Studies in Universal Logic

Yvon Gauthier

Towards an Arithmetical Logic

The Arithmetical Foundations of Logic

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Yvon Gauthier
University of Montreal
Montreal
Québec, Canada

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Dedicated to the memory of the great arithmetician André Weil

Foreword

The project of an arithmetical logic has been in the making for many years and the present work is the continuation of my 2002 book *Internal Logic. Foundations of Mathematics from Kronecker to Hilbert* (Kluwer, Dordrecht). In the intervening years, I have pursued the programme and I have published many scientific papers and a book in French on the subject. The progress made towards an arithmetical logic is here recorded, but the idea of an internal logic of arithmetic has not been altered. It is still the inner structure of classical arithmetic or number theory, which is the objective of the foundational enterprise. I have baptized that arithmetic the Fermat-Kronecker (FK) arithmetic and I have constantly opposed it to Peano arithmetic. What I have been trying to show is that there is no set-theoretic element in pure arithmetic, while Peano or Dedekind-Peano formalized arithmetic is embedded in a transfinite set-theoretic framework. Kronecker's finitist stand in mathematics extends from Hilbert to contemporary constructive mathematics, e.g. Bishop's constructive analysis and Nelson's predicative arithmetic. Gödel's ‹extended finitism› of the *Dialectica Interpretation* could be counted as a mitigated reappropriation of Kronecker's radical constructivism via Hilbert's introduction of functionals inherited from Kronecker's higher-order forms (polynomials). This is one of the main themes I have proposed in the recent years.

The central thesis of this book has been expanded to cover the constructivist insights in physics and mathematical physics, from relativity theory to quantum physics and cosmology where I have attempted to explore the ramifications of the constructivist-finitist motives. My objective here has been to elaborate on the foundational aspects of arithmetical logic—the proper name of which I have dubbed modular polynomial logic—with incursions in probability theory and theoretical and mathematical physics. At the same time, I have been trying to see what is conceptually (and technically) going on in contemporary ‹real› mathematics from the constructivist viewpoint of arithmetical foundations, without too much prejudice as to what constitutes mathematical practice with or without foundational concerns. Still, needless to say that constructivist foundations are inherently critical of mathematical (and logical) practice in classical logic and classical mathematics, but the critique comes from within, that is without invoking principles that are alien to mathematical activity in its historical, epistemological and rational pursuits.

In that endeavour, the main source of my inspiration remains André Weil with whom I discovered both Fermat and Kronecker in the 1980s. Early on, André Weil had encouraged me in correspondence to explore further the mathematical virtues of Fermat's method of infinite descent and I discovered at the same time the importance of Kronecker's general arithmetic in Weil's original writings on algebraic geometry (see his *Œuvres scientifiques. Collected Works*, Springer-Verlag, 3 vols, 1980)—see my review Gauthier (1994b) of Weil (1992). Weil has put Kronecker's theory of forms or homogeneous polynomials and his divisor theory (moduli systems) at the very beginning of algebraic-arithmetic geometry with the emphasis on finite fields where Fermat's infinite descent is at work. I must also acknowledge the beneficial exchanges I have had over a period of years, either in personal contacts or in correspondence with Henri Margenau, A. Wheeler, E.P. Wigner, I.M. Segal, G. Chew, René Thom, N.A. Shanin, H.M. Edwards, Ed Nelson, G. Kreisel, Y. Gurevich, U. Kohlenbach, H. Putnam, D. van Dalen, A. Urquhart, A. Joyal, Bas van Fraassen either for scientific counsels, critical assessments or friendly approvals. All have contributed to my understanding of the many facets of foundations, may they be logical, mathematical, physical or philosophical.

In the writing of this opus, I have drawn freely from previous work, my two books on the subject *Internal Logic* mentioned above and *Logique arithmétique. L'arithmétisation de la logique* (PUL, Québec, 2010) and numerous papers that have appeared in recent years in a variety of scientific journals, *Synthese*, *Logique et Analyse*, *Revue internationale de philosophie*, *Foundations of Science*, *International Studies in the Philosophy of Science*, *Reports on Mathematical Logic*, *International Journal of Theoretical Physics*, *Journal of Physical Mathematics*, *International Journal of Pure and Applied Mathematics* and *Reports on Mathematical Physics*. Some of the ideas that are still on the forefront here have appeared in earlier publications in *Modern Logic*, *Zeitschrift für mathematische Logik und Grundlagen der Mathematik* and *Archiv für mathematische Logik und Grundlagenforschung*, *Zeitschrift für allgemeine Wissenschaftstheorie*, *Notre-Dame Journal of Formal Logic*, *Dialectica* and *Philosophy of Science*, but those ideas have taken on new clothes in my up-to-date synthesis. I have completed the writing of this work in the summer and fall of 2014, not without the assistance of my two LaTeX men, David Montminy and Benoit Potvin. Benoit Potvin has made it possible for me to be up to the requirements of scientific journals by diligently latexizing my papers in the last five years. He is here thanked for his expertise as a computer scientist. I also wish to thank the Canadian Research Council (SSHRC) for funding my research in the last four years (and many years before!). Finally, I am grateful to Jean-Yves Béziau who has had a sympathetic ear and a friendly reception to my work over the years.

Montreal
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Yvon Gauthier

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Chapter 1
Introduction: The Internal Logic of Arithmetic

The idea of an internal logic of arithmetic or arithmetical logic is inspired by a variety of motives in the foundations of mathematics. The development of mathematical logic in the twentieth century, from Hilbert to the contemporary scene, could be interpreted as a continuous tread leading to arithmetical logic. Arithmetization of analysis with Cauchy, Weierstrass and Dedekind and arithmetization of algebra with Kronecker have lead to the foundational inquiries initiated by Hilbert. Frege's logical foundations of mathematics, mainly arithmetic, have contributed to clarify philosophical motives and although Frege's logicism has not achieved its goals, it has given birth to Russell's theory of types and to some extent to Zermelo's set-theoretic cumulative hierarchy while launching philosophical logic and philosophy of language. But the "arithmetism" I have in mind here is mostly anti-Fregean in that in turns logicism upside down and asks the question: "how far can we go into logic with arithmetic alone" rather than the Fregean question: "how far can we go into arithmetic with deductive logic alone?" It is Kronecker's polynomial arithmetic that guides here and the purpose of this book is to see how far a Kroneckian constructivist program can go in the arithmetization (and algebraization) of logic in the twenty-first century. The present work has been conceived has a sequel to my 2002 book *Internal Logic, Foundations of Mathematics from Kronecker to Hilbert* (Kluwer) and as a continuation of my efforts towards an arithmetical logic.

Hilbert is the starting point in the arithmetization process of logic; he is the one who introduced logic into mathematics and if he wanted to provide foundations for logic and arithmetic at the same time, he realized later on that logic should be arithmetized in order to provide a consistency proof for arithmetic and he adopted finally a finitist foundational stance reminiscent of his former teacher, Kronecker. Skolem wanted also a finitist arithmetic and Brouwer's theory of choice sequences with his assignment of natural numbers to members of a species (set) and sequences was a predecessor to Gödel's arithmetization of syntax (formal system). The birth certificate of model theory, Tarski's method of quantifier elimination, can be traced to Kronecker's theory of substitution-elimination of indeterminates. The proof theory of first-order arithmetic and its subsystems is another witness to the arithmetization program and finally the idea of algorithm and

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ramifications in complexity theory, in computer algebra and in theoretical computer science counts certainly as one of the major achievements of such a programme. Cantor's transfinite arithmetic is also part of the arithmetization of analysis and we could pinpoint the reasons why the Cantorian enterprise has ended in transarithmetical transfinite set theory. And can analysis be constructivized? Brouwer said yes, to a certain extent. Bishop thought that it could be arithmetized, that is, have numerical content. Non-standard analysis with Robinson's hyperreals goes even further in trying to give some content to infinitesimals. Cantorian set theory has nevertheless become the standard semantics of classical logic and Peano arithmetic and it is not easy to disentangle the arithmetical syntax from the set-theoretical semantics of contemporary logical theories, may they be Martin-Löf's constructive or intuitionistic type theory and other alternatives like reverse mathematics and Feferman's predicative mathematics. Only Nelson's predicative arithmetic stands out as a radical program, but it is considered too restrictive to serve as a foundational guideline.

From a more philosophical viewpoint, Yessenin-Volpin's ultrafinitism is out of touch with actual practice and Wittgenstein's finitist views do not seem to have grasped the full extent of mathematical logic in the twentieth century. What is at stake is not the existence of mathematical or logical entities, the ideal elements Hilbert had introduced to eliminate them afterwards as a mere detour, nor the objectivity of public mathematical construction, but the arithmetical, ultimately computational, content of logical or mathematical theory. The objectual reality of mathematics can be summarized as the different ways it counts physical objects and non-physical entities and how it accounts for the uncountable, if at all. Kronecker's attitude in that matter has been largely misinterpreted and it must be emphasized that Kronecker's polynomial arithmetic, which he called general arithmetic "*allgemeine Arithmetik*", has been a central theme in the advent of abstract algebra. One must keep in mind that contemporary algebraic or arithmetic geometry has originated in Kronecker's work on elliptic functions, as has been pointed out by André Weil, and Kronecker's arithmetical ideal could be tracked in present-day grandiose programmes of Langlands and Grothendieck. Algebraic number theory has also benefited immensely from Kronecker's arithmetic theory of algebraic quantities—that is the subject matter of his 1882 seminal paper dedicated to Kummer—and Hermann Weyl has stressed the advantages of the Kroneckerian theory of domains of rationality over Dedekind's ideal theory.

Elementary or ordinary number theory was the arena of the first axiomatisations by Dedekind and Peano. Recursive functions and the idea of recurrence made an early appearance. Dedekind is supposed to have introduced that mode of thought in his "*What are and what should be numbers*"—"*Was sind und was sollen die Zahlen*" but the procedure is implicit in Kronecker's theory of forms (homogeneous polynomials). Dedekind's and Peano's axiomatisations are based on Cantorian set theory and they are impregnated in a set-theoretic semantics which obnubilate the arithmetical motivations. Hilbert will be more aware of that arithmetical background in his metamathematics devoted to formalized mathematics in a finite system of arithmetical operators and polynomials equations (and inequations).

Kronecker's work has been ignored, except by mathematicians. Logicians and philosophers know only his "*On the concept of number*" (*Über den Zahlbegriff*), not his most important work by far. Frege mentioned Kronecker only once and characterized him as an empiricist, in the like of Helmholz—which Kronecker quotes at the end of his "*On the concept of number*"—then discrediting him as a non-logician. Most people associated Kronecker with his dictum stating that "Integers are the creation of God, the rest is the creation of man", which is not to be found in Kronecker's writings, but is probably attributed to him by association! Hilbert, maybe with some resentment towards his former professor, depicted him as a "*Verbotsdiktator*" or dictator of interdiction.

Arithmetization of logic took place without the contribution of the algebra of logic before Tarski. De Morgan, Boole, Peirce, Schröder and Löwenheim were not part and parcel of the process, except that Boole was influential in the algebrization of classical elementary logic, which he linked to the theory of probability in his 1854 *An Investigation of the Laws of Thought*. But what should be emphasized at this point is the separation of arithmetic from number theory. The tradition of number theory from Fermat, Euler, Gauss, Legendre, Lagrange, Dirichlet up to Kummer, Kronecker's cherished professor, has developed autonomously, as the queen of sciences, as Gauss would name it. Classical number theory is not on the same footage as Dedekind-Peano arithmetic and one should make it clear that there is no historical connection between number theory and formal arithmetic of logical descent. However arithmetical logic is aimed at bridging the gap between logic and arithmetic (the "true" arithmetic of number theory) and this is why I shall be using extensively Fermat's method of infinite descent still prominent in the work of contemporary number-theorists from Mordell to Weyl. Infinite descent combined with Kronecker's polynomial arithmetic will reveal an essential ingredient of the program of an arithmetical logic that I want to confront here with alternative programs in the foundations of mathematics. In so doing, I hope to connect historical, philosophical, logical, mathematical and foundational questions—foundations being the synthesis of all—into a unified treatment and propose a new framework for the philosophical question and the mathematical problem of the consistency of arithmetic. The programme of arithmetical logic finds its final justification in the internal logic of arithmetic, an internal logic that shows that arithmetic has to be self-consistent if it is to be the building stone of logic and mathematics.[1]

[1] In the following, all translations from French, German, Russian, Italian and Latin are mine.

Chapter 2
Arithmetization of Analysis and Algebra

Arithmetization of analysis evokes at once the names of Cauchy, Weierstrass, Cantor and Dedekind and to a lesser degree those of Dirichlet, Abel or Bolzano; the process of arithmetization illustrates the need to instill rigour in analysis through what Cauchy called algebraic analysis in order to overcome the intuitive limitations of the geometer's method of proof. The story of arithmetization needs not to be retold here (see Grattan-Guinness 1970); it is not a one-sided history, for rigour had a different meaning then and the tools used for rigorization (e.g. quadratic forms or homogeneous polynomials) were partly available in the nineteenth century. The algebraic symbolism (Descartes, Fermat and Leibniz) was already invading geometry and number theory (Diophantine equations) from Fermat on was to become the queen of arithmetical sciences.

2.1 Cauchy and Weierstrass

In the introduction of his 1821 *Cours d'analyse*, Cauchy warns that: "It would be a serious error to think that certainty can be found only in geometrical demonstration or in the testimony of the senses", but he begins with an algebraic analysis of real functions and their limits, polynomials as continuous functions and of convergent and divergent series. Cauchy still talks of infinitively small quantities or infinitesimals, but he equals them with limits approached by real-valued functions. He introduces there the convolution product—also named Cauchy product—for convergent series and recursive series ("*séries récurrentes*") or polynomials in the increasing or decreasing order of their powers. Cauchy achievement resides in his precise definitions of limits 0 and ∞ for an infinite quantity.

> When the successive values given to the same variable approach indefinitely a fixed value in such a way as to differ from it as little as one wishes, this fixed value is called the limit of all the other values (Cauchy 1847, p. 14)

Weierstrass has not been satisfied with Cauchy's notion of a variable approaching a limit. In his text "*Theorie der Maxima und Minima von Functionnen einer und mehrerer*

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Veränderlichen" or *Theory of maxima and minima of functions of one and many variables*, Weierstrass innovates with his idea of the $\epsilon - \delta$ method.

> For a function $f(x)$, one says that its value at $x = a$ is a minimum when it is smaller for $x = a$ than all the neighboring values of x, that is when a positive quantity δ can be determined such that
>
> $$f(a+h) - f(a) > 0$$
>
> with $|h| < \delta$. And the values of a function $f(x)$ for $x = a$ is called a maximum, when for all values under the restriction $h < \delta$, one has
>
> $$f(a+h) - f(a) < 0$$
>
> (Weierstrass 1894–1927, p. 7)

Weierstrass then shows that for the derivative

$$f(a+h) - f(a) = hf'(a + \epsilon h)$$

a quantity is needed with the sufficient condition

$$0 < \epsilon < 1$$

and the first necessary condition for the existence of a minimum or a minimum for $f(x)$ at $x = a$ is that $F'(a) = 0$; the second necessary condition is that in the sequence of derivatives the first non-vanishing derivative for $x = a$ must be of even order. Weierstrass goes on and generalizes easily these conditions to many-variable functions (Weierstrass 1894–1927, p. 7).

The interesting fact here is that Weierstrass couches his ideas in the language of quadratic forms (homogeneous polynomials): these are the arithmetical foundations by excellence in the number-theoretic tradition since Gauss. Of course Bolzano has already proven that polynomials are continuous and Weierstrass had only to define a vanishing positive definitive form as one whose variable have all the value 0 (Weierstrass 1894–1927, p. 20). Further considerations have to do with Sturm's theorem on changed signs in the real roots of an algebraic equation; Kronecker had also dealt with refinements of Sturm's theorem as he has criticized Bolzano's intermediate value theorem for being imprecise, since the interval for the values was not defined, although Bolzano had believed to arithmetize the geometrical intuitive proofs for continuous functions. But Cantor and Dedekind will give a new twist of the history of arithmetization in extending the more or less constructive methods of their predecessors into a new array of non-constructive techniques.

2.2 Dedekind and Cantor

From Dedekind's side, arithmetization is a logical enterprise to the extent that proof and provable are interwoven in mathematical constructions, and the logic he has in mind is internal to arithmetic or the science of number ("*Wissenschaft der Zahlen*"). Irrational numbers, Dedekind insists, must be grounded arithmetically, but foundations have to be general or abstract, for the science of number is *a priori*, independent of space and time; this is why Dedekind speaks of things and systems of things as the building blocks of arithmetic (Dedekind 1965). Soon enough, those concrete building blocks will be assembled by abstract concepts like mappings (*Abbildungen*) and images (*Bilder*). Chains (*Kette*) remain the concrete links for the assembly-line of things and systems. In Dedekind chain of thought however, the highest point is his unification of an infinite system as a system which is in bijection with a proper part of itself, that is excluding itself; otherwise, a system is finite. The proof of the statement has been diversely appreciated—some have downgraded it as a metaphysical or psychological proof—it involves "the world of my thoughts" ("*meine Gendankenwelt*") as an actual infinite system, while Dedekind expressly says that the world of my thoughts, that is the totality of things that can be objects of my thought, is infinite, which points out to a potential infinite at most (Dedekind 1965, p. 14). It is not without reason that Dedekind notes that there is a similar consideration in the Platonist Bolzano. One could say there is a clash here between the arithmetical potentialities and the set-theoretical actuality of the transarithmetical or transcendental world.

In Dedekind's hands, the construction of the natural number system with complete induction as a simply infinite bijection system $N \to N$ is straightforward. In a sharp contrast, the construction of the rational and irrational numbers in his "*Stetigkeit und irrational Zahlen*" (Continuity and irrational numbers) is completed in a purely arithmetical style, as if we are entering the real mathematical world with rational and irrational numbers. Dedekind's writes:

> I see the whole arithmetic as a necessary or at least as a natural consequence of the simplest arithmetical act of counting, and counting itself as nothing more than the successive creation of the unending sequence of positive whole numbers (integers)... (Dedekind 1965, II, p. 5)

The idea of a chain is born out of the step by step unfolding of numbers in their natural succession, rational numbers when compared to a point or straight line, are situated on the right or on the left of a given point and on the continuous line there is an infinity of points which do not correspond to any rational number. The essence of continuity lies in the principle:

> If we cut the straight line in two segments (or classes) in such a manner as to put all the points of the first segment to the left of any point of the second segment, then there is only one point that divides all the points in two segments of the straight line. (Dedekind 1965, II, p. 10)

A cut ("*Schnitt*") (A_1, A_2) has the meaning that every rational number in A_1 is smaller than any number in A_2 and also that there is either a largest number in A_1 or a smallest number in A_2. It follows immediately that there is an infinity of cuts in the straight line that are not generated by rational numbers which therefore do not exhaust the whole continuous line. So an irrational number like $\sqrt{2}$ is represented by an irrational cut. Total order and Dedekind-completeness of the real line are properties directly following from the cut construction, since supremum and infimum are consequences of the theorem that there is only one number that cuts the system of all real numbers in two. Dedekind also saw his theory of cuts as applying easily to differential calculus or infinitesimal analysis because of the centrality of the concept of continuity.

Of greater importance to us is Dedekind emphasis on the arithmetic construction of the real continuum; for him, the main guiding principle, despite the set-theoretic overtones, is the arithmetic process that extends in a natural way ordinary arithmetic into higher arithmetic (number theory and algebra) and analysis. In the same arithmetical spirit, Cantor went even further into a transfinite arithmetic that would exhaust not only the real line, but also the real arithmetic spirit itself by immersion in the set-theoretic universe!

At the beginning, Cantor was interested in number theory and algebra, as his first writing reveals. Indeed, his 1867 dissertation deals with quadratic forms (homogeneous polynomials) and one of his thesis is that purely arithmetical methods prevail over analytical ones. These writings seem to be mere exercises on problems raised by number-theorists from Gauss to Kronecker (his professor in Berlin) until he reaches the rich terrain of trigonometric functions where his mathematical talent finds its start. Cantor inherits from Riemann and Heine the problem of how to represent a real-valued function $f(x)$ by a convergent trigonometric series for all the values of x and show that there is no other series of the same form which is convergent and uniquely represent the said function $f(x)$. In other words, it is the problem of the canonical representation of a real-valued function by a trigonometric series. In the course of his work on the canonical representation, Cantor came to benefit from Kronecker's comments for a simplification of his proof; the simplification amounted to evince infinitesimals by using two arithmetical expressions $y + x$ and $y - x$ where y is a constant in order to cancel or make vanish the infinitesimal coefficients

$$\lim(c_n nx) = 0 \quad \mathit{for} \;\; n = \infty$$

It is interesting to note that Cantor's rejection of infinitesimals originates in Kronecker's arithmetization program. But Cantor was quick to find another use for his finite limits in his theory of derived sets of points. As I said, Cantor did not rest content with the arithmetical simplifications (or restrictions) suggested by Kronecker and went on extending his results on trigonometric series in 1872. Here Cantor develops a theory of limit points ("*Grenzpunkte*") or accumulation points ("*Häufungspunkte*"), which is of geometric inspiration (with Cartesian coordinates) even if it looks like Weierstrass' arithmetical theory from the outside. Cantor's infinitary inclination is apparent in his propension to include the infinite as well as the finite in his mathematical research.

The limit point of a set of points P is a point on the straight line such that every neighborhood of that point contains an infinity of points, but in the interior ("*in seinem Innern*") of the (real) interval. We have here the first lineaments of point-set topology. Cantor denotes P' the set of limit points as the first derived set of points of P, if that derived set of points contains an infinite number of points, P has a second derived set of points denoted by P'' and so on till $P^{(v)}$ for the vth derived set of points. The construction of an unending series of derived sets of points opens the way for a theory of fundamental sequences, defined by

$$\lim_{v \to \infty} (\alpha_{v+\mu} - \alpha_v) = 0$$

and in general

$$\lim_{v \to \infty} \beta_v = \beta.$$

Those sequences remind us of Cauchy sequences, but where Cauchy and his followers saw a limit, Cantor took that limit symbol as a radical departure from traditional views. He wrote:

> One should pay attention to this cardinal point, the significance of which could easily be misunderstood in the third definition of the real number—with the help of fundamental sequences as in
>
> $$\lim_{v \to \infty} \alpha_v = b$$
>
> number b is not defined as the bound ("*Grenze*") of the parts of a fundamental sequence (α_v), it would be a logical mistake similar to our discussion of the first definition—of a limit as a sum—where the existence of the bound
>
> $$\lim_{v \to \infty} \alpha_v$$
>
> would only be presumed; it is much more the other way around, since from our former definitions of the concept with its properties and relations in the rational number system we can conclude with certainty that the limit exists and is equal to b. Forgive my insistence on this apparent trifle ("*Kleinigkeit*")... (Cantor 1966, p. 187)

Cantor goes on and declares that the irrational numbers derive the same status of determinate reality in our mind ("*bestimmte Realität*") as do the rational numbers. Then

$$\lim b_v = b$$

also exists and arbitrary high orders of fundamental sequences exist as well. I see in this passage the birth certificate of transfinite set theory with the unlimited generation of orders of fundamental sequences becoming infinite ordinals of the second number class

$$\omega, \omega + 1, \ldots, \omega^2, \ldots, \omega^\omega, \omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega}}}}}, \epsilon_0$$

defined by

$$\lim_{n \to \omega} \left.\omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega}}}}}\right\}^n = \epsilon_0$$

and more explicitly

$$\begin{aligned}
\omega &= \lim < 0, 1, 2, \ldots n > \\
\omega \cdot 2 &= \lim < \omega n > \\
\omega^2 &= \lim < \omega \cdot n > \\
\omega^\omega &= \lim < \omega^n > \\
\omega^{\omega^\omega} &= \lim < \omega^{\omega^n} > \\
\epsilon_0 &= \lim < \left.\omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega}}}}}\right\} n >
\end{aligned}$$