of the book use different approaches. We will, however, strive to make each part self-contained. Nor will we try to unify the notation, although some of these works deal with closely related topics. As a matter of fact, the developments we report here have evolved, relatively independently, over more than a decade. As a result, they have developed independent, consistent notation systems. Merging them at this late stage would have been close to impossible. We will provide a concise "dictionary" between the notations of Parts II–V.

Part I is simply an introduction that aims to review, for the sake of reference, two of the most classic results of dynamic portfolio management: Merton's optimal portfolio and Black and Scholes' pricing theory, each with a flavor more typical of this volume than classic textbooks. The Cox–Ross–Rubinstein model will be presented in detail in Part II, together with the interval model.

Parts II and III mostly deal with the classic problem of hedging one option with an underlying asset. Part II tackles the problem of incompleteness of the interval model, introducing the fair price interval, and an original problem of maximizing the best-case profit with a bound on worst-case loss. Part III only deals with the seller's price – the upper bound of the fair price interval – but adding transaction costs, continuous and discrete trading schemes, and the convergence of the latter to the former, for both plain vanilla and digital options. Both parts deal in some respect with the robustness of the interval model to errors in the estimation of price volatility. Both use a detailed mathematical analysis of the problems at hand: portfolio optimization under a robust risk constraint in Part II, classic option pricing in Part III, to provide a "fast algorithm" that solves with two recursions on functions of one variable a problem whose natural dynamic programming algorithm would deal with one function of two variables.

It is known that in the approach of Cox, Ross, and Rubinstein, the risk-neutral probability associated with the option pricing problem spontaneously appears in a rather implicit fashion. Part V elucidates the deep links between the minimax approach and risk-neutral probability and exploits this relationship to solve the problem of pricing so-called rainbow options and credit derivatives such as credit default swaps.

Part V uses the tools of viability theory and, more specifically, the guaranteed capture basin algorithm to solve the pricing problem for complex options. A remarkable fact is that, as opposed to the fast algorithm of Part III, which is specifically tailored to the problem of pricing a classic option, the algorithm used here is general enough that, with some variations, it solves this large set of problems.

There obviously is no claim of unconditional superiority of one model over others or of our theories over the classic ones. Yet, we claim that these theories do bring new insight into the problems investigated. On the one hand, they are less isolated now than they used to be in the early 2000s, as a large body of literature has appeared since then applying robust control methods to various fields including finance, a strong hint that each may have a niche where it is better suited than more entrenched approaches. On the other hand, and more importantly, we share the belief that uniform thinking is not amicable to good science. In some sense, two different – sensible – approaches to the same problem are more than twice as good as one, as they may enlighten each other, be it by their similarities or by their contradictions.

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Notation Dictionary

II	III	IV	V	Part number
T	T	T	T	Exercise time
X	$\mathscr{K}$	K	K	Exercise price
F	M	f	U	Terminal payment
0	$C^{\pm}, c^{\pm}$	β	δ	Transaction cost rates
	$S_0 = RS_0(T)$		S_0	Riskless bond price
		$j \in \{1, \ldots, J\}$		Asset (upper) index
	Continuous time			
	Constants			
	μ_0		r_0	Riskless return rate
	$\tau^- + \mu_0$		$r^\flat$	Min risky asset return
	$\tau^+ + \mu_0$		$r^\sharp$	Max risky asset return
	Time functions			
$t \in [0, T]$	$t \in [0, T]$	$t \in [0, T]$	$t \in [0, T]$	Current time
	$R = S_0/S_0(T)$			End-time discount rate
S	$S = Ru$	S	S	Risky asset price
	$\tau + \mu_0$		r	Risky asset return rate
	$XS = Rv$		E	Portfolio exposure
	$v = \varphi^\star(t, u)$		$E = E^\heartsuit(t, W)$	Optimal hedging strategy
	Y		p_0	Number of bonds in portfolio
	X		p	Number of shares of risky stock
	Rw		W	Portfolio worth
	RW		$W^\heartsuit$	Optimal portfolio worth
	(Control)	Impulses	(Triggered)	
	t_k		t^n	Impulse times
	ξ_k		$\psi(x) - x$	Impulse amplitudes
	Discrete time			
	Constants			
h	h	τ	ρ	Time step
n	K	n	N	Total number of steps
	$e^{\mu_0 h}$	$\rho = 1 + r\tau$	$1 + \rho r_0$	One-step riskless ratio
d	$1 + \tau_h^-$	d^j	$1 + \rho r_d$	Min one-step S ratio
u	$1 + \tau_h^+$	u^j	$1 + \rho r_u$	Max one-step S ratio
	Time functions			
$t_j = jh$	$t_k = kh$	m	$t_n = n\rho$	Current time
S_j	$S_k = R_k u_k$	S_m^j	S^n	Risky asset price
v	$1 + \tau_k$	ξ^j	$1 + r_\rho^n$	One-step S ratio
γ_j	X_k	γ_m^j		Risky shares in portfolio
$\gamma_j = g_j(S_j)$	$v_k = \varphi_k(u_k)$			Hedging strategy
	$R_k w_k$	X_m	W^n	Portfolio worth

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Part I
Revisiting Two Classic Results in Dynamic Portfolio Management

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The material presented in this part was developed for a course in portfolio management while the author was a professor at École Polytech'Nice, a department of the University of Nice Sophia Antipolis.

In this two-chapter part, we revisit the two most classic results in the theory of dynamic portfolio management: the Merton optimal portfolio in Chap. 1 and the famous Black and Scholes option pricing theory in Chap. 2.

In both cases we recall the classic result, to be used also as a reference for the remainder of the volume, but with nonclassic developments in the spirit of this volume attached to them. The "Merton" optimal portfolio problem is investigated with two different models: the classic one and a uniform-interval model. The Black and Scholes option pricing theory is dealt with in a robust control – or game theoretic – probability-free approach. A third very classic result is the discrete-time theory of Cox, Ross, and Rubinstein. It will be presented in the next part and fully revisited in Part IV.

Notation

- a^t: For any vector or matrix a, *a transposed*.

Universal constants

- $\mathbb{R}$: The real line.
- $\mathbb{N}$: The set of natural (positive) integers.
- $\mathbb{K} = \{0, 1, \ldots, K-1\}$.
- $\mathbb{1}$: A vector of any dimension with all entries equal to 1.

Main variables and parameters

- T: Horizon of finite horizon problems [Time]
- h: Time step of discrete trading theory [Time]
- $S_i(t)$, $i = 1,\ldots,n$: Market price of risky asset i (without index if only one risky asset is present) [Currency]
- $S_0(t)$: Price of riskless asset [Currency]
- $R(t) = S_0(t)/S_0(T)$: End-time value coefficient [Dimensionless]
- $u_i(t) = S_i(t)/R(t)$: Normalized market price of asset i [Currency]
- μ_i: Drift coefficient in model for S_i [Time^{-1} (continuous), dimensionless (discrete)]
- μ_0: Expected return of riskless asset [Time^{-1} or dimensionless]
- $\lambda_i = \mu_i - \mu_0$ Excess expected return of asset i over riskless asset [Time^{-1} (continuous) or dimensionless (discrete)]
- σ_i: A line of coefficients defining the variability of S_i around its expected value [Time$^{-1/2}$ (continuous), dimensionless (discrete)]
- σ: Matrix whose lines are the σ_i [Time$^{-1/2}$ (continuous), dimensionless (discrete)]
- $\Sigma = \sigma\sigma^t$: Covariancelike matrix [Time^{-1} (continuous), dimensionless (discrete)]
- $X_i(t)$: Number of shares of asset i in portfolio [Dimensionless]
- $W(t)$: Portfolio worth [Currency]
- $w(t) = W(t)/R(t)$: Portfolio normalized worth. [Currency]
- $\varphi_i = X_i S_i/W = X_i u_i/w$: Fraction of portfolio invested in asset i [Dimensionless]
- $C(t)$: Rate of portfolio consumption [Currency$\times$Time^{-1}] (continuous) or stepwise consumption [Currency] (discrete)
- $c(t) = C(t)/R(t)$
- $\chi(t) = C(t)/W(t) = c(t)/w(t)$: Relative rate of withdrawal of funds from portfolio for consumption [Time^{-1} (continuous), dimensionless (discrete)]
- Π: Coefficient of bequest utility function [Currency]
- π: Coefficient of running utility function [Currency$\times$time^{-1} (continuous), currency (discrete)]
- P: Coefficient of Bellman function [Currency]
- $\gamma \in (0,1)$: Exponent of c and w in utility functions [Dimensionless]
- α ($\alpha^{1-\gamma}$ in discrete theory): Maximum (normalized in continuous theory) return rate of a portfolio [Dimensionless]
- δ: Discount rate for infinite horizon problem [Time^{-1} or dimensionless]
- $\beta = \delta - \mu_0$ (continuous) or $\delta^{h/(1-\gamma)}$ (discrete): Normalized discount rate [Time^{-1} or dimensionless]

Chapter 1
Merton's Optimal Dynamic Portfolio Revisited

1.1 Merton's Optimal Portfolio Problem

The problem considered here is that of managing a dynamic portfolio over a period of time $[0,T]$ – or over $[0,+\infty)$; we shall consider this infinite horizon case in a separate subsection – in a market where several assets are available, with differing and varying returns. The portfolio manager is allowed to sell parts of his portfolio to obtain an immediate utility. He is also interested in having sufficient wealth at the end of the period considered. In this section, we deal with the classic continuous trading formulation of [118].

1.1.1 Problem and Notation

1.1.1.1 The Market

We consider a market with n risky assets and one riskless asset. The riskless asset will always be referred to by the index 0, the risky assets by indices 1 to n. Let $S_i(t)$ be the market price of asset i, $i = 0,1,\dots,n$. We need a model of the market. We extend the model (0.1) to n assets in the following way: we assume that each risky asset obeys the Itô equation

$$\frac{\mathrm{d}S_i}{S_i} = \mu_i \mathrm{d}t + \sigma_i \mathrm{d}b. \tag{1.1}$$

Here, μ_i are known constants and σ_i are *lines* of coefficients, all of the same length $\ell \le n$. (But to make things simple, we may take $\ell = n$.) Accordingly, b is a standard Brownian motion of dimension ℓ (or n), i.e., a vector whose entries are independent standard Brownian motions. We let σ be a matrix whose line number i is σ_i, and

$$\Sigma := \sigma\sigma^t. \tag{1.2}$$

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The riskless asset satisfies

$$\frac{\mathrm{d}S_0}{S_0} = \mu_0 \mathrm{d}t.$$

In the finite horizon case, we shall rather use the dimensionless end-time value coefficient

$$R(t) = S_0(t)/S_0(T) = \mathrm{e}^{\mu_0(t-T)}.$$

We shall assume that a reasonable portfolio does not contain assets whose expected return μ_i is less than the riskless return μ_0 since they would bring no value or risk alleviation. We shall call

$$\mu_i - \mu_0 = \lambda_i$$

the excess of expected return over the riskless rate. And we shall use the ratios

$$u_i(t) = \frac{S_i(t)}{R(t)},$$

which, in view of (1.1), satisfy the Itô equation

$$\frac{\mathrm{d}u_i}{u_i} = \lambda_i \mathrm{d}t + \sigma_i \mathrm{d}b. \tag{1.3}$$

1.1.1.2 The Portfolio

A portfolio shall be defined by $n+1$ functions $X_i(t)$, $i = 0, 1, \ldots, n$, giving the number of shares of each asset in the portfolio at time t. As a model simplification, we consider that the X_i are not restricted to being integers. They shall take their values in $\mathbb{R}$. Thus we also allow the portfolio to be "short" in some assets. Its worth is therefore

$$W(t) = \sum_{i=0}^{n} X_i(t) S_i(t).$$

We shall instead use

$$w(t) = \frac{W(t)}{R(t)} = X_0(t) S_0(T) + \sum_{i=1}^{n} X_i(t) u_i(t).$$

Transactions will be variations $\mathrm{d}X_i$ of the number of shares. Together with the variations in market prices, they produce variations in the worth of the portfolio:

$$\mathrm{d}W = \sum_{i=0}^{n} (S_i \mathrm{d}X_i + X_i \mathrm{d}S_i) \qquad \text{or} \qquad \mathrm{d}w = S_0(T)\mathrm{d}X_0 + \sum_{i=1}^{n} (u_i \mathrm{d}X_i + X_i \mathrm{d}u_i).$$

We allow such transactions to yield some excess cash, which the manager may want to use for immediate consumption. Let, therefore, $C\mathrm{d}t = cR\mathrm{d}t$ be the cash taken from the portfolio by the transactions $\mathrm{d}X_i$. (If we wanted to allow discontinuous X_i, we could let $\mathrm{d}X_i$ be finite and $C(\cdot)$ contain a Dirac impulsion. But we will not need to do this because the solution we shall find has continuous X_i.) This is obtained through transactions satisfying $\mathrm{d}W + C\mathrm{d}t = 0$, or, dividing through by R,

$$S_0(T)\mathrm{d}X_0 + \sum_{i=1}^{n} u_i \mathrm{d}X_i + c\mathrm{d}t = 0.$$

Hence, we find that

$$\mathrm{d}w = \sum_{i=1}^{n} X_i \mathrm{d}u_i - c\mathrm{d}t,$$

or, using (1.3),

$$\mathrm{d}w = \sum_{i=1}^{n} X_i u_i (\lambda_i \mathrm{d}t + \sigma_i \mathrm{d}b) - c\mathrm{d}t.$$

It is customary to simplify this expression using the *fractions* φ_i and χ of the portfolio defined as

$$\varphi_i = \frac{X_i S_i}{W} = \frac{X_i u_i}{w}, \qquad \chi = \frac{C}{W} = \frac{c}{w}.$$

Since we allow short positions for the portfolio, φ is unconstrained in $\mathbb{R}^n$. Consumption, on the other hand, is assumed to be nonnegative; hence so must χ be.

We also use the vector notation $\lambda \in \mathbb{R}^n_+$ and $\varphi \in \Delta_n$ (the simplex of $\mathbb{R}^n$) for the n-vectors of the λ_i and φ_i, $i \geq 1$. We finally obtain

$$\mathrm{d}w = \left[(\varphi^t \lambda - \chi)\mathrm{d}t + \varphi^t \sigma \mathrm{d}b\right] w. \tag{1.4}$$

1.1.1.3 Utility

The utility derived by the manager, which he wants to maximize, is supposed to be the sum of a running utility (which we may, of course, write directly in terms of c instead of C; it is just a matter of lowering the discount rate by μ_0)

$$\int_0^T U(t, c(t))\,\mathrm{d}t$$

and a *bequest* utility $B(w(T))$. Hence, he seeks to maximize

$$J = \mathbb{E}\left[B(w(T)) + \int_0^T U(t, c(t))\,\mathrm{d}t\right]. \tag{1.5}$$

The utility functions U and B should be chosen to be increasing concave to model risk aversion and satiation effects. It turns out that a decision that will lead to a simple solution of the optimization problem can be made by choosing them to be the same fractional powers of c and w, respectively. Let $p(\cdot)$ be a given nonnegative function and Π a given nonnegative constant. Let us thus choose $\gamma \in (0,1)$ and

$$U(t,c) = p(t)^{1-\gamma}c^{\gamma} = p(t)^{1-\gamma}\chi^{\gamma}w^{\gamma}, \qquad B(w) = \Pi^{1-\gamma}w^{\gamma}. \tag{1.6}$$

We may, for instance, wish to have the future running utility of consumption in the form $\pi^{1-\gamma}C^{\gamma}\exp[\delta(T-t)]$, i.e., discounted by a factor $\delta = \mu_0 + \beta$. In that case, we just take

$$p(t) = \pi e^{\tilde{\beta}(T-t)}, \qquad \tilde{\beta} = \mu_0 + \frac{\beta}{1-\gamma}. \tag{1.7}$$

We are finally led to investigate a simple stochastic control problem of optimizing (1.5) (1.6) under the scalar dynamics (1.4).

1.1.2 *Solution*

We investigate first the finite horizon problem.

1.1.2.1 Finite Horizon

We apply a standard dynamic programming technique. The Bellman equation for the Value function $V(t,w)$ is, making use of the notation (1.2),

$$\begin{aligned} &\forall (t,w) \in [0,T] \times \mathbb{R}, \\ &\frac{\partial V}{\partial t} + \max_{\varphi \in \mathbb{R}^n, \chi \in \mathbb{R}_+} \left[\frac{\partial V}{\partial w}(\varphi^t \lambda - \chi)w + \frac{w^2}{2}\varphi^t \Sigma \varphi \frac{\partial^2 V}{\partial w^2} + U(t,\chi w) \right] = 0, \\ &\forall w \in \mathbb{R}, \quad V(T,w) = B(w). \end{aligned} \tag{1.8}$$

We replace U and B with (1.6) and look for a solution of the form

$$V(t,w) = P(t)^{1-\gamma}w^{\gamma}.$$

The simplifying fact is that now all individual terms in the equation have a coefficient w^{γ}, so that we may divide through by it, obtaining an ordinary differential equation for $P(t)$:

$$\begin{aligned} \forall (t,w) \in [0,T] \times \mathbb{R}, \qquad &(1-\gamma)P(t)^{-\gamma}\dot{P}(t) + \max_{\varphi \in \mathbb{R}^n, \chi \in \mathbb{R}_+} \Big[\gamma P(t)^{1-\gamma}(\varphi^t \lambda - \chi) \\ &+ \frac{1}{2}\varphi^t \Sigma \varphi \gamma(\gamma - 1)P(t)^{1-\gamma} + p(t)^{1-\gamma}\chi^{\gamma} \Big] = 0, \end{aligned}$$

$$P(T) = \Pi.$$

Moreover, the maximizations in φ on the one hand and in χ on the other hand separate and yields further simplifications, giving with extremely simple calculations the optimal $\varphi^\star$ and $\chi^\star$ as

$$\varphi^\star(t) = \frac{1}{1-\gamma}\Sigma^{-1}\lambda, \qquad \chi^\star(t) = \frac{p(t)}{P(t)}. \tag{1.9}$$

And, finally, the equation for $P(t)$ is

$$\dot{P} + \frac{\gamma}{2(1-\gamma)^2}\lambda^t\Sigma^{-1}\lambda P + p = 0. \tag{1.10}$$

Let, therefore,

$$\alpha := \frac{\gamma}{2(1-\gamma)^2}\lambda^t\Sigma^{-1}\lambda, \tag{1.11}$$

to get

$$P(t) = \mathrm{e}^{\alpha(T-t)}\Pi + \int_t^T \mathrm{e}^{\alpha(s-t)}p(s)\,\mathrm{d}s. \tag{1.12}$$

Notice also that if $p(\cdot)$ is differentiable, then $\kappa = 1/\chi^\star$ can be directly obtained as the solution of a linear differential equation:

$$\dot{\kappa} + \left(\frac{\dot{p}}{p} + \alpha\right)\kappa + 1 = 0, \qquad \kappa(T) = \frac{\Pi}{\pi}. \tag{1.13}$$

Many comments are in order. More complete discussions of this classic result can be found in textbooks. We will make a minimum number of remarks.

Remark 1.1. 1. Formula (1.9) for $\varphi^\star$ yields a constant composition of the portfolio. Withdrawals for consumption should be made proportionally.
2. This formula is reminiscent of the corresponding formula in Markowitz's theory of static portfolio optimization (where S is a covariance matrix). See [110].
3. A "small" covariance matrix S tends to produce a "large" φ^*, leaving a smaller share $\varphi_0^\star = (1 - \sum_{i=1}^n \varphi_i^\star)$ for the riskless asset. Specifically, if $\langle 1\!\!1, \Sigma^{-1}\lambda\rangle \geq 1-\gamma$, then the prescription is to borrow cash to invest in risky assets.
4. $\chi^\star$, though not constant, is also exceedingly simple, given by (1.9), (1.11), (1.12) or by $\chi^\star = 1/\kappa$ and (1.13) if p is differentiable.
5. It readily follows from (1.10) that $P(\cdot)$ is always decreasing, so that $P(t) \geq \Pi$ for all t. Hence if $\Pi > 0$, then $\chi^\star(t) \leq p(t)/\Pi$.
6. If $p(t)$ is chosen according to (1.7), then $\dot{p}/p = -\tilde{\beta}$ is constant, and $\chi^\star$ is obtained via (1.13) as a closed form. Let $\beta' = \tilde{\beta} - \alpha$. We shall see in the next subsection that it is desirable that it be positive. Then one obtains $\chi^\star = [(\Pi/\pi - 1/\beta')\exp[-\beta'(T-t)] + 1/\beta']^{-1}$. Noticeably, if $\beta' > \pi/\Pi$, then this ensures that $\pi/\Pi < \chi^\star < \beta'$.

7. A "large" Π and a small p make for a smaller $\chi^\star$. If one cares about the bequest to the next period, he should be parsimonious. In contrast, if $\Pi = 0$, then in the end, as $t \to T$, $\chi^\star(t) \to \infty$. The entire portfolio is sold for consumption.

1.1.2.2 Infinite Horizon

The concern for long-run wealth, represented by the bequest function, may be addressed by a utility performance index of the form

$$J = \mathbb{E}\pi^{1-\gamma} \int_0^\infty \mathrm{e}^{-(\beta+\mu_0)t} C^\gamma \,\mathrm{d}t.$$

(The coefficient $\pi^{1-\gamma}$ is there for the sake of preserving the dimension of J as a currency amount.)

To deal with that case, in the portfolio model of Sect. 1.1.1.2, we set $T = 0$. And we write the new criterion using $\tilde{\beta}$ as in (1.7):

$$J = \mathbb{E}\pi^{1-\gamma} \int_0^\infty \mathrm{e}^{-(1-\gamma)\tilde{\beta}t} c^\gamma \mathrm{d}t.$$

Equation (1.8) is now replaced by its stationary form:

$$(1-\gamma)\tilde{\beta}V = \max_{\varphi\in\mathbb{R}^n,\chi\in\mathbb{R}_+} \left[\frac{\partial V}{\partial w}(\varphi^t\lambda - \chi)w + \frac{w^2}{2}\varphi^t\Sigma\varphi\frac{\partial^2 V}{\partial w^2} + \pi^{1-\gamma}\chi^\gamma w^\gamma \right].$$

Calculations completely similar to those of the previous paragraph show that the optimum exists if and only if $\tilde{\beta} > \alpha$. (Otherwise, the portfolio may yield an infinite utility.) We find $P = \pi/(\tilde{\beta} - \alpha)$, so that we finally get

$$\varphi^\star = \frac{1}{1-\gamma}\Sigma^{-1}\lambda, \qquad \chi^\star = \tilde{\beta} - \alpha. \tag{1.14}$$

Similar remarks can be made as above. We leave them to the reader.

1.1.3 Logarithmic Utility Functions

Other forms of the utility functions lead to closed-form solutions. Such an instance is $U(C) = -\exp(-\gamma C)$, $\gamma > 0$. Yet it is considered less realistic in terms of representing the risk aversion of the portfolio manager. We refer to [118] for further discussion. We propose here a different extension.

Adding to the criterion to be maximized a number independent of the controls φ and χ clearly does not change the choice of optimal controls. The same holds if we multiply the criterion by a positive constant. Thus the criterion

$$\tilde{J}_\gamma = \mathbb{E}\left[\Pi^{1-\gamma}\frac{w^\gamma(T)-1}{\gamma} + \int_0^T p^{1-\gamma}(t)\frac{w^\gamma(t)-1}{\gamma}\,\mathrm{d}t\right]$$

leads to the same optimal controls as the original one. However, this new criterion presents the added feature that, as $\gamma \to 0$, it has a limit

$$\tilde{J}_0 = \mathbb{E}\left[\Pi \ln w(T) + \int_0^T p(t)\ln w(t)\,\mathrm{d}t\right].$$

We therefore expect that the same formulas for $\varphi^\star$ and $\chi^\star$, but with $\gamma = 0$, should hold for the criterion $\tilde{J}_0$ with logarithmic utility functions. This is indeed correct. However, the Value function is less simple. It is nevertheless a simple exercise to check that the following Value function $\widetilde{V}$, with

$$P(t) = \Pi + \int_t^T p(s)\,\mathrm{d}s,$$

satisfies the Bellman equation with the same formulas $\varphi^\star = \Sigma^{-1}\lambda$, $\chi^\star = p/P$. The Value function is

$$\widetilde{V}(t,w) = P(t)\ln w + \frac{\lambda^t\Sigma^{-1}\lambda}{2}\left[(T-t)\Pi + \int_t^T (s-t)p(s)\,\mathrm{d}s\right] + \int_t^T p(s)\left(\ln\frac{p(s)}{P(s)} - 1\right)\mathrm{d}s.$$

1.2 A Discrete-Time Model

We follow essentially the same path as in the continuous trading problem, but with a discrete-time model, generalizing somewhat Samuelson's solution [135]. But it will be convenient to postpone somewhat the description of the market model.

1.2.1 Problem and Notation

1.2.1.1 Dynamics and Portfolio Model

Market

We want to allow discrete transactions, with a fixed time step h between transactions, an integer submultiple of T. We set $T = Kh$ and $\mathbb{K} := \{0, 1, \dots, K-1\}$. Let, therefore, $t_k = kh$, $k \in \mathbb{K}$ be the trading instants.

As previously, the index 0 denotes a riskless asset for which

$$S_0(t_k) = \exp(\mu_0(k-K)h)S_0(T) = R(t_k)S_0(T).$$

For $i = 1, \dots, n$, let, as previously, $u_i(t_k) = S_i(t_k)/R(t_k)$, and let

$$\tau_i(t) = \frac{u_i(t+h) - u_i(t)}{u_i(t)}$$

be the relative price increment of asset i in one time step,[1] so that we have

$$u_i(t_{k+1}) = (1 + \tau_i(t_k))u_i(t_k).$$

The n-vector of τ_i is as usual denoted by τ.

Portfolio

Our portfolio is, as in the continuous trading theory, composed of X_i shares of asset number i, $i = 0, 1, \dots, n$. Its worth is again

$$W(t_k) = \sum_{i=0}^{n} X_i(t_k)S_i(t_k).$$

We prefer to use

$$w(t_k) = \frac{W(t_k)}{R(t_k)} = \sum_{i=0}^{n} X_i(t_k)u_i(t_k).$$

We allow the portfolio manager to change the X_i at each time t_k. Hence, we must distinguish values before and after the transactions. We let $X_i(t_k)$, $W(t_k)$, and $w(t_k)$ denote the values *before the transactions* of time t_k and, when needed, $X_i(t_k^+)$, $W(t_k^+)$, and $w(t_k^+)$ be their values *after the transactions* of time t_k [with $X_i(t_k^+) = X_i(t_{k+1})$]. One exception to this rule is that $\varphi_i(t_k)$ will denote the fractions *after the transactions*. The transactions of time t_k may decrease the worth of the portfolio by an amount $C(t_k) = R(t_k)c(t_k)$ available for immediate consumption. Hence

$$\sum_{i=0}^{n}(X_i(t_k^+) - X_i(t_k))u_i(t_k) + c(t_k) = 0$$

and

$$W(t_k^+) = W(t_k) - C(t_k), \qquad w(t_k^+) = w(t_k) - c(t_k).$$

Let

[1] We choose to consider τ_i as dimensionless, but this is an increment *per time step*, so that it might be considered the inverse of a time. Avoiding that ambiguity complicates the notation.

$$\varphi_i(t_k) = \frac{X_i(t_k^+)S_i(t_k)}{W(t_k^+)} = \frac{X_i(t_k^+)u_i(t_k)}{w(t_k^+)} \quad \text{and} \quad \chi(t_k) = \frac{C(t_k)}{W(t_k)} = \frac{c(t_k)}{w(t_k)}$$

be the decision variables of the manager. As previously, φ may lie anywhere in $\mathbb{R}^n$, while now, χ is constrained to be nonnegative and no more than one.

One easily obtains the dynamics of the portfolio as

$$w(t_{k+1}) = [1+\varphi^t(t_k)\tau(t_k)][1-\chi(t_k)]w(t_k). \tag{1.15}$$

1.2.1.2 Utility

We assume that the portfolio manager wants to maximize a weighted sum of the expected utility of future consumption and of the expected utility of the portfolio worth at final time $T = t_K = Kh$; hence a performance index of the form

$$J = \mathbb{E}\left[B(w(T)) + \sum_{k=0}^{K-1} U(t_k, c(t_k))\right]. \tag{1.16}$$

And as in the continuous trading theory, we shall specialize the analysis to fractional power utility functions. Let Π be a given nonnegative constant and $\{p_k\}_{k\in\mathbb{K}}$ a given sequence of nonnegative numbers. We set

$$U(t_k, c) = p_k^{1-\gamma}c^\gamma = p_k^{1-\gamma}\chi^\gamma w^\gamma, \qquad B(w) = \Pi^{1-\gamma}w^\gamma. \tag{1.17}$$

In formula (1.16), p_K is not used. It will be convenient to define it as

$$p_K = \Pi. \tag{1.18}$$

We shall consider the logarithmic utility in Sect. 1.2.2.3.

1.2.2 Solution

1.2.2.1 Finite Horizon

We have to optimize criterion (1.16), (1.17) with the dynamics (1.15). We do this via dynamic programming. Bellman's equation reads

$$\forall (k,w) \in \mathbb{K}\times\mathbb{R}_+, V(t_k, w)$$
$$= \max_{\varphi\in\mathbb{R}^n, \chi\in[0,1]}\left\{\mathbb{E}V\Big(t_{k+1}, (1+\varphi^t\tau(t_k))(1-\chi)w\Big) + p_k^{1-\gamma}\chi^\gamma w^\gamma\right\}, \tag{1.19}$$

with the terminal condition

$$\forall w \in \mathbb{R}_+, \qquad V(T,w) = \Pi^{1-\gamma} w^\gamma. \tag{1.20}$$

Assume that, for some number P_{k+1},

$$V(t_{k+1},w) = P_{k+1}^{1-\gamma} w^\gamma,$$

which is true for $k = K-1$ with $P_K = \Pi$. Then (1.19) yields

$$V(t_k,w) = \max_{\varphi\in\mathbb{R}^n,\chi\in[0,1]} \left\{ \mathbb{E}P_{k+1}^{1-\gamma}\Big([1+\varphi^t\tau(t_k)](1-\chi)\Big)^\gamma + p_k^{1-\gamma}\chi^\gamma \right\} w^\gamma,$$

hence $V(t_k,w) = P_k^{1-\gamma} w^\gamma$, with

$$P_k^{1-\gamma} = \max_{\varphi\in\mathbb{R}^n,\chi\in[0,1]} \left\{ \mathbb{E}P_{k+1}^{1-\gamma}\Big([1+\varphi^t\tau(t_k)](1-\chi)\Big)^\gamma + p_k^{1-\gamma}\chi^\gamma \right\}.$$

This recurrence formula for P_k may be simplified as follows. Notice first that it can be written as

$$P_k^{1-\gamma} = \max_{\chi\in[0,1]} \left\{ P_{k+1}^{1-\gamma} \max_{\varphi\in\mathbb{R}^n} \mathbb{E}[1+\varphi^t\tau(t_k)]^\gamma(1-\chi)^\gamma + p_k^{1-\gamma}\chi^\gamma \right\}.$$

In the preceding equation, on the right-hand side, the market parameters enter only the term

$$L(\varphi) := \mathbb{E}[1+\varphi^t\tau(t_k)]^\gamma. \tag{1.21}$$

Let α be defined as

$$\alpha^{1-\gamma} := \max_{\varphi\in\mathbb{R}^n} L(\varphi). \tag{1.22}$$

This is a characteristic of the market. With this notation, the recursion for P_k becomes

$$P_k^{1-\gamma} = \max_{\chi\in[0,1]} [P_{k+1}^{1-\gamma}\alpha^{1-\gamma}(1-\chi)^\gamma + p_k^{1-\gamma}\chi^\gamma].$$

We now use the following "little lemma."

Lemma 1.2. *Let p, q, and r be positive numbers, and $\gamma \in (0,1)$. Then*

$$\max_{x\in[0,r]} \{p^{1-\gamma}x^\gamma + q^{1-\gamma}(r-x)^\gamma\} = (p+q)^{1-\gamma}r^\gamma,$$

it is obtained for

$$x = \frac{p}{p+q}r, \qquad 1-x = \frac{q}{p+q}r.$$

Proof. It suffices to equate the derivative with respect to x to zero,

$$\gamma p^{1-\gamma}x^{\gamma-1} - \gamma q^{1-\gamma}(r-x)^{\gamma-1} = 0,$$

to get

$$\frac{r-x}{x} = \frac{q}{p},$$

hence $x = rp/(p+q)$, which lies in $(0,r)$, and place this back in the quantity to maximize. We check the second derivative:

$$\gamma(\gamma-1)[p^{1-\gamma}x^{\gamma-2} + q^{1-\gamma}(r-x)^{\gamma-2}]$$

is negative for all $x \in [0,1]$ since $\gamma - 1 < 0$. □

As a consequence, we find that

$$P_k = \alpha P_{k+1} + p_k, \qquad P_K = \Pi = p_K;$$

hence, recalling (1.18) and (1.20),

$$P_k = \sum_{\ell=k}^{K} \alpha^{(\ell-k)} p_\ell.$$

We also obtain that the optimal consumption ratio $\chi^\star$ is

$$\chi^\star(t_k) = \frac{p_k}{P_k}.$$

The optimal $\varphi^\star$, as well as the precise value of α, depends on the probability law we adopt in the market model. We shall consider that question hereafter, but we may nevertheless make some remarks similar to those for the continuous-time theory.

Remark 1.3. 1. $\varphi^\star$, maximizing $L(\varphi)$, is constant, depending only on the market model.
2. α is a measure of the efficiency of the market. The P_k are increasing in α. Hence $\chi^\star$ is decreasing in α. As α goes to 0, χ^* goes to 1.
3. If Π is large and the p_k, $k < K$, are small, then χ^* is small. In contrast, if $\Pi = 0$, then $\chi^\star(t_{K-1}) = 1$. The entire portfolio is sold for consumption in the last step.

1.2.2.2 Infinite Horizon

We investigate now the formulation in an infinite horizon, which is another way of addressing the long-run worth of the portfolio. Therefore, let a discount constant δ be given, and let the performance index be

$$J = \mathbb{E}\sum_{k=0}^{\infty} \delta^{-kh} c(t_k)^\gamma = \mathbb{E}\sum_{k=0}^{\infty} \delta^{-kh} \chi(t_k)^\gamma w(t_k)^\gamma.$$