THE LEARNER'S PERSPECTIVE STUDY

Algebra Teaching around the World

Frederick K.S. Leung, Kyungmee Park, Derek Holton and David Clarke (Eds.)

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SERIES PREFACE

The Learner's Perspective Study provides a vehicle for the work of an international community of classroom researchers. The work of this community is reported in a series of books of which this is the fifth. International comparative and cross-cultural research has the capacity to inform practice, shape policy and develop theory. Such research can reflect regional, national or global priorities. Cross-cultural comparisons of social practice in settings such as classrooms can lead us to question our assumptions about what constitutes desirable learning or effective instruction. International comparative research offers us more than insight into the novel, interesting and adaptable practices employed in other school systems. It also offers us a new perspective on the strange, invisible, and unquestioned routines and rituals of our own school system and our own classrooms. In addition, a cross-cultural perspective on classrooms can help us identify common values and shared assumptions across geographically disparate social settings, which in turn can facilitate the adaptation of practices from one classroom for use in a different cultural setting. The identification of structure and recurrence within cultural diversity can help us to distinguish between fundamental commonalities and local conventions. The topic of algebra provides a watershed in the school careers of many students and a common curricular referent for any comparisons of classroom practice in 8th grade mathematics classrooms. By focusing on a single mathematical domain: algebra, we allow other aspects of the mathematics classroom to become more visible through their variation. The chapters of this book document just how different are the forms in which algebra is employed, modeled, and experienced in mathematics classrooms around the world.

David Clarke
Series Editor

FREDERICK K.S. LEUNG, DAVID CLARKE, DEREK HOLTON & KYUNGMEE PARK

CHAPTER 1

How is Algebra Taught around the World?

INTRODUCTION

Algebra is a major component of the mathematics curriculum in all countries around the world. There is ample research on the algebra curriculum and on algebra teaching reported in the literature, mainly on the importance of algebra (Edwards, 1990; Moses & Cobb, 2001; Kaput, Blanton & Moreno, 2008; National Mathematics Advisory Panel, 2008; Watson, 2009) and the difficulties students face in learning algebra (Van Ameron, 2002; Kieran, 2007; Harel, Fuller & Rabin, 2008; Linsell, 2009; Caglayan & Olive, 2010). However, research in international comparison of classroom algebra teaching is hampered by limits on comparability, due to differences in the grade levels of the classrooms studied. This book offers an opportunity to compare algebra teaching at eighth grade level across many countries that are both geographically and culturally distant. As a result this book will be of value to researchers with a focus on algebra, pedagogy or international comparisons of education. In addition, various research methods are on view that will be of value to young researchers and graduate students. Because of the pedagogical variations noted here, there is a great deal of material that will be of interest to both teachers and teacher educators.

THE LEARNER'S PERSPECTIVE STUDY (LPS)

The LPS provides a common platform for documenting eighth grade mathematics teaching in a large number of countries around the world. In addition to a standardized way of videotaping mathematics classrooms, the LPS collected data on the reflective accounts of participant students and teachers, adding 'complementarity' and richness to the data set (Clarke, 2000, 2006).

Not all of the LPS lessons are on algebra, but it is possible to select algebra lessons specifically and to study issues related to its teaching in eighth grade classrooms around the world. This is exactly what is being done in this book on algebra teaching based on the LPS data.

F.K.S. Leung et al. (eds.), Algebra Teaching around the World, 1–15.

What are algebra and algebraic activities?

In this book, we examine algebraic activities in classrooms from different countries. But what is algebra? According to Usiskin (1988), algebra is "generalized arithmetic; a way to solve certain types of problems; a study of relationships among quantities; and a study of structures" (Kaur uses this definition in chapter 5). Stacey, Chick and Kendal (2004), on the other hand, define algebra as "a way of expressing generality; a study of symbol manipulation and equation solving; a study of functions; a way to solve certain classes of problems; a way to model real situations; and a formal system involving set theory, logic, and operations on entities other than real numbers" (Novotná & Hošpesová use this definition in chapter 4). These definitions are not the same, but neither are they incompatible. Each emphasizes particular aspects of the mathematical domain of algebra.

Adopting an inclusive approach and accepting this multi-faceted characterization of algebra, algebraic activities can be understood as comprising three core activities: generational activity, transformational activity, and global/meta-level activity. "The generational activities involve the forming of expressions and equations that are the objects of algebra. The transformational activities include, for instance, collecting like terms, factoring, expanding, substituting, adding and multiplying polynomial expressions, solving equations, simplifying expressions, working with equivalent expressions and equations, etc. The global/meta-level mathematics activities refer to those for which algebra is used as a tool but which are not exclusive to algebra. They include problem solving, modeling, noticing structure, studying changes, generalizing, analyzing relationship, justifying, proving, and predicting, etc." (Kieran, 2004, quoted in Chapter 10 by Huang et al.).

In the definition of algebraic activities above, there seems to be an assumption that algebra is universal. But is it understood in the same way in different countries around the world? Do different countries expect students to learn the same knowledge and skills in algebra? And, more central to the focus of this book, are there differences in how algebra is taught in different countries? In this book, access to classroom data from several countries around the world allows us to address these related questions.

CONTENT OF THIS BOOK

Issues of concern pertaining to algebra teaching and learning may be different from country to country, and a number of different issues are discussed in this book. In Chapter 2, Anthony and Burgess describe how a New Zealand teacher used a "balance model" (Warren & Cooper, 2005) in teaching linear equations in order to occasion learning opportunities that support students' adjustments in the learning transition from arithmetic to algebra. The authors argue that the teacher's planning, instruction and interactions within the classroom are themselves a series of balancing acts that afford or constrain opportunities for students to make the necessary adjustments.

In Chapter 3, Pepin and Sør-Trøndelag report a Norwegian classroom where the concept of equality and the meanings and use of the equal sign were explored. The authors utilize the concept of "orchestration of signs" (Trouche, 2004) to explain the teacher's pedagogic practice in the algebra classroom, following a semiotic and 'instrumental' approach. In Chapter 4, Novotná and Hošpesová report two Czech teachers' approaches to teaching the topic of linear equations and their systems. The authors point out that the two teachers' approaches differed in terms of the pupils' expected output competencies and what is regarded as the most adequate tools for reaching them.

In Chapter 5, Kaur reports a case study of how a Singaporean teacher engaged her grade eight students in developing procedural fluency in algebraic structures through the use of varied learning tasks. In Chapter 6, Park and Leung report the perceptions of a Korean teacher and his students in relation to a series of algebra lessons. They show how the same lessons were perceived differently by the teacher and his students in terms of the lesson objectives and what is considered important in a lesson, because of their different levels of competence and their differing values with regard to mathematics.

In Chapters 7 and 8, Ohtani and Fujii report separately on algebra lessons in two different Japanese schools. Ohtani investigates factors which affect the connections among a series of lessons on simultaneous equations, and utilizes the construct of "Construction Zone" (Newman, Griffin & Cole, 1989) in analyzing students' progressive understanding of new mathematical concepts. He conceptualizes the "generic task" as a mediating tool between students and the teacher, which constitutes a construction zone where people with different levels of understanding can interact with each other, which in turn opens up a learning opportunity for deeper understanding. Fujii reports a series of seven algebra lessons, which were devoted to solving one mathematics problem in order to understand the concept of the range of variables. Fujii demonstrates how the Japanese teacher involved the students as a "community of mathematical inquiry" (Lewis, 1995) to deepen their understanding of mathematics concepts.

Chapters 9 to 11 are on algebra lessons in China. In Chapter 9, Huan et al. examine how algebra teaching in a Beijing classroom is aligned with the guiding principles of the official Mathematics Curriculum Standards in China. They report not only that the official curriculum standards exerted a strong influence on algebra teaching in this Beijing classroom, but also that the senior high school entrance examination in Beijing had a strong influence on the teaching. However, the authors point out that the spirit of the new curriculum was only followed superficially by the teacher. In Chapter 10, Mok shows a Shanghai teacher's pedagogical delineation of the graphical method of solving linear equations in two unknowns by an analysis of a series of algebra lessons. Through analysis of the lessons and interviews with the teacher, Mok characterizes both the teacher's beliefs and his instructional practice. In Chapter 11, Huang et al. utilize the theory of variation (Marton & Booth, 1997) to study the process of forming and developing algebraic concepts in algebra classrooms in Hong Kong, Macau and Shanghai. The authors report that the Chinese lessons followed a pattern of four phases, which provided learning opportunities for students to develop new concepts through exploring inductive/broadening variations of problems and to consolidate the concepts through practicing with deepening variations of problems and reflecting on the learning experiences.

In Chapter 12, Huang and Li compare algebra teaching in China and the USA, focusing on how the teaching promoted the understanding of algebraic concepts. The goals and contents of the algebraic lessons are compared, as well as the students' perceptions of the classroom instruction with regard to understanding in the two countries. In Chapter 13, Häggström utilizes the concept of "opportunity to learn" (Hiebert & Grouws, 2007) to analyse algebra teaching in Sweden and China (Shanghai and Hong Kong) based on the theory of variation. Häggström's purpose is to illustrate the methodological development in the field of research on mathematics teaching and the development of the concept "opportunity to learn" as a powerful tool to analyse and compare mathematics instruction.

ALGEBRA TEACHING IN DIFFERENT COUNTRIES

As can be seen from the description above, the intention of the authors in most of the chapters in this book is not just to describe how algebra is taught in their countries. Each chapter has a theme or focus of its own. Certainly each chapter offers a particular perspective on algebra teaching in a particular country and Chapters 12 and 13 make country-specific features explicit through the comparison of algebra teaching in classrooms in China and the USA and in China and Sweden respectively. But since the focus of all of the studies in all of the chapters is algebra and its teaching, the combination of sites and perspectives provides a rich and complex picture of similarity and variation in the teaching of algebra around the world.

The range of countries in which the LPS classrooms are situated is sufficient for us to make a comparison between two commonly-invoked cultural clusters: the Confucian-Heritage Culture (CHC) countries (Biggs & Watkins, 1996) and "Western" countries. For the sample of classrooms studied in this book, representatives of the CHC cluster of countries include China (Beijing, Shanghai, Hong Kong and Macau), Japan, Korea, and Singapore; while "Western" classrooms are represented by the Czech Republic, New Zealand, Norway, Sweden, and the USA. The significance of these two clusters of countries rests on the suggested cultural coherence within each cluster and the possibility that cultural coherence might translate into pedagogical coherence. While there may be substantial cultural differences between China and Japan and between the USA and the Czech Republic, the clustering of countries in this fashion has the capacity to provide insight into patterns of pedagogical similarity within a cluster and pedagogical distinctiveness between clusters. Such patterns are amenable to empirical investigation, given a data set such as that available within LPS. Equally, as will be seen, interesting differences are evident within clusters and even the designation of classrooms as "Chinese" can conceal interesting variations in practice between those as proximate as Hong Kong, Macau and Shanghai.

Naively, it might be asked, "How much difference can there be between one algebra class and another?" The chapters in this book demonstrate the variation possible in different classrooms around the world, even when the common focus is as specific as algebra.

Similarities and differences among countries

There are many commonalities in the ways in which algebra is taught among countries as revealed by the chapters in this book. But there are also striking differences. In discussing differences among countries, reference will be made to individual countries and to clusters such as the CHC countries and the Western countries. But we need to bear in mind that the LPS research design targeted sequences of at least ten lessons in the classrooms of three competent mathematics teachers in each city. The focus was the detailed documentation of competent practice, and the selected teachers and their classrooms should not be taken as representative of the countries concerned as a whole. Also, when discussing between-country differences, we should note that, even for this focused LPS sample, the detailed data set for each classroom revealed within-country differences in addition to the between-country differences being discussed.

Take the Czech Republic as an example. In chapter 4, Novotná and Hošpesová report two teachers teaching the same topic and trying to meet the same requirements of that country. Yet they taught in very different ways, one stressing more "how" while the other placed more emphasis on "why". Both linked new content to what students already knew, but the order of presentation of the content differed, the level of algorithmisation differed, the level of complexity and difficulty of the problems discussed differed, and the ways in which word problems were used differed. So even for a small sample of teachers in just one country, there are within-country differences in how algebra is taught. In the case of China, even greater differences were recorded between the classroom practices of the selected mathematics teachers in Hong Kong, Shanghai, Macau and Beijing. Notwithstanding these within-country differences, when we look at how algebra is taught in different countries, as revealed in this book, we can see that there are even wider between-country differences.

How is algebra taught differently in different countries?

Aims and objectives of the lesson. From the chapters in this book, when the aims of algebra lessons are described, it seems that what students are expected to learn in algebra (or in mathematics in general) is basically an understanding of algebraic concepts and procedural fluency in dealing with algebraic manipulations (Hiebert & Carpenter, 1992). But the ways in which these aims are formulated differ in different countries. The data reported in this book show that some teachers set very general and ambitious aims for algebra teaching, while other teachers tend to set concrete and detailed goals. For example, in Chapter 12, Huang and Li report a USA teacher setting very general goals, while a Chinese teacher set very specific and concrete aims for the lesson.

The content of the lesson. With respect to the content of the algebra lessons, the situations in the Western countries varied. Some lessons dealt with rather conceptual content, while the teaching in others was rather procedural. A common feature seemed to be a stress on everyday life problems, much more so than in CHC countries.

The lessons from these classrooms in the CHC cluster shared some common features, although the emphasis of the lessons varied from highly conceptual (Japan) to highly procedural (Hong Kong). A particular common feature among the CHC classrooms was the abundance of practice problems and exercises, which were generally of high cognitive demand. The problems were usually carefully planned: they varied systematically and the content of the problems was interconnected.

The way lessons are conducted. There is clear evidence from the descriptions in this book that there was a strong influence of public examination on the way in which algebra was taught in the classrooms in Beijing (see Huan et al., Chapter 9) and this seems characteristic of the CHC cluster in general. The way in which the lessons were conducted in the CHC classrooms suggests a prescribed syllabus that was followed closely, with a focus on the efficient delivery of mathematics content. Students in the CHC classrooms were, in general, serious about their learning. In contrast, the algebra lessons in the Western countries were more interactive (see also Kaur, Anthony, Ohtani, & Clarke, 2013), and the atmosphere was more relaxed.

Teachers in the classrooms from both clusters of countries said that it is important for students to participate in lessons, but the nature of that participation varied in interesting ways. For example, while small group work occurred in the classrooms in both clusters, collaboration among students during the group work was given greater emphasis in the Western classrooms. Group work, when it occurred, was monitored and controlled more closely by the teachers in the CHC classrooms than in the Western classrooms.

Conceptual understanding

Many lessons in the classrooms in both Western and CHC countries stressed conceptual understanding. For example, in Chapters 2 and 3, the New Zealand lessons (on solving linear equations based on the “balance” model, where the teacher’s goal was to help the students eventually to solve equations detaching from the model) and the Norwegian lessons (on understanding the equality sign) both used the ’balance‘ or ’scale‘ model to discuss equality and linear equations. The concepts dealt with were fairly basic, but the approach was conceptual. Both series of lessons aimed at helping the students to understand important concepts in algebra.

The lessons from the CHC countries reported in this book also stressed conceptual understanding, and the lessons shared some similarities in structure. Huang et al. (Chapter 11) identify four phases of a typical Chinese lesson as follows:

1. Introducing the concept,
2. Explaining the meanings,
3. Discriminating the concept with varying exercises, and
4. Summarization.

Huang et al. illustrate this structure in Chapter 11with a Shanghai lesson on systems of linear equations in two unknowns. The authors argue that a distinctive characteristic of Chinese lessons is teaching with variation. It is worth considering the implications of teaching with variation for the students' learning of algebra.

Theory of variation

In Chapter 13, Häggström draws on the theory of variation (Marton & Booth, 1997) and uses the concept of "opportunity to learn" to explain what students might learn in lessons taught with variation. Marton's and Booth's theory of variation is applicable to a wider field of learning, but since algebra is about generalization and transformation, it can certainly be argued that algebra can be learned effectively through teaching with variation. Variations of various components within a concept (for example, various components of the concept of system of linear equations in two unknowns) help students to understand the concept, and variations between different concepts (for example, the concepts of system of linear equations in two unknowns and system of other types of equations) help students to connect one concept to others (which will also help their understanding of the concept itself). Häggström compares six Swedish, Hong Kong and Shanghai lessons, and examines the different dimensions of variation opened in the lessons. The analysis of how the content was handled in the six classrooms shows a difference in students' 'opportunities to learn' in terms of the dimensions of variation (Marton & Pong, 2005).

VARIATIONS IN ALGEBRA TEACHING AMONG THE CLASSROOMS FROM THE CHC CLUSTER OF COUNTRIES

As pointed out above, in referring to the CHC or Western cluster of countries, we need to bear in mind within-cluster differences. But since nearly all of the CHC countries are represented in this book, the detail provided by the chapters reporting algebra teaching in these CHC classrooms is sufficient to justify a detailed discussion of the variations in practice among CHC mathematics classrooms.

Despite the identified similarities of the algebra lessons in the CHC countries, some major differences are evident. It is acknowledged widely that developing conceptual understanding and procedural fluency are important in mathematics learning (Hiebert & Carpenter, 1992; NCTM, 2000; NMAP, 2008), and the differences in algebra lessons among the CHC countries can be summarized in terms of the different emphases on these two aspects. Three different models can be identified: **The Procedural Model, the Conceptual Model and the Blended Model**. The classrooms in Korea, Singapore, Hong Kong and Macau seem to align most closely to the Procedural Model, while the Japanese classrooms exemplify the Conceptual Model. The classrooms in Shanghai and Beijing seem to implement a Blended Model that combines aspects of both procedural and conceptual emphases. The differences between these models illustrate how much variation is possible, even with those classrooms situated within the CHC cluster.

Similar distinct pedagogical differences have been discussed by Clarke, Xu and Wan (2013a and 2013b) from the perspective of patterns of classroom discourse. In fact, it can be argued that the differences in emphasis identified in the algebra lessons provide part of the rationale for the differences in discourse patterns. The emphasis on conceptual understanding corresponding to the promotion of fluency in the spoken use of mathematical terms is in contrast to an emphasis on procedural fluency, which does not necessarily require sophisticated mathematical language. Even where two classrooms appear to place the same emphasis on conceptual or procedural understanding, the instructional approach can still be quite different (for example, Korea compared with Singapore or the classrooms in Shanghai compared with those in Tokyo (see Xu and Clarke, 2013)).

Fujii (Chapter 8) points out that a typical Japanese problem-solving oriented lesson consists of four phases or components:

Phase 1: Presenting one problem for the day (understanding the problem),
Phase 2: Problem solving by students,
Phase 3: Comparing and discussing (students present solutions), and
Phase 4: Summing up by the teacher.

Fujii illustrates these four phases with a series of lessons on a paper-folding problem. It can be seen that there was a strong emphasis on conceptual understanding, as is evident from the fact that seven lessons were devoted to tackling just one mathematics problem. One might ask how, then, do the Japanese students attain their procedural fluency, given that so much lesson time is spent on the elaboration of mathematics concepts.

The answer may lie in the parallel *juku* system in Japan (Harnisch, 1994). So students acquire deep conceptual understanding through the classroom teaching in the regular schools, and then after school they attend *juku* schools and further practise their mathematics skills and consolidate their understanding (Leung, Park, Shimizu & Xu, 2012). This 'distribution of labour' in the Japanese system is an area of interest in itself, but is beyond the scope of this book. Such 'shadow education' systems are common across the cluster of CHC countries and are becoming more prominent in the Western cluster of countries as well. The Japanese example serves to illustrate the possibility that mathematics classroom practices in CHC countries may develop in symbiosis with the practices of the increasingly popular after-hours practice classes.

In Singapore, Hong Kong, Macau and Korea, content was typically covered relatively quickly, and students were given the opportunity to do a lot of practice exercises. This is quite a different pedagogy from that found in the LPS classrooms in Tokyo, Shanghai or Beijing. The other CHC countries also have a private tutorial school system that plays a similar role to the *juku* system in Japan (Kwok, 2004), but, based on the data reported in the chapters of this book, algebra teaching in the regular classrooms in these CHC countries seems to have a different emphasis from the regular classroom teaching in Japan. But this does not necessarily mean that the teaching of algebra in the classrooms in other CHC countries is not conceptual. As mentioned above, Huang et al. point out in Chapter 11 that the promotion of conceptual understanding in the Hong Kong, Macau and Shanghai classrooms was built into the varied examples practised by students. The difference between the Conceptual Model and the Procedural Model lies in whether the concepts are addressed explicitly (as in the case of the Tokyo classrooms and. to some extent, in the case of the classrooms in Shanghai and Beijing as well), or whether the concepts are embedded in well-designed, systematically varied exercises that students practice (as in the case of Singapore, Hong Kong and Korea). It can be argued that what we have called the Blended Model, as practised in Shanghai and Beijing, seems to capture both the emphasis on conceptual understanding of the Japanese model and the stress on procedural fluency identified in Singapore, Hong Kong and Korea. It is probably fair to say that the difference is best seen to be one of emphasis, with each classroom striking a different balance between conceptual understanding and procedural fluency.

COMPARING CHARACTERISTICS OF ALGEBRA TEACHING IN CHC AND WESTERN CLASSROOMS

From what is reported in this book, it seems that a common emphasis of algebra teaching in CHC countries is the 'linkage' or 'coherence' of mathematics concepts (Gu, Huang & Marton, 2004; Leung, 2001; Shimizu, 2007). This linkage of concepts is both within a topic (e.g., concepts within the topic of system of linear equations in two unknowns) and between topics (relation between system of linear equations in two unknowns and other topics).

Contemporary algebra teaching in many Western school systems places increasing emphasis on the use of algebra in mathematical modeling in 'real world' contexts (e.g., Czech Republic, Chapter 4). Other emphases in the teaching of algebra in Western classrooms include the instructional use of metaphors, such as "balance" (reported in both New Zealand and Norway, Chapters 2 and 3). These two examples can both be seen as being driven by a desire to assist students in the construction of meaning in relation to algebra, but the nature of that meaning is differently purposed. In each case, meaning construction is assisted by invoking contexts outside the domain of algebraic manipulation. The clear intention is to help students to form connections between algebra and other aspects of their experience.

Irrespective of whether they are classrooms akin to the Conceptual Model, the Procedural Model or the Blended Model, all of the algebra lessons in the CHC countries reported in this book look rather teacher-directed in comparison with the lessons from the Czech Republic, New Zealand, Norway, Sweden and the USA. As has been argued by Mok (2006), this appearance of teacher domination is misleading and it seems that conceptual understanding, as well as procedural fluency, can be achieved through teacher-directed lessons. These teacher-directed lessons include a lot of listening to explanations by the teacher and practice of mathematics problems. Of course, practising mathematics problems is an active learning experience, even when done under the dominating direction and close supervision of the teacher, but listening to the teacher's explanation can be an active mode of learning as well (refer to the notion of 'listening-oriented learning' proposed by Cortazzi and Jin, 2001, quoted in Chapter 11 by Huang et al.). However, the pre-requisite for listening to be an active learning experience is that there should be high quality content to be 'listened' to. It may also be advantageous for the content to be delivered in a manner that 'engages' the students' attention. Other research on the differences in the ethos of Asian and Western classrooms has suggested that a key difference is the location of responsibility: where Western practice requires the teacher to engage the student's attention, Asian practice places the responsibility for engagement with the student (Stevenson & Stigler, 1992). The phenomena of students listening to the teacher's explanation and engaging in a lot of practice may give the impression that the students are very passive. But the quality of the teacher's explanation, the strategic selection of the problems (with systematic variation) that students practise, and the informed use of worked examples are all essential aspects of competent practice in CHC classrooms and all have been connected by recent research to high quality learning (see Sweller & Cooper, 1985).

Contrasts such as implicit versus explicit, deep versus surface, or closed versus open may provide useful ways of characterising mathematics teaching in ways that reveal culturally-specific practices and beliefs. The contrasts point to the possibility of achieving effective teaching based on different philosophies and following different traditions (Clarke, 2013). In the final analysis, it is always the teacher who sets the agenda, and one may argue that the expression of student agency is a relatively infrequent occurrence. We always 'teach' students the teacher's version of mathematics or mathematics learning, be it the CHC 'didactic' mathematics and mathematics learning or the Western constructivist approach.

IS ALGEBRA "UNIVERSAL"?

Returning to a question posed at the beginning of this chapter, based on the different ways of teaching algebra reported in this book, and with reference to the discussion in the preceding sections, is algebra 'universal'? Is 'what algebra is' understood in the same way in different countries around the world? Further, there is the question of whether different countries expect students to learn the same knowledge and skills.

These are difficult questions to answer, partly because this book did not set out to address them specifically. It is worth noting, though, that in Chapter 4, Sutherland (2000), who has made a study of the curricula of 12 countries, is quoted as saying that “more research needs to be carried out in order to understand the implications of this [curriculum] on classroom practice”. This research is not reported explicitly in this book.

It should also be noted that, in Chapter 8, Fujii makes the point that, in the Japanese Course of Study for elementary and lower secondary school mathematics, there is no specific mention of algebra. Fujii writes “instead of being an independent domain in the curriculum, algebra is included systematically in various parts of the mathematics curriculum, particularly at the elementary level” (p. 129). Fujii goes on to discuss the different conceptions of algebra proposed by Usiskin (1988) and the way in which these are aligned with algebra within the Japanese mathematics curriculum. It is clear that algebra is conceived, named and situated differently within mathematics curricula in different countries. These curricular differences are then compounded by differences in the ways in which algebra is taught. These differences in the teaching of algebra constitute the primary focus of the chapters in this book. However, it is clear that a number of countries see linear equations and their applications to be necessary (Chapters 2, 4, 5, 6, 7, 8, 9 and 11). There is a certain amount of prerequisite knowledge required for this topic. For instance, almost all of the material that is required for Year 8 students in Singapore, shown in Figure 1 of Chapter 5, must be needed by other countries too. It would be surprising if the Binomial Theorem and quadratic equations were omitted from that list. As has been said above, many countries also require understanding to support students’ skill ability.

Given the different ways in which algebra is taught in different countries, it is reasonable to expect that students’ conceptions of it might be different. The different ways algebra is taught reflect the fact that it is conceived of differently by teachers in different countries. But one may argue that algebra is algebra, no matter whether it is in China or Korea or Sweden or the USA! This touches on an absolutist versus a fallibilist view of mathematics. If we accept a fallibilist view, then the conceptions of algebra may indeed be different in different countries. The curricular encryption of algebra, as is evident from the Japanese example, mediates between algebra as a domain of mathematical knowledge and enquiry and algebra as a school subject and the object of classroom teaching and learning. The chapters of this book document just how different are the forms in which algebra is employed, modeled, and experienced in mathematics classrooms around the world.

Notwithstanding these different conceptions of algebra and algebra teaching, identifying good practices and explicitly formulating the practices and the reasons for their effectiveness systematically (in the form of a theory) should help reinforce and promulgate the effective practices and reduce the occurrence of the ineffective ones. This is a major aspect of what research in the field of comparative study in mathematics teaching and learning attempts to achieve. The chapters in this book are intended to support such comparisons by identifying key features in the practices of competent mathematics teachers in many different school systems.

CONCLUDING REMARKS

Different issues in algebra teaching are the primary focus of this book. They range from the question of the alignment between algebra teaching and the official mathematics curriculum; how procedural fluency is achieved through the use of varied tasks; a teacher's pedagogical delineation of a certain method in teaching an algebraic topic; to how students are involved as a 'community of inquiry' in algebra lessons. Different theories (including the construct of the 'construction zone'; the theory of variation; the notion of 'opportunity to learn'; and the concept of 'orchestration of signs') are used to characterize algebra lessons or to compare algebra teaching in different countries. Algebra teaching is perceived as a series of balancing acts for the teacher; discrepancies between students and their teacher in their perceptions of the algebra lessons are identified; how teachers in the same country approached the same algebraic topic differently is discussed; algebra teaching approaches in different countries are compared, and approached the same algebraic topic differently is discussed; and algebra teaching approaches in different countries are compared.

Many commonalities in algebra teaching around the world can be identified from what is reported in this book, but there are also striking and deep-rooted differences. The different ways algebra was taught in the mathematics classrooms of different countries points to how algebra teaching may be embedded in the culture and the general traditions of mathematics education of the countries concerned. The different classroom practices resulting from the different cultures may also help to explain the achievements of students from different countries in international studies of mathematics achievement. It is hoped that the following chapters provide useful insights into the classroom practices of mathematics teachers in different cultures and school systems and that the various accounts of the teaching of algebra enrich the didactical knowledge base of mathematics teachers and teacher educators internationally.

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CHAPTER 2

Solving Linear Equations: A Balanced Approach

INTRODUCTION

Concerted efforts at improving student performance in algebra demonstrate that "children throughout the elementary grades are capable of learning powerful unifying ideas of mathematics that are the foundation of both arithmetic and algebra" (Carpenter, Franke, & Levi, 2003, p. xi). In New Zealand, Britt and Irwin's (2005) investigation of the Numeracy Development Project found that those students who acquired flexibility in using a range of general arithmetical strategies also developed the ability to express the structure of those strategies in symbolic forms. Such a foundation should bode well for students learning about solving linear equations—a process that requires a sound understanding of mathematical equality, and the commutative, distributive and inverse properties. However, a large scale study by Linsell (2009) found that many New Zealand secondary students have limited facility with solving and understanding linear equations. In common with students across other countries (Chazan, 2008), these difficulties include "grasping the syntax or structure of algebraic expression" and understanding "procedures for transforming equations or why transformations are done the way they are" (National Mathematics Advisory Panel (NMAP), 2008, p. 32).

Some of the difficulties that arise when learning to solve equations are associated with the move to solving them algebraically as opposed to arithmetically. Kieran (2004) argued that, in negotiating this transition, a student must make a series of adjustments as follows:

A focus on relations and not merely on the calculation of numeric answers;
A focus on operations as well as their inverses and on the related ideas of doing and undoing;
A focus on both representing and solving a problem instead of merely solving it;
A focus on both numbers and letters rather than on numbers alone; and
A refocus on the meaning of the equal sign.

In this chapter, we look closely at one New Zealand teacher's sequence of lessons involving the solving of linear equations in a Year 9 (Grade 8) classroom. In focusing our attention on the lessons involving the shift from arithmetic linear equations—those with the unknown on one side only—to algebraic linear equations—those with

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the unknown on both sides, we seek to understand the ways in which the teacher sought to occasion learning opportunities that support these adjustments. We argue that the teacher's planning for learning, his choice of instructional activities, and his in-the-moment (inter)actions within the classroom can be viewed as a series of balancing acts—acts that variously afforded or constrained opportunities for students to make the necessary adjustments and associated understandings concerning solving linear equations. Before turning our attention to the New Zealand data, we briefly review the research literature on teaching linear equations.

TEACHING LINEAR EQUATIONS

Over the years, researchers (e.g., Caglayan & Olive, 2010; Filloy & Rojano, 1989; Filloy & Sutherland, 1996; Johnson, 1989; Lima & Tall, 2008; Sfard & Linchevski, 1994; Vlassis, 2002) have experimented with various situations in which students learn to solve equations. Their studies highlight a range of teaching approaches and illustrate a variety of significant cognitive challenges associated with learning algebra. For example, we are aware from the work of Sfard and Linchevski that "algebraic symbols do not speak for themselves" (p. 191). In different contexts the expression $3(x + 5) + 1$ can be read as a computational process, a certain number, a function, or a mere string of symbols which represent nothing—an algebraic object in itself. Moreover, the one representation may sometimes be interpreted operationally and at other times structurally. Vlassis (2002) reported confusion with students' interpretations of the negative sign within an equation; was it to be viewed as an operation (to subtract) or as a negative quantity? Caglayan and Olive's (2010) study of the use of a representational metaphor (cups and tiles) for writing and solving equations of one unknown noted the existence of a disconnect between students' mental and physical operations. All of these studies concur that developing productive ways of thinking about algebraic symbols takes time and is one of the big goals of learning algebra (Kaput, Blanton, & Moreno, 2008).

Given the range of cognitive obstacles for symbolizations and representations, and the series of adaptations needed in thinking about solving equations algebraically, what teaching approaches are commonly available to the teacher? Pirie and Martin (1997) presented a useful summary of two general approaches that dominate classroom instruction:

> One method is to repeatedly change the equation, by 'doing the same to both sides' until one has an equation that directly gives the answer. Understanding is assumed to come through the notion of 'undoing' a series of operations to get back to the original value…an alternative approach is that of 'change sides, change signs' based on the concept of inverse operations. Here, understanding is built on the assumption that the integrity of the original equation is preserved. (p. 161)

Students do not, however, need to employ these formal methods of mathematical reasoning when solving one-step or two-step linear equations with ‘x’ on one side only. These equations can be solved intuitively using purely arithmetic means such as known facts, substitution, cover-up, or backtracking. However, when solving the likes of ax ± b = cx ± d, informal arithmetic methods do not suffice. Research has provided confirming evidence that students experience major difficulties in solving these ‘non-arithmetical’ equations (Linsell, 2009; Vlassis, 2002). Filloy and Rojano (1989) suggested that this type of equation demands teacher intervention, ‘the didactic cut’. That is, they argued that the student requires assistance from the teacher in the form of some sort of device to negotiate access to the more complex domain of algebraic equations. Filloy and Sutherland (1996) summarised two extreme positions regarding a suitable device: (1) modelling the new operations and objects in some concrete context which is familiar to the students, so that they become endowed with meaning; and (2) beginning at the syntactic level, learning the appropriate syntactical rules and, later on, applying them to the resolution of problem solving and equations.

Herscovics and Linchevski (1994), in a similar exploration of the transition from arithmetic to algebra, introduced the notion of a ‘cognitive gap’ which “is characterized by the students’ inability to operate with or on the unknown” (p. 75). Findings from further research (Linchevski & Herscovics, 1996) suggested ways to ‘cross the gap’, including specific instruction in ‘grouping like terms’, development of the balance model, and decomposing into a difference to facilitate cancelling subtracted terms.

A third perspective on the transition was offered by Pirie and Martin (1997). They argued that this ‘cut’, “this implied cognitive difficulty, is, in reality a notion imposed by the observer, with hindsight, to explain an artifact of particular methods of teaching” (p. 161). Their research described a teaching approach built on the metaphoric notion of equality as a ‘fence’. Using this notion of equality, they researched an instructional sequence of activities that involved beginning with unknowns on each side. Despite the absence of formalization, or models, the students in their study built “a sound and secure basic image for the solution of linear equations from which they could work with confidence” (p. 177).

With all these approaches the underlying aim is for students to develop deep and conceptual understanding. As Kaput et al. (2008) suggest:

> If the algebraic system is not introduced via a well-grounded symbolization process in the context of both expressing generality and the lifting out of previously established actions, then it is based only in rules about itself and is symbolically and conceptually isolated from the foundations in what the student knows and can do. (p. 46)