
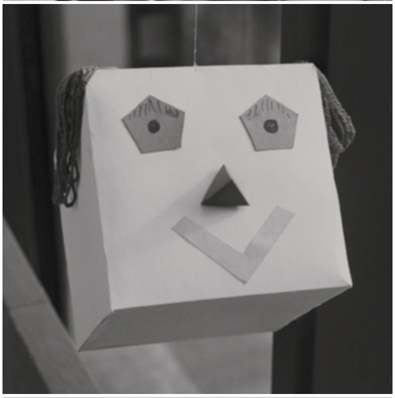
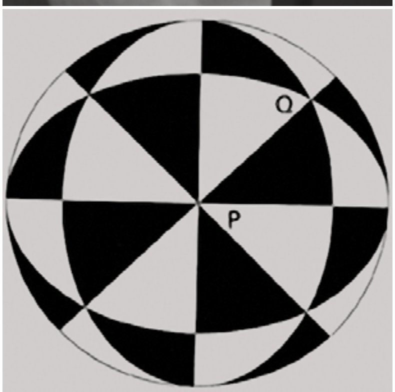


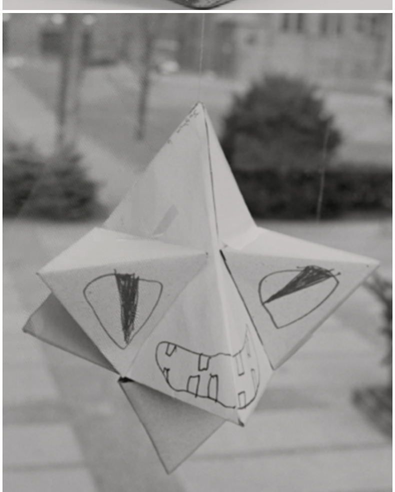

Marjorie Senechal *Editor*

SHAPING SPACE

Exploring Polyhedra in Nature, Art, and the Geometrical Imagination

Springer

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Editor

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Exploring Polyhedra in Nature, Art, and the Geometrical Imagination

with George Fleck and Stan Sherer

Springer

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Part III Polyhedra in the Geometrical Imagination

Preface

Molecules, galaxies, art galleries, sculpture, viruses, crystals, architecture, and more: *Shaping Space: exploring polyhedra in nature, art, and the geometrical imagination* is an exuberant survey of polyhedra in nature and art. It is at the same time hands-on, mind-turned-on introduction to one of the oldest and most fascinating branches of mathematics. In these pages you will meet some of the world's leading geometers, and learn what they do and why they do it. In short, *Shaping Space* is as many-faceted as polyhedra themselves.

Shaping Space is a treasury of ideas, history, and culture. For students and teachers, from elementary school to graduate school, it is a text with context. For the multitude of polyhedra hobbyists, an indispensable handbook. *Shaping Space* is a resource for professionals—architects and designers, painters and sculptors, biologists and chemists, crystallographers and physicists and earth scientists, engineers and model builders, mathematicians and computer scientists. If you are intrigued by the exquisite shapes of crystals and want to know how nature builds them, if you marvel at domes and wonder why most stay up but some fall down, if you wonder why Plato thought earth, air, fire and water were made of polyhedral particles, if you wonder what geometry is and are willing to try it yourself, this book is for you. In *Shaping Space* you will see that polyhedra are as new as they are old, and that they continue to shape our spaces in new and exciting ways, from computer games to medical imaging.

The computer revolution has catalyzed new research on polyhedra. A quarter century ago, discrete and computational geometry (the branch of mathematics to which polyhedra belong) was less a field in its own right than—in the eyes of many people, even many mathematicians—a grab-box of mathematical games. Today an international journal, *Discrete and Computational Geometry*, publishes six issues a year with the latest research on configurations and arrangements, spatial subdivision, packing, covering, and tiling, geometric complexity, polytopes, point location, geometric probability, geometric range searching, combinatorial and computational topology, probabilistic techniques in computational geometry, geometric graphs, geometry of numbers, and motion planning, and papers with a distinct geometric flavor in such areas as graph theory, mathematical programming, real algebraic geometry, matroids, solid modeling, computer graphics, combinatorial optimization, image processing, pattern recognition, crystallography, VLSI design, and robotics.

Figure 1. An icosahedron built and decorated by elementary school children. Photograph by Stan Sherer.

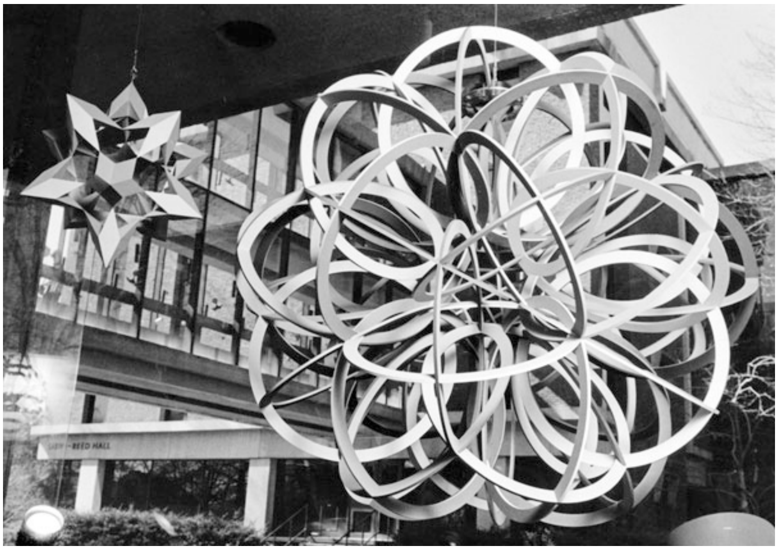

Figure 2. Sculptures by Morton Bradley. Photograph by Stan Sherer.

Yet, it is also true, as the saying goes, *plus ça change, plus c'est la même chose*. The more things change, the more they stay the same, especially in school mathematics curricula. Despite its central importance in the sciences, the arts, in mathematics and in engineering, solid geometry has all but vanished from the schools, plane geometry is being squeezed to a minimum, and model-building is relegated to kindergarten. Reasons for this unfortunate (and, unfortunately, long-term) trend include a lack of teacher training and pressures to teach testable skills. But educators will realize, sooner rather than later, that "technology in the classroom" is more than clicking the latest gadgets, it means understanding our technological world. Geometry will reappear as a blend of model-building, engineering and fundamental math and science.

Meanwhile, the internet is helping to bring geometry back to life and with it a community of geometers. You can explore polyhedra in nature, art, and the geometrical imagination on the world wide web by yourself, with *Shaping Space* as your guide, and share your findings and frustrations with the like-minded through chat groups. Keep pencils, paper, a ruler, scissors, and tape handy: Confucius got it right 2500 years ago:

> I hear and I forget,
> I read and I remember,
> I do and I understand.

Shaping Space will evolve as the subject grows. The notes and references at the end of this book are also on my website, http://www.marjoriesenechal.com. Authors will post updates there; you will also find links to instructional and recreational materials, and to websites of polyhedra-minded scientists, artists and hobbyists. Visit often!

Indeed, *Shaping Space* has grown already. Its ancestor, *Shaping Space: a polyhedral approach* was inspired by a three-day festival of workshops, exhibitions and lectures on polyhedra held at Smith College in 1984. *Shaping Space: Exploring Polyhedra in Nature, Art, and the Geometrical Imagination* includes the best of the past and new chapters by Robert Connelly, Erik Demaine (with Martin Demaine and Vi Hart), George Hart, Joseph O'Rourke, Ileana Streinu, and Günter Ziegler (with Moritz Schmidt).

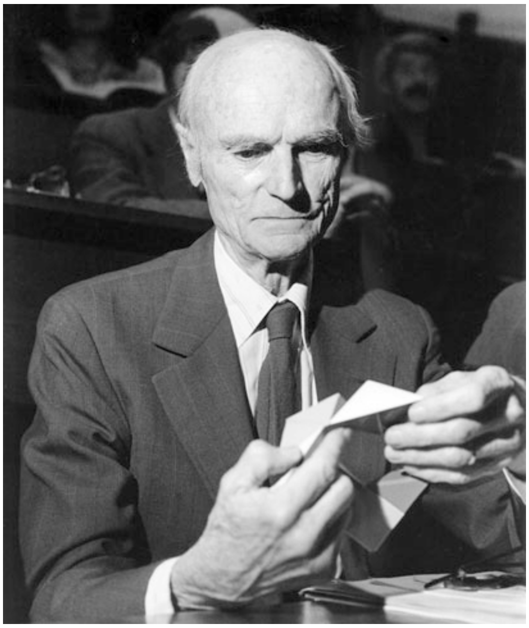

Figure 3. H. S. M. Coxeter (1907–2003). Photograph by Stan Sherer.

Figure 4. Arthur L. Loeb (1923–2002). Photograph by Stan Sherer.

Shaping Space: exploring polyhedra in nature, art, and the geometrical imagination is dedicated to the memory of two friends and colleagues, the legendary geometer H. S. M. Coxeter and the many-faceted design scientist Arthur L. Loeb. Without their enthusiasm, encouragement, support and participation, the Shaping Space Conference could not have been held and the first edition of this book might never have appeared. They continue to inspire us.

Northampton, MA, USA Marjorie Senechal

Part I

First Steps

1

Introduction to the Polyhedron Kingdom

Marjorie Senechal

What is a polyhedron? The question is short, the answer is long. Although you may never have heard of the Polyhedron Kingdom before, it is nearly as vast and as varied as the animal, mineral, and vegetable kingdoms (and it overlaps all three of them). There are aristocrats and workers, families and individuals, old polyhedra with long and interesting histories and young polyhedra born yesterday or the day before. In this kingdom you can take a walking tour of polyhedral architecture, visit a nature preserve and an art gallery and an artisans' polyhedra fair. As you stroll along you may even glimpse polyhedral ghosts from four-dimensional space.

The boundaries of the Polyhedron Kingdom are in dispute (as are those of most kingdoms) but it is safe to visit the border areas. You need not worry about the nature of the disputes until the last part of this book.

The language of the Polyhedron Kingdom is mathematics but, for this brief first visit, you can get by if you learn three important words: face, edge, and vertex. The word *polyhedron* comes from the Greek word for "many" and an Indo-European word for "seat." To geometers, it means an object with many faces. In Figures 1.1 and 1.2 we see polyhedra with faces. But this is not what we mean when we speak of the faces

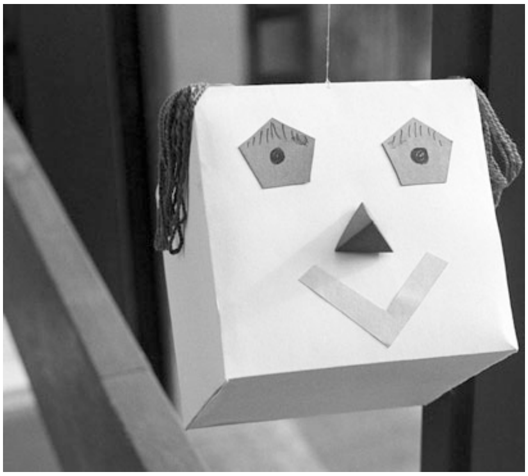

Figure 1.1. Cube with face, by a fifth-grade student at the Smith College Campus School.

of a polyhedron. For our purposes the *faces* of a polyhedron are the polygons from which its surface is constructed. The *edges* of a polyhedron are the lines bounding its faces; its *vertices* are the corners where three or more faces (and thus three or more edges) meet (Figure 1.3). You will see as we go along that these terms can have more general meanings, but these definitions will do for the moment. As you tour the Polyhedron Kingdom you will become more comfortable with an increasing vocabulary and a wider range of common usages.

We begin our tour, of course, with a visit to the rulers of the Kingdom.

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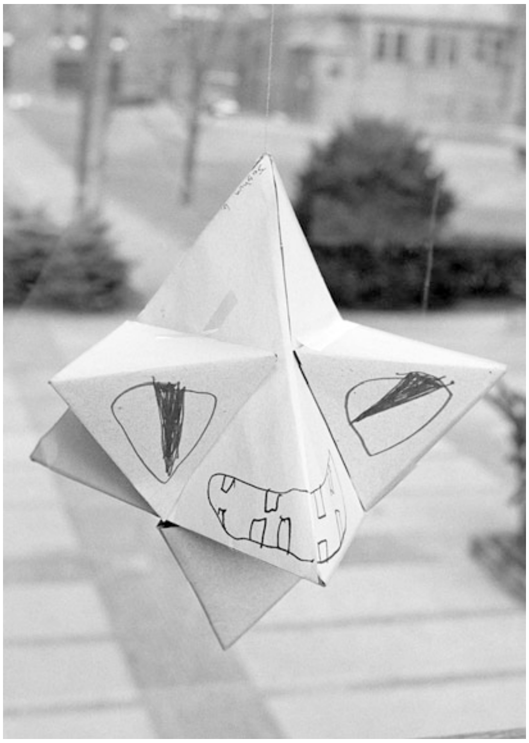

Figure 1.2. A polyhedral monster, also by Campus School student.

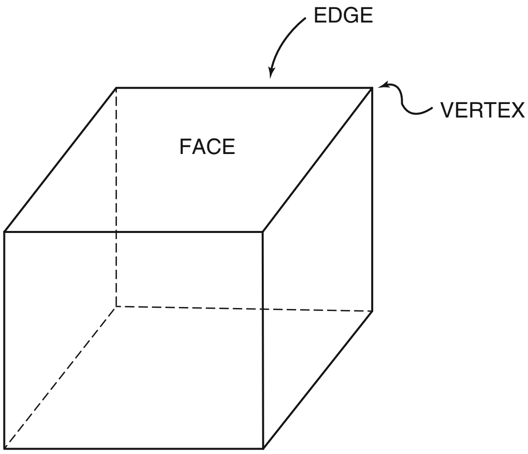

Figure 1.3. The cube has six faces, twelve edges, and eight vertices.

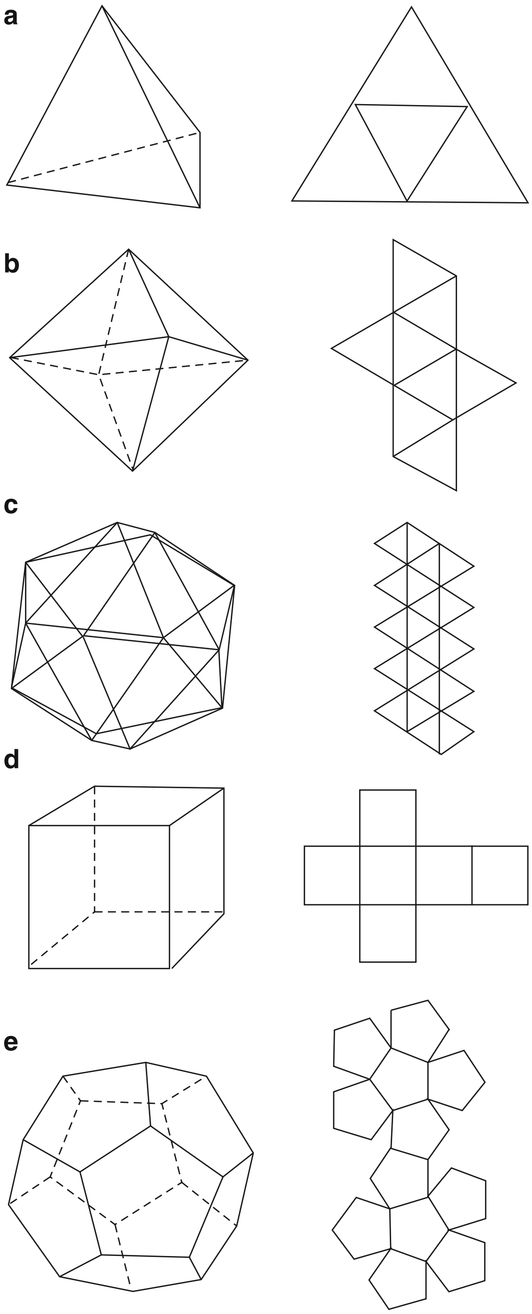

Figure 1.4. *Left*: the five regular polyhedra. *Right*: the same, "unfolded" into planar nets (For more about nets for polyhedra and some unsolved problems concerning them, see Chapters 6 and 22).

The Regular "Solids"

At the gates of the Kingdom live its rulers, the famous and venerable regular "solids" pictured in Figure 1.4. Each of these polyhedra is called *regular* because of certain very special properties: its faces are identical regular polygons, and the same number of polygons meet at each vertex.

(Remember that regular polygons are polygons whose edges have equal lengths and whose angles have equal measure: a regular polygon of three edges is an equilateral triangle, of four edges a square, and so on.) So the faces of each regular polyhedron are all alike and their vertices (or, more precisely, the arrangements of polygons at their vertices) are all alike too.

If we try to build polyhedra with the regularity property just described, we will quickly find that there are only five possibilities. We start by constructing polyhedra whose faces are equilateral triangles. First, we can put three triangles together to form one vertex of a polyhedron. If we continue this pattern at all the other corners, we obtain a pyramid that has four triangular faces, four vertices, and six edges; this is the regular *tetrahedron* (Figure 1.4a). If we put four triangles at each vertex, we can build an *octahedron* (Figure 1.4b); if we put five together then we get the *icosahedron* (Figure 1.4c). Six equilaterial triangles fit together around a point to form a flat surface, so that arrangement is out. And if we try to fit seven or more together—well, try it and see what happens! So these three polyhedra are the only regular ones that can be built out of equilateral triangles.

Now let us try to build a regular polyhedron out of squares. We see that there is just one possibility, the *cube* (Figure 1.4d), in which three faces meet at each vertex, because four squares in a plane lie flat around a point. (What happens if we try to fit five?) If we use regular pentagons, we can again build just one solid, the *dodecahedron* (Figure 1.4e). We cannot continue this procedure with regular polygons with a greater number of sides because three regular hexagons lie flat, three or more heptagons or octagons buckle, and so forth. We conclude that there are no other regular polyhedra.

The regular polyhedra are also known as the "Platonic solids" because the Greek philosopher Plato (427–347 B.C.E.) immortalized them in his dialogue *Timaeus*. In this dialogue Plato discussed his ideas about the "elements" of which he believed the universe to be composed: earth, air, fire, and water. Today when we think of "element," we usually think of the chemical elements in the Periodic Table. (We recognize the solid, gas, plasma, and liquid *states* of matter.) But notice that we still speak of needing protection from the "elements," and when we say this we mean snow, wind, lightning, and rain. In *Timaeus*, Plato argued that the geometric forms of the smallest particles of these elements are the cube, the octahedron, the tetrahedron, and the icosahedron, respectively. (The fifth regular solid, the dodecahedon, was assigned to the Great All, the cosmos.) This association of the regular solids with the elements captured the imagination of many people from Plato's time to our own. The twentieth century artist M.C. Escher presented them in various ways; Figure 1.5 might be subtitled "Platonic Puzzle," because all of the five Platonic solids appear in it in one form or another! Figure 1.6 shows an icosahedral candy box decorated by Escher.

Plato aside, do the regular polyhedra have any special significance outside the Polyhedra Kingdom? Maybe not. The astronomer Johannes Kepler (1571–1630) believed that he had at last discovered their true meaning: the spheres in which they can be inscribed, nested one inside another, are the divine model for the orbits of the six planets! This explained why there could be only six! (Kepler's ideas are discussed in detail by H.S.M. Coxeter in Chapter 3.) The beauty of the regular polyhedra has led scientists astray in our own time as well. In 1936 Dorothy Wrinch

Figure 1.5. *Reptiles*. Woodcut by M.C. Escher.

Figure 1.6. *Icosahedron with Starfish and Shells*, a candy box by M.C. Escher.

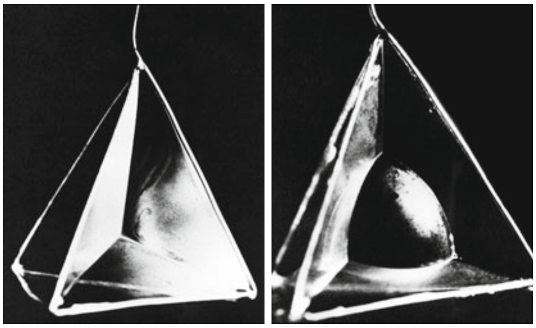

Figure 1.7. Soap films, made by dipping a tetrahedral wire frame into a soapy solution. Notice that the tetrahedral bubble has curved faces.

proposed a patterned octahedron as the first model for the molecular structure of proteins (Figure 1.8); unfortunately the structures of proteins have turned out to be much less elegant.

The regular polyhedra may not solve the riddle of the universe or reveal the secret of life, but they do crop up in the most unexpected places: for example, in the soap films shown in Figure 1.7 (if we agree that a polyhedron can have curved faces and edges), in decorative ornament (Figure 1.9), and as the shapes of many viruses.

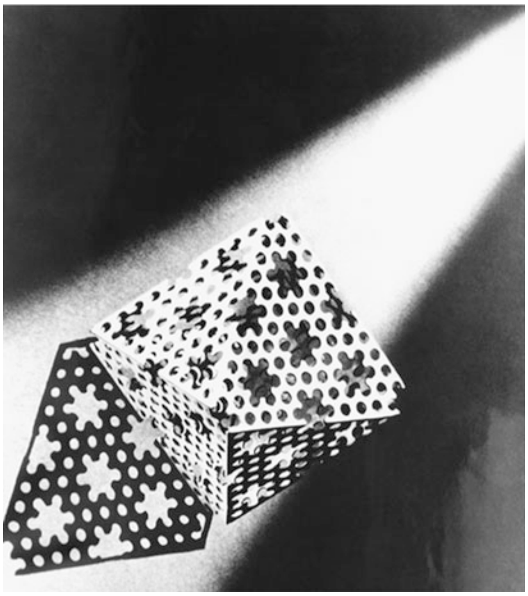

Figure 1.8. The model for protein structure proposed by Dorothy Wrinch in 1936.

Figure 1.9. The icosahedron and other polyhedra often appear as decorative elements in Baroque architecture; here, the church of Santissimi Apostoli by Borromini.

The shapes of many molecules are thought to be closely related to the regular polyhedra (Figure 1.10). Many crystals have cubic, octahedral, or dodecahedral forms; others are tetrahedral or icosahedral. But most dodecahedral (and icosahedral) crystals, like the pyrite crystals in Figure 1.11, are not regular. (Indeed, until November 1984, it was believed that regular dodecahedral and icosahedral crystals could not exist, because their symmetry is theoretically impossible for a crystal. Then some crystals with this symmetry were discovered, posing some challenging problems for symmetry theory!) Perhaps to make up for its limited role in the mineral kingdom, the regular dodecahedron with its twelve faces has been used by people in imaginative ways, such as street corner recycling bins in France (Figure 1.12).

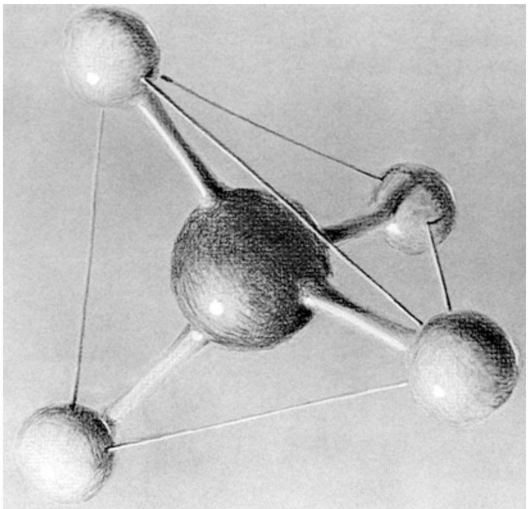

Figure 1.10. An artist's conception of a methane molecule.

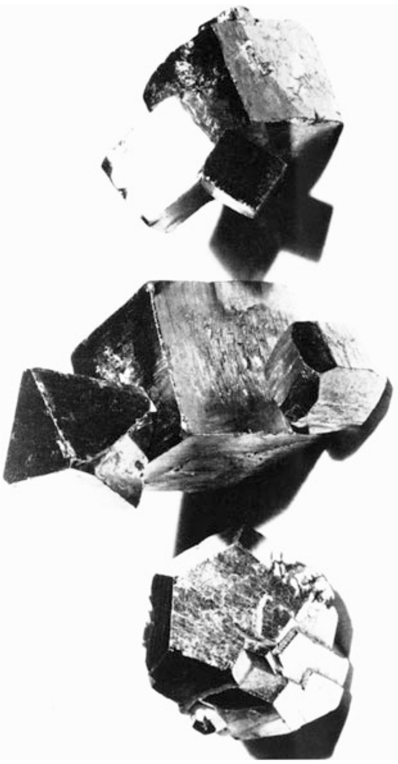

Figure 1.11. Pyrite crystals.

Today we believe that it is not the classical form of the regular polyhedra that is significant: instead it is the high degree of order which they represent. Indeed, as Figure 1.4 suggests, the regular "solids" are not always found in solid form. In some contexts, they have hollow interiors; in others, they have perforated surfaces; in yet others they have no faces, but appear as skeletons made of edges and vertices. Still, they are usually recognizable because of their high degree of *symmetry*. For example, all of the regular polyhedra have *mirror symmetry:* they can be divided into mirror-image halves in many different ways. They also have *rotational symmetry*: there are many ways in which they can be rotated without changing their apparent position. Both mirror symmetry and rotational symmetry are due to the fact that, for each of these polyhedra, every face, every vertex, and every edge is like every other. In other words, they are repetitively organized; this is one of the reasons that they are found so often in nature. This organization is also

Figure 1.12. Dodecahedral recycling bin for glass, on a street corner in Paris, France. Photograph by Marjorie Senechal.

aesthetically pleasing, and it is largely because of their symmetry that they are considered to be beautiful. The regular solids have the highest possible symmetry among polyhedra that are finite in extent. This is one reason why we can justly say that the regular solids are the rulers of the Polyhedron Kingdom. As you read through this book you will learn a great deal about symmetry.

Direct Descendants

There are many variations on the theme of the regular polyhedra. First let us meet the eleven (in Figure 1.13) which can be made by cutting off (*truncating*) the corners, and in some cases the edges, of the regular polyhedra so that all the faces of the faceted polyhedra obtained in this way are regular polygons. These polyhedra were first discovered by Archimedes (287–212 B.C.E.) and so they are often called Archimedean solids. Notice that vertices of the Archimedean polyhedra are all alike, but their faces, which are regular polygons, are of two or more different kinds. For this reason they are often called *semiregular*. (Archimedes also showed that in addition to the eleven obtained by truncation, there are two more semiregular polyhedra: the snub cube and the snub dodecahedron. (Also shown in Figure 1.13.)

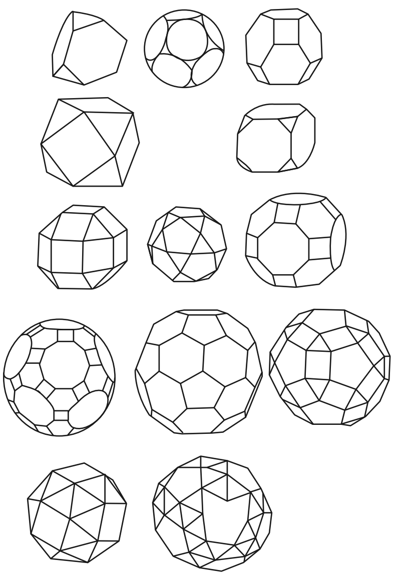

Figure 1.13. The Archimedean or semiregular polyhedra; The first eleven can be obtained from the regular polyhedra by truncation.

By this definition, *prisms* (see Figure 1.14) with regular polygonal bases and square sides are semiregular solids too. Prisms are quite common in nature and in architecture, as we will see later (Chapter 7). *Antiprisms* also have two identical polygonal faces, but the "top" face is rotated relative to the "bottom" one, so that the two polygons are joined by triangles (see Figure 1.15); when its faces are regular polygons, an antiprism is a semiregular polyhedron.

Perhaps the most elaborate variations on the theme of the regular polyhedra are those of the sixteenth-century Nuremberg goldsmith Wenzel Jamnitzer, who engraved a fascinating and extensive series of polyhedra in honor of Plato's theory of matter. In his book *Perspectiva Corporum Regularium*, published in 1568, each of the five regular solids is presented in exquisite variation. Can you tell which solid is being varied in Figure 1.16? Jamnitzer's figures show us that polyhedra need not be *convex*; that is, they can have indentations. Regular *polygons* that are not convex, such as the famous pentagram (Figure 1.17), are familiar to most of us. Such "star polygons"

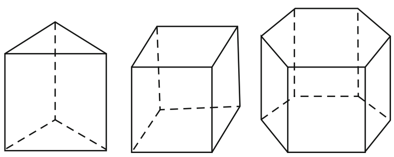

Figure 1.14. The three semiregular prisms.

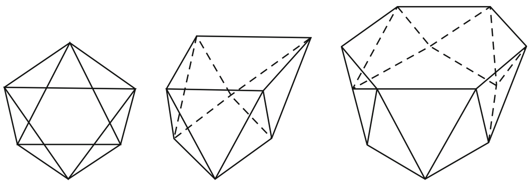

Figure 1.15. Three semiregular antiprisms.

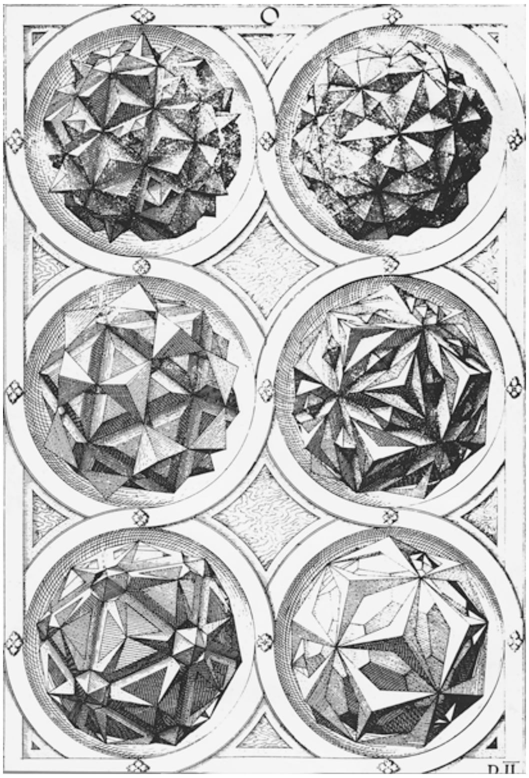

Figure 1.16. Plate D.II. from Wenzel Jamnitzer, *Perspectiva Corporum Regularium*, 1568.

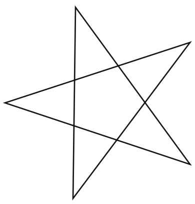

Figure 1.17. The pentagram has equal sides and equal angles.

can be used to build regular "star polyhedra." There are exactly four regular star polyhedra (see Figure 1.18). Notice that all their faces are regular polygons and the same number of faces meet at each vertex. In this case, however, either the faces or the vertex arrangements are pentagrams. The lineage of these polyhedra can be traced to fifteenth-century Venice (see Figure 1.19), but no general theory seems to have been developed at that time. Later Kepler investigated regular star polyhedra and found two of them; after that

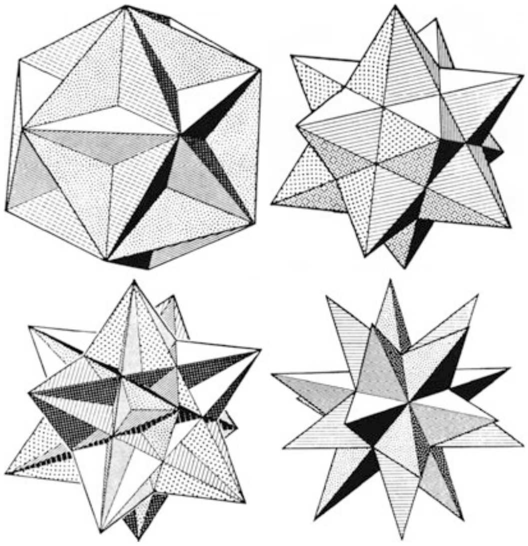

Figure 1.18. The four regular star polyhedra.

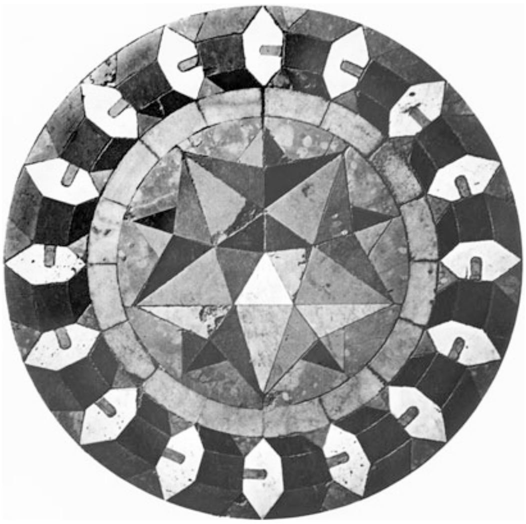

Figure 1.19. Marble tarsia (1425–1427) in the Basilica of San Marco, Venice, attributed to Paolo Uccello.

star-shaped polyhedra (not necessarily regular) became ubiquitous (see for instance Figure 1.20). But it was not until the early nineteenth century that two more regular star polyhedra were found and the French mathematician Augustin-Louis Cauchy (1789–1857) showed that there are no others (see Chapter 4).

The *uniform* polyhedra are polyhedra, star or otherwise, whose vertices are all symmetrically

Figure 1.20. Courtyard of Borromini church.

equivalent. (They are generalizations of the Archimedean polyhedra.) Perhaps the most spectacular uniform polyhedron is the Yog-Sothoth, shown in Figure 1.21. Although its existence had been predicted (on theoretical grounds) for many years, no one had ever seen it before Bruce Chilton's was presented to society for the first time at the Shaping Space Conference. The debut was a spectacular success. The Yog-Sothoth has 112 faces: 12 are pentagrams, 40 are triangles of one type, and 60 are triangles of another. Yet despite its complexity its symmetry is that of the icosahedron and dodecahedron, no more no less!

There are many other interesting lines of descent from the regular solids. For example, there are polyhedra whose faces are all alike but whose vertices are not. Closely related to the semiregular solids, these polyhedra are especially important in the study of crystal forms.

Impossible Polyhedra

Despite its diversity, the Polyhedra Kingdom is exclusive. You will not find polyhedra with any number of faces, edges, and vertices you might think up; only certain combinations are permitted. In the eighteenth century a Swiss Mathematician named Leonhard Euler discovered why. He found a curious relation among the numbers of faces, edges, and vertices of any convex polyhedron. (*Convex* means that the surface has no bumps or dents.) For example, a cube has six faces, twelve edges, and eight vertices; a tetrahedron has four faces, six edges, and four vertices. In both cases, the sum of the numbers of faces and vertices is two more than the number of edges. If we write F for the number of faces of a given polyhedron, V for the number of its vertices, and E for the number of its edges, we have a simple formula: $E = F + V + 2$.

This means, for example, that there is no polyhedron with four faces, six vertices, and nine edges. Nor—though this is harder to prove—can you build a "soccer ball" out of hexagons.

Next Steps

But before losing yourself in contemplation of the impossible, you should build some *possible* polyhedra with your own hands.

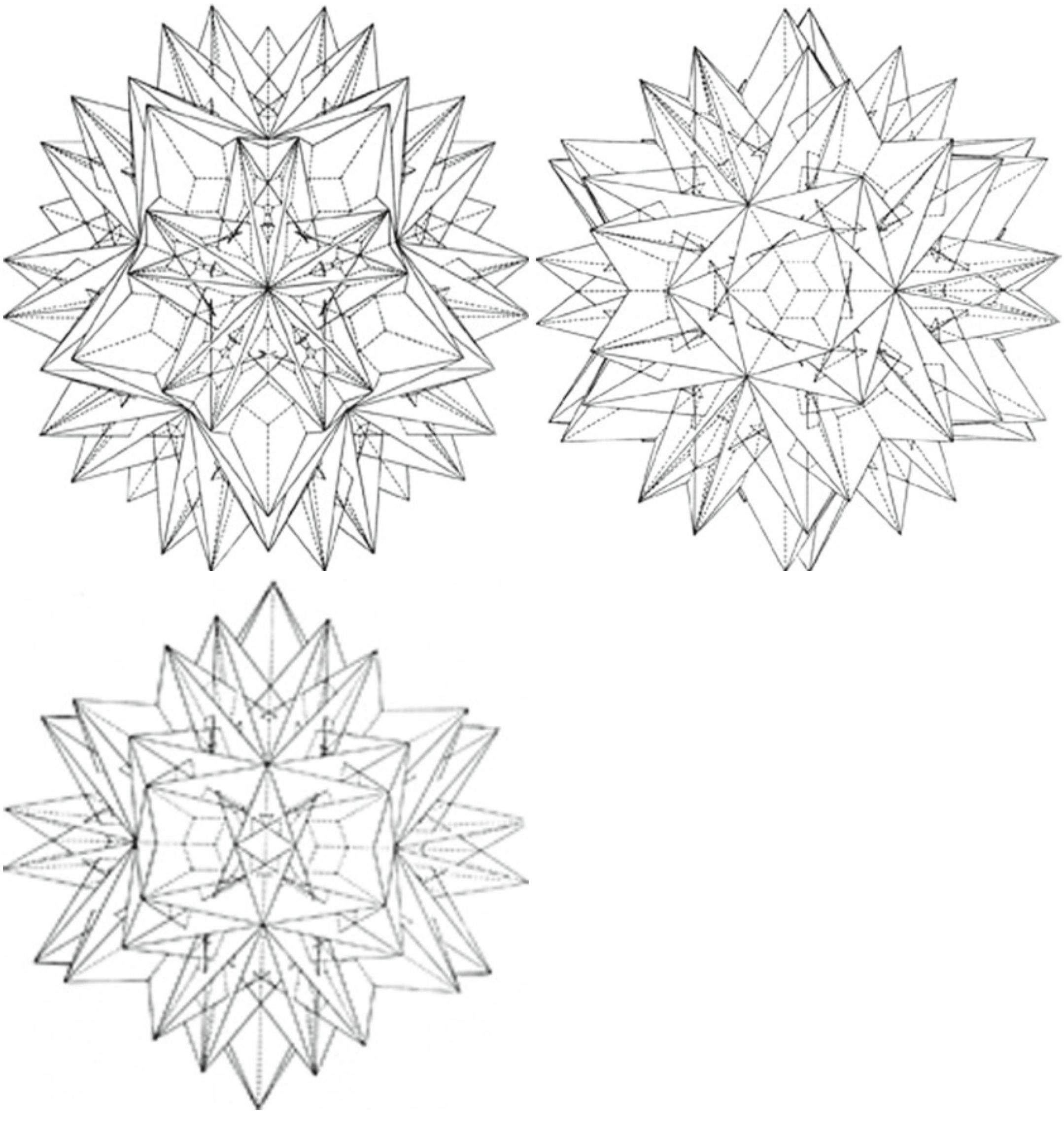

Figure 1.21. Three plan views of the Yog-Sothoth, along five-, three-, and twofold axes, drawn by Bruce L. Chilton.

2

Six Recipes for Making Polyhedra

Marion Walter, Jean Pedersen, Magnus Wenninger, Doris Schattschneider, Arthur L. Loeb, Erik Demaine, Martin Demaine, and Vi Hart

This chapter includes six "recipes" for making polyhedra, devised by famous polyhedrachefs. Some recipes are for beginners, others are intermediate or advanced. You can use these recipes, or devise your own. Building models is fun, and will give you a deeper understanding of the chapters that follow.

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Constructing Polyhedra Without Being Told How To!

Marion Walter

Getting Started: How to Attach Polygons

Put some cut-out regular polygons on a table. Put a little glue on a flat tile, a plastic lid, or a piece of plastic, and spread out the glue a little so that you can dip a whole edge of a polygon into the glue.

Choose two polygons that you want to glue together along an edge, and dip one of these edges in the glue. Dip lightly; if polygons don't stick well it is usually because there is too much glue (Figure 2.1).

Hold the two edges together firmly. The joint will remain flexible but the polygons will stick together (Figure 2.2).

If you find later that you need extra glue on an edge of a polygon that you have already attached, you can (lightly) dip a toothpick or applicator stick in the glue to smear some along an edge.

What Shape Are You Going to Make?

It is most fun and most rewarding to make a shape you yourself create rather than following someone else's plans. How can you do this?

M. Senechal (ed.), *Shaping Space*, DOI 10.1007/978-0-387-92714-5_2,

Figure 2.1.

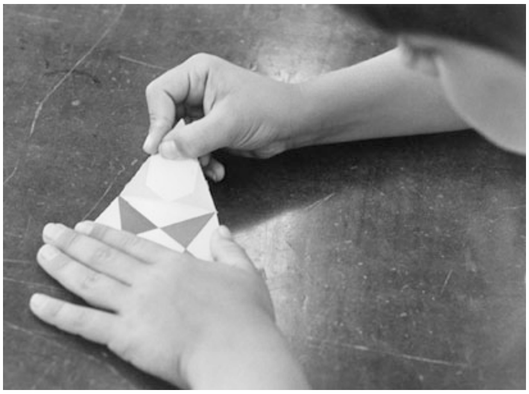

Figure 2.2.

Figure 2.3.

There are many ways to start. One way is to limit yourself to using only one or two different shapes — say triangles, or triangles and pen-

Figure 2.4.

Figure 2.5.

tagons, or triangles and squares. What shapes can you make using triangles and only one pentagon? (See Figure 2.3).

The first shape the boy shown in Figure 2.4 made has a pentagon for its base and triangles for sides. It is called a pentagonal pyramid. Now make up another question of your own. What will your first shape look like? When you experiment freely, you may get a few surprises and you will learn a lot. For example, six triangles lie flat.

What a surprise: the shape in Figure 2.5 lies flat too! Notice that the twelve triangles that surround the hexagon help to make a bigger hexagon. The student shown in the photograph

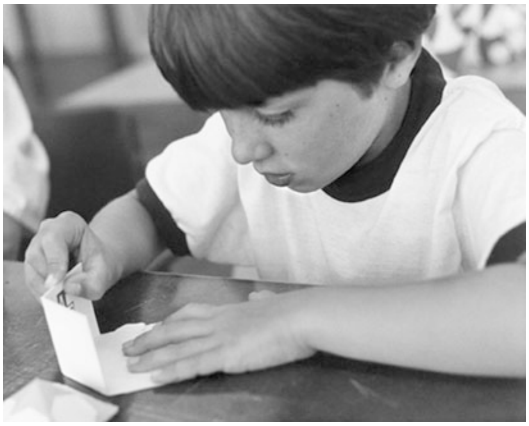

Figure 2.6.

Figure 2.7.

Figure 2.8.

Figure 2.9.

also had a surprise after she attached only six triangles to the hexagon. Do you think it will make a pyramid with a hexagonal base?

What shapes can you make with hexagons and squares? (See Figures 2.6 and 2.7).

Making shapes requires thinking ahead. Try to make a shape using only pentagons. What a relief: the two edges in Figure 2.8 really do seem to meet! How will the boy shown go on? Do the girls in Figures 2.9 and 2.10 seem to be making the same shape?

The shape in Figure 2.11 is made entirely of pentagons: how many of them were used? Turn it around and look at it. How many edges does it have? How many corners? How many edges meet at one corner? How many faces meet at a corner? This shape is a *dodecahedron*.

When you are experimenting, don't expect that your shape will always close! (Figure 2.12). Some shapes may have holes that you cannot fill with the shapes that we have; remember that we are using only regular polygons.

Shapes You Can Make with Triangles

The shape in Figure 2.13 is only one of the many you can make using just triangles. It is an *icosahedron*. Look at it from many sides. How many faces, edges, and corners does it have? Compare these numbers to the corresponding numbers you found for the dodecahedron.

In Figure 2.14 the girl is placing one five-sided pyramid over the base of another one. How many

Figure 2.10.

Figure 2.11.

Figure 2.12.

faces will this polyhedron have? What other shapes can you make with triangles?

A Note to the Teacher

Every problem leads to new observations and questions. For example, even the simple problem "Make all possible convex shapes using only equilateral triangles" is very rich in possibilities. These shapes are called *deltahedra*, after the triangular Greek letter Δ. Usually after some experimentation, students will discover the tetrahedron, the octahedron, the triangular and pentagonal bipyramids, and the icosahedron. Later the search also yields the 12-, 14-, and 16-sided deltahedra. Figure 2.15 shows a 14-sided deltahedron made of applicator sticks. Use sticks all of the same length. Some drugstores sell applicator sticks which are ideal; be sure to get the kind without cotton at each end. Hobby and craft stores often sell small-diameter wooden dowel rods which work well. Put a small amount of contact glue on the ends of the sticks and let it dry for about 15 minutes, until the glue is tacky. Then the sticks will join well and yet stay flexible. Don't be surprised if a cube or dodecahedron made of applicator sticks won't stand up, however. Unlike structures built entirely of triangles, these structures are nonrigid.

Figure 2.13.

Figure 2.14.

The observation that each deltahedron has an even number of faces leads to the question of why this should be so. The reason is straightforward once one sees it! Each triangle has three edges. If the shape has F faces, then there are $3F$ edges altogether. These $3F$ edges are glued in pairs, so there must be an even number of edges. Hence $3F$ and therefore F must be be even. Noticing that there exist 4-, 6-, 8-, 10-, 12-, 14-, 16-, and 20-sided deltahedra immediately sets off a search for an 18-sided one. Can an 18-sided deltahedron

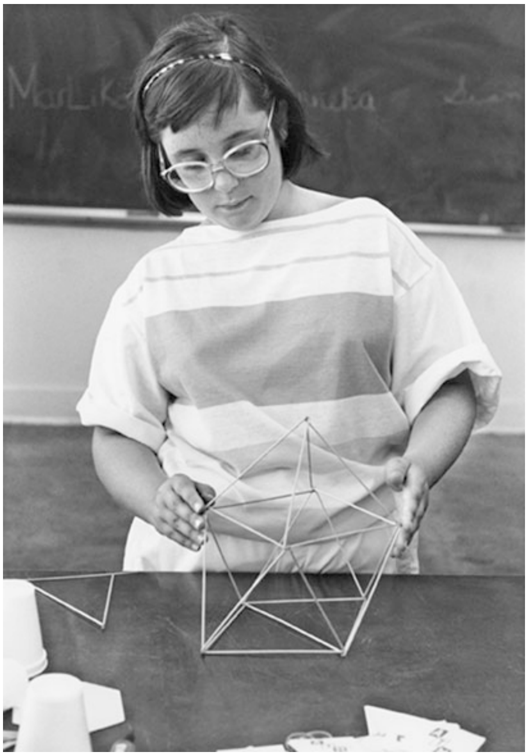

Figure 2.15.

be made? It was not until 1947 that the answer was proved to be *no*.

Looking at deltahedra is one thing; visualizing them without models is quite another. I found it difficult to close my eyes and visualize the 12-, 14-, 16-sided deltahedra. One day while I was looking at a cube made from applicator sticks and glue, I decided to pose problems by using the What-If-Not Strategy. The idea is that one starts with a situation, a theorem, a diagram, or in our case an object, lists as many of its attributes as one can, and then asks, "What if not?" For example, among the many attributes (not necessarily independent) of a cube that I had listed were the following:

1. All edges are equal.
2. All faces are squares.
3. The object is not rigid.
4. The top vertices are directly above the bottom ones.
5. Opposite faces are parallel.

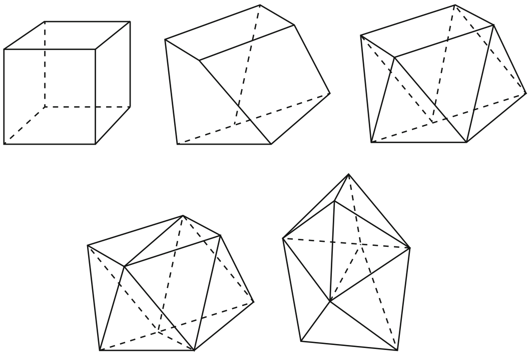

Figure 2.16.

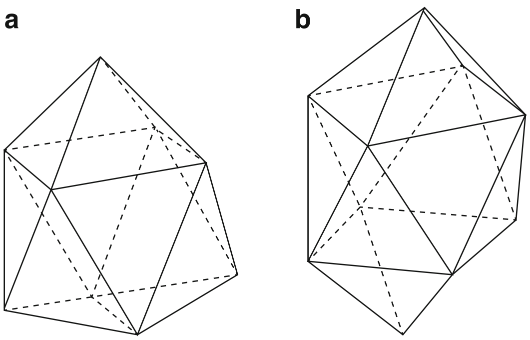

Figure 2.17.

Figure 2.18. Alice Shearer beginning construction of a model.

While working on attribute 4, I asked myself: "What if the top vertices were *not* directly above the bottom ones?" And because the contact glue gives movable joints, it was easy to give the top square a twist. As my twist approached 45°, I began to see an antiprism emerge. I attached sticks to complete the antiprism, but the shape wasn't rigid. The obvious thing to do to make it rigid was to add diagonals to the top and bottom squares. Since all the applicator sticks are of the same length, I had to squeeze the squares into "diamonds." The resulting shape was rigid-and was built of 12 equilateral triangles! (See Figure 2.16).

How else could I have made the antiprism rigid? I hastily removed the top diagonal, and added four sticks that meet above the square to form a square pyramid (Figure 2.17a.) Lo and behold, I had made a 14-sided deltahedron! From there it was a quick step to remove the bottom diagonal also, build another four-sided pyramid, and thus obtain the 16-sided deltahedron (Figure 2.17b).

Not only have these deltahedral "villains" now become friends, I see now that they are closely related to one another. One can also place the icosahedron in this family, since it is a pentagonal antiprism capped with two pentagonal pyramids. (Indeed the octahedron itself is an antiprism, and the tetrahedron can be viewed as an antiprism in which the two bases have shrunk to an edge. Two opposite edges may be considered degenerate polygons, which are here in antiprism orientation.) That leaves us only with the 6- and 10-sided deltahedra as "odd ones out," but they are both bipyramids and are easy to visualize.

Figure 2.19. Jane B. Phipps contemplating a polyhedron constructed from MATs.

A Word About Materials

Cardboard always works well; you should experiment with different weights. I prefer MATs, described in the next paragraph. All the polygons shown in these photographs are MATs. A glue used for carpets, such as Flexible Mold Compound – Mold It® is excellent, as is the English Copydex.

Adrien Pinel found that hexagonal cardboard beer mats (used in English pubs) were excellent for making polyhedra with holes and, when augmented by triangles and squares cut from the hexagons, became even more useful. It was not long before the Association of Teachers of Mathematics of Great Britain had regular polygons of three, four, five, six and eight sides produced from the same easy-to-glue material as the beer mats. They call them Mathematics Activity Tiles (MATs for short). They also produce rectangles and isosceles triangles. The polygons may be ordered separately or in two different kits: Kit A has 100 each of equilateral triangles, squares, pentagons, and hexagons, and Kit B has 200 each of triangles and squares and 50 each of pentagons, hexagons and octagons.

Constructing Pop-Up Polyhedra

Jean Pedersen

Required Materials

- One 22 × 28 inch piece of brightly colored heavyweight posterboard
- Six rubber bands
- One yard stick or meter stick
- One ballpoint pen
- One pair of scissors

General Instructions for Preparing the Pattern Pieces

Begin by drawing the pattern pieces on the posterboard as shown in Figure 2.20. Press hard with the ballpoint pen so that the posterboard will fold easily and accurately in the final assembly. Label the points indicated. Be certain to put the labels on what will become the cube (or octahedron) when the model is finished — not on the paper that surrounds it. Cut out the pattern pieces and snip the notches at A and B (but not the notches at C and D).

Constructing the Cube

1. Crease the pattern piece with square faces on all of the indicated fold lines, remembering that the unmarked side of the paper should be on the outside of the finished cube. Thus each individual fold along a marked line should hide that marked line from view.
2. Position the pattern piece so that it forms a cube with flaps opening from the top and the bottom, as shown in Figure 2.21.
3. Temporarily attach the two rectangles together inside the cube with paper clips. Then, with the cube still in its "up" position, cut through both thicknesses of paper at once to produce the notches at the positions which you already labeled C and D.
4. Connect three rubber bands together, as shown in Figure 2.22.
5. Slide one end-loop of this chain of rubber bands through the slot which you labeled A,

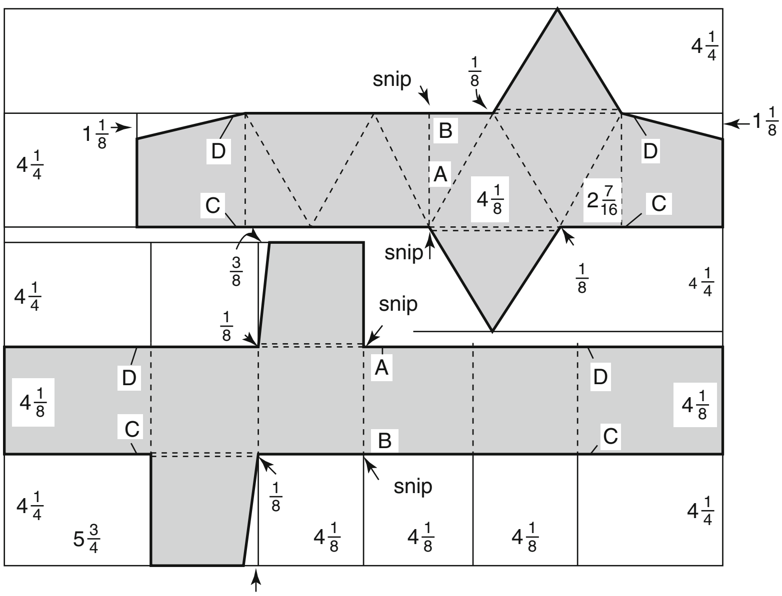

Figure 2.20.

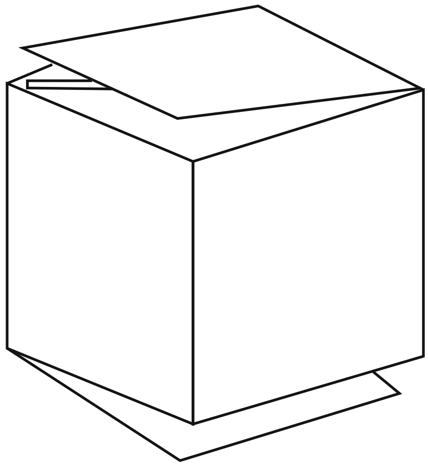

Figure 2.21.

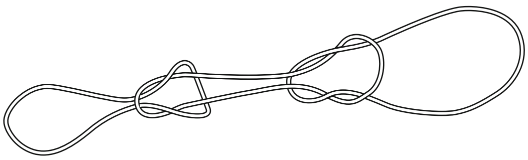

Figure 2.22.

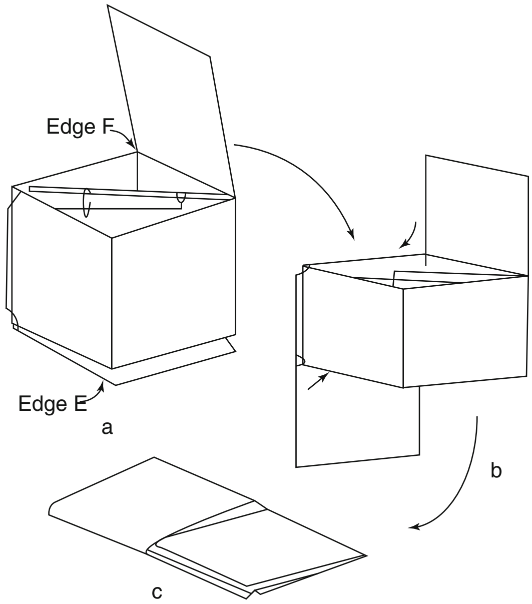

Figure 2.23.

and the other end-loop through the slot labeled B, leaving the knots on the outside of the cube.

6. Stretch the end loops of the rubber bands so that they hook into slots C and D, as shown in Figure 2.23. The bands must produce the right amount of tension for the model to work. If they are too tight the model will not go flat and if they are too loose the model won't pop up. You may need to do some experimenting to obtain the best arrangement.
7. Remove the paper clips when you are satisfied that the rubber bands are performing their function.
8. To flatten the model push the edges labeled E and F toward each other as shown in Figure 2.23b and wrap the flaps over the flattened portion as in Figure 2.23c.
9. Holding the flaps flat, toss the model into the air and watch it *pop up*. If you want it to make a louder noise when it snaps into position, glue an additional square onto each visible face of the cube in its "up" position. This also allows you to make the finished model very colorful.

Constructing the Octahedron

1. Crease on all the indicated fold lines so that the marked lines will be on the inside of the finished model.
2. Position the pattern piece so that it forms an octahedron with triangular flaps opening on the top and bottom, as shown in Figure 2.24a. Don't be discouraged by the complicated look of the illustration; the construction is so similar to the cube that once you have the pattern piece in hand, it becomes clear how to proceed.
3. Secure the quadrilaterals inside the octahedron with paper clips and cut through both thicknesses of paper to make the notches at C and D. Angle these cuts toward the center of the octahedron (so that the rubber bands will hook more securely). Gluing the quadrilaterals inside the model to each other in their proper position produces a sturdier model.