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# L'Hôpital's Analyse des infiniments petits

An Annotated Translation with Source  
Material by Johann Bernoulli



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# L'Hôpital's *Analyse des infiniments petits*

An Annotated Translation with Source  
Material by Johann Bernoulli

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## Translators' Preface

*Analyse des infiniment petits, pour l'intelligence des lignes courbes* was the first textbook of the differential calculus. The title translates as *Analysis of the infinitely small, for the understanding of curved lines*. It was published anonymously in Paris in 1696, although members of the French mathematical community were well aware that the author was Guillaume François Antoine de l'Hôpital,<sup>1</sup> the Marquis of Saint-Mesme (1661–1704). The textbook was successful, as evidenced by the appearance of a posthumous second edition (L'Hôpital 1716), which identified the author.<sup>2</sup> Pierre Varignon (1646–1722), who was professor of mathematics at Collège des Quatre-Nations in Paris and a friend of l'Hôpital, created a collection of clarifications and additions to the *Analyse*. These were published posthumously (Varignon 1725), a few years after the 1716 edition of the *Analyse*. Later editions of the *Analyse* included similar commentary and continued to appear throughout the 18th century (L'Hôpital 1768, 1781).

Differential and integral calculus are generally considered to have their origins in the works of Sir Isaac Newton (1642–1727) and Wilhelm Gottfried von Leibniz (1646–1716)<sup>3</sup> in the late 17th century, although the roots of the subject reach far back into that century and, arguably, even into antiquity. Leibniz first described his new calculus in a cryptic article more than a decade before the publication of the *Analyse* (Leibniz 1684). For all practical purposes, Leibniz' early papers were not

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<sup>1</sup>L'Hôpital spelled his name "Hospital" and this was the spelling used in the posthumous second edition of the text in 1715/1716. Fontenelle used the spelling "Hôpital," which is the standard modern spelling of the name, in his Eulogy (p. 295). There is no consensus among English-language authors of the early 21st century as to which spelling ought to be used.

<sup>2</sup>Technically, this was the third edition, despite the words "Seconde Edition" on the title page. There was a "Seconde Edition" one year earlier (L'Hôpital 1715), with many typographical errors, of which the 1716 edition is a corrected version (Bernoulli 1955, pp. 499–500).

<sup>3</sup>For more on the priority dispute over the discovery of the calculus, which is not a matter of interest for this volume, see Hall (1980).

understood, until Jakob Bernoulli (1654–1705) and his younger brother Johann<sup>4</sup> (1667–1748) began studying them in about 1687 and making discoveries of their own using his techniques.

Bernard de Fontenelle (1657–1757) became the secretary of the *Académie des Sciences* in Paris in 1697 and wrote the eulogy of l'Hôpital for the academy's journal. He said that in 1696,

... the Geometry of the Infinitely small was still nothing but a kind of Mystery, and, so to speak, a Cabalistic Science shared among five or six people. They often gave their Solutions in the Journals without revealing the Method that produced them, and even when one could discover it, it was only a few feeble rays of this Science that had escaped, and the clouds immediately closed again.<sup>5</sup> (Fontenelle 1708, pp. 133–134)

Later on, Jean Etienne Montucla (1725–1799) went one step further and listed the only people that he believed understood Leibniz' calculus before 1696: Leibniz himself, Jakob and Johann Bernoulli, Pierre Varignon and l'Hôpital (Montucla 1799, p. 397). L'Hôpital's *Analyse* changed all of this and for much of the 18th century, his book served aspiring French mathematicians as their first introduction to the new calculus.

For all that the *Analyse* was a popular and successful introduction to the differential calculus, it's remarkable that there is no account of the integral calculus in the book. In his Preface, l'Hôpital explained why

In all of this there is only the first part of Mr. Leibniz' calculus, ... The other part, which we call *integral Calculus*, consists in going back from these infinitely small quantities to the magnitudes or the wholes of which they are the differences, that is to say in finding their sums. I had also intended to present this. However, Mr. *Leibniz*, having written me that he is working on a Treatise titled *De Scientiâ infiniti*, I took care not to deprive the public of such a beautiful Work ... [p. liii].

Unfortunately, Leibniz never completed this book *On the Science of the Infinite*.

The *Analyse* consists of ten chapters, which L'Hôpital called "sections." We consider it to have three parts. The first part, an introduction to the differential calculus, consists of the first four chapters:

1. In which we give the Rules of this calculus.
2. Use of the differential calculus for finding the Tangents of all kinds of curved lines.
3. Use of the differential calculus for finding the greatest and the least ordinates, to which are reduced questions *De maximis & minimis*.
4. Use of the differential calculus for finding inflection points and cusps.

Taken together, these chapters provide a thorough introduction to the differential calculus in about 70 pages. The next five chapters are devoted to what can only be described as an advanced text on differential geometry, motivated in part by what were then cutting-edge research problems in optics and other fields. The final

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<sup>4</sup>Often referred to as Johann (I) Bernoulli to distinguish him from his son Johann and grandson Johann, who were also successful mathematicians.

<sup>5</sup>This quotation from p. 299 of this text.

chapter is mildly polemical, demonstrating the superiority of Leibniz' new calculus, when compared to the methods of René Descartes (1596–1660) and Johann van Waveren Hudde (1628–1704).

## The Role of Johann Bernoulli

Most biographical information known about the Marquis de l'Hôpital comes from Fontenelle's eulogy (Fontenelle 1708, pp. 116–146), a translation of which is included in Chapter 13 of this volume. However, Fontenelle knew little or nothing at the time of l'Hôpital's death about the role of Johann Bernoulli in the composition of the *Analyse*. L'Hôpital himself acknowledged a debt to Johann Bernoulli in his preface:

I acknowledge having received much from the illuminations of Messrs. Bernoulli, particularly those of the younger, presently Professor at Groningen. I have made plain use of their discoveries and those of Mr. Leibniz. This is why I grant that they may claim as much of this as they may wish, being content with that which they are willing to leave for me [p. liv].

In light of this, it seems somewhat strange that Montucla would write “We may only find fault in that Mr. de l'Hôpital did not make well enough known the debt he owed to Mr. Bernoulli” (Montucla 1799, p. 397), but the record shows that Johann Bernoulli's influence on the structure and content of the *Analyse* was much more significant than these words of recognition would suggest.

Among the few details known about l'Hôpital's early life, Fontenelle recounted that he solved one of Pascal's problems involving the cycloid at the age of 15. The Marquis became a cavalry officer, but had only attained the rank of captain when he resigned his commission due to poor eyesight. He devoted all of his energy to mathematics from that point onward. Some time around 1690, he joined Nicolas Malebranche's (1638–1715) circle, which was engaged, among other things, in the study of higher mathematics. It was there in November 1691 that he met the 24-year-old Johann Bernoulli, who was visiting Paris and had been invited by Malebranche to present his construction of the catenary at the salon. Although Fontenelle made no mention of this meeting, it is documented by Spiess in his introduction to the Bernoulli-l'Hôpital correspondence, which contains what may be considered a definitive biography of the Marquis de l'Hôpital (Bernoulli 1955, pp. 123–130).

There is no contemporary account of this meeting. Bernoulli wrote of the encounter in his autobiography, which he composed in 1741, but Spiess considers an earlier account that he gave in a letter to Pierre Rémond de Montmort (1678–1719) to be more reliable; see Bernoulli (1955, p. 135–137). In May 21, 1718, Bernoulli told Montmort that upon meeting the Marquis, he soon found him to be a good enough mathematician with regard to ordinary mathematics, but that he knew nothing of the differential calculus, other than its name, and had not even heard of the integral calculus. L'Hôpital had apparently mastered Fermat's method of finding maxima and minima and told Bernoulli that he had used it to invent a rule for determining the radius of curvature for arbitrary curves. The method was



unwieldy and actually could only be used at local extrema of algebraic curves. Bernoulli showed him the formula for the radius of curvature that he had developed with his brother Jakob, which employed second-order differentials. Apparently, this so impressed the Marquis that he visited Bernoulli the very next day and engaged him as his tutor in the differential and integral calculus.



The Marquis de L'Hôpital (1661–1704) – portrait in walnut by Susan Petry, 2014, 25 × 19 × 3 cm.; photograph by the artist

Bernoulli tutored the Marquis in his Paris apartment four times a week from late 1691 through the end of July 1692. We are fortunate that l'Hôpital insisted that Bernoulli commit his lessons to paper. Bernoulli typically composed each lesson, which he wrote in Latin, the night before he gave them to the Marquis. Fortunately, his friend and later colleague at the University of Basel, Johann Heinrich von



Johann Bernoulli (1667–1748) – portrait in cherry by Susan Petry, 2014, 26 × 19 × 4 cm.; photograph by the artist

Stähelin (1668–1721), was rooming with him in Paris. Stähelin made copies of the lessons before Bernoulli handed them over to the Marquis.

In the summer of 1692, Bernoulli accompanied the Marquis to his estate in Oucques, near the French city of Blois, where he continued giving him tutorials until some time in October. Bernoulli's lessons from this period have not survived, although he told Montmort that the Marquis' valet made copies of some of them, one of which appears to have survived (Bernoulli 1955, p. 137). It is also possible that some of the Bernoulli's lessons on the integral calculus, which he later reported to have been given in Paris, were actually given at Oucques.

In any case, Bernoulli kept copies of his lessons to the Marquis throughout his long and productive career. The first part, on the differential calculus, was incorporated by l'Hôpital into the first four chapters of the *Analyse*. Bernoulli himself published the much larger second part, concerning the integral calculus, in his collected works (Bernoulli 1742, pp. 385–558). Titled *Lectiones mathematicae de methodo integralium*, (Mathematical Lectures Concerning the Method of Integration), this treatise bears the subtitle “written for the use of the Illustrious Marquis de l'Hôpital while the author spent time in Paris in the years 1691 & 1692.” The first sentence of this work makes reference to what was seen “in the preceding.” A footnote explained that Bernoulli meant the lectures in differential calculus, which had preceded this but which he had omitted, because all of it had appeared in the *Analyse*, “which is in everyone's hands.” (Bernoulli 1742, p. 387) That is, he left out the portion of his Paris lessons that l'Hôpital had incorporated into his introductory chapters. Because Bernoulli chose not to publish this part, it was impossible in the 18th century to say how closely l'Hôpital's textbook coincided with Bernoulli's lessons.

A comparison finally became possible when Paul Schafheitlin discovered a manuscript copy of the full set of lessons, on both the differential and integral calculus, in the library of the University of Basel in 1921. Schafheitlin published the first portion as *Lectiones de calculo differentialium* (Schafheitlin 1922) and argued in his introduction that the manuscript was a copy made in 1705 by Bernoulli's nephew Nikolaus (1687–1759), who had been living with him in Groningen. Because the latter part was a near-perfect match to what Bernoulli had published in 1741, he could be quite certain that the first part was essentially the same set of lessons l'Hôpital had used when composing the *Analyse*.

In this volume, we have brought together both l'Hôpital's *Analyse* and Bernoulli's *Lectiones* for the first time, in English translation. We have cross-referenced the texts in order to facilitate a comparison of Bernoulli's original contributions with l'Hôpital's final version. Since the appearance of the *Lectiones*, various authors have characterized the *Analyse* as having essentially been written by Bernoulli. Indeed, Bernoulli himself, in an angry letter to Varignon of February 26, 1707, said that “to speak frankly, Mr. de l'Hôpital had no other part in the production of this book than to have translated into French the material that I gave him, for the most part, in Latin . . .” (Bernoulli 1992, p. 215). The truth is much more nuanced. The superstructure of l'Hôpital's first four chapters is certainly due to Bernoulli and many of the details are essentially the same in both texts. However,



l'Hôpital added much, in both quantity and quality. For one thing, Bernoulli's *Lectiones* occupied 37 manuscript pages, compared to 70 typeset pages for the first four chapters of the *Analyse*, but the Marquis added much more than mere verbiage to Bernoulli's lesson. He was a very talented pedagogue. He organized his material very well, extracting general propositions where Bernoulli gave examples, and explained matters clearly and in some detail. Furthermore, he frequently included many illustrative examples, gradually increasing in difficulty, generally providing an appropriate level of detail, but always leaving some things for readers to work out for themselves.

To the best of our knowledge, our English translation of the *Lectiones* is the first to be published. Strictly speaking, our translation of the *Analyse* is the second to appear. L'Hôpital published only one other book, a posthumous textbook on conic sections (L'Hôpital 1707), which also went through multiple editions. Edmund Stone (ca. 1700–1768) published an English translation of this more elementary text (Stone 1723). The son of a gardener of the Duke of Argyll and a self-taught mathematician, Stone subsequently published an English translation of L'Hôpital (1696). However, it was a re-writing of l'Hôpital's book in the sense that Stone translated every statement about differentials into the language of Newton's fluxions. This translation made up the first part of his calculus textbook (Stone 1730), which concluded with an original text on the integral calculus.

## The L'Hôpital-Bernoulli Correspondence

Bernoulli departed Oucques and returned to Basel in the fall of 1692. He began a correspondence with l'Hôpital in November 1692, which continued until 1702. He completed a doctorate in medicine, for which the thesis was really a work of applied mathematics, in 1694 and ascended to the Chair of Mathematics at the University of Groningen in Holland late in 1695. He returned to Basel in 1705 and took over the Chair of Mathematics at the University of Basel, which had been occupied by his brother Jakob until his death the same year. As father of Daniel Bernoulli (1700–1782) and mentor to Leonhard Euler (1707–1783), his influence on Continental mathematics was even more significant than his impressive list of publications would suggest.

Bernoulli's estate contained 60 letters from the Marquis, two from his wife, the Marquise de l'Hôpital, and copies Bernoulli had made of 25 of his letters to the Marquis. Spiess infers that Bernoulli wrote at least 28 other letters to the Marquis. None of Bernoulli's original letters are known to have survived. In fact, all of l'Hôpital's grandchildren died childless and none of the Marquis' mathematical papers seem to have been preserved. Bernoulli's papers, including the correspondence with the Marquis, went to his son and subsequently to his grandson Johann (III) Bernoulli (1744–1807), who found them a home in the archives of the Royal Swedish Academy of Sciences. The letters remained essentially unknown until they were rediscovered by Gustav Eneström (1852–1923) in 1879. Although

Eneström and others published bits and pieces in the late 19th and early 20th centuries, e.g. Eneström (1894), it was not until much later that a critical edition of the entire correspondence was eventually published (Bernoulli 1955) as part of the *Bernoulli Edition*.

We have included significant excerpts from this correspondence in Chapter 12 of this volume. The editors of the *Bernoulli Edition* numbered the letters in Johann Bernoulli's correspondence and we have used their numbering system to refer to the letters here. Numbers were only assigned to letters that are actually in the archives. For example, l'Hôpital's letters of December 8, 1692 and January 2, 1693 were numbered 6 and 7, respectively. In letter 7, l'Hôpital makes mention of Bernoulli's letter of December 18, and the symbol (6, 1) was assigned to this lost letter. In this volume, when we skip from letter 7 to letter 11, for example, you will know that we have skipped over three extant letters, which can be found in Bernoulli (1955), and that there are possibly some missing letters in between as well. The letters themselves were written in French, with occasional Latin phrases, and Spiess' accompanying modern editorial information in Bernoulli (1955) was written German.

We have included translations of some of the letters in their entirety, for some others we have only included certain portions, and many others have been omitted. The entire correspondence is quite large (about 225 pages in Bernoulli 1955), so we have needed to be selective. Some of the discussion in the correspondence has little or no relevance to the *Analyse* and its composition. The portions that we have chosen to include are generally either relevant to the contents of the book, or they shed light on some aspect of the personal or professional relationship between l'Hôpital and Bernoulli. We have also chosen to include letter 5, from l'Hôpital to Malebranche, which shows that the Marquis was writing about the "Arithmetic of the Infinities" before he met Bernoulli.

Between Bernoulli's first letter (6, 1) of November 1692 and his letter (15, 1) of late September 1693, letters between the two men were written in almost perfect alternation. Then a number of l'Hôpital's letters went unanswered. Bernoulli was apparently unhappy that l'Hôpital had published one of the results that he had given him in his lessons. L'Hôpital's paper,<sup>6</sup> which gave the solution to a problem that had originally been proposed by Florimund de Beaune, was published under the pseudonym "Mr. G\*\*\*," but Bernoulli would undoubtedly have recognized his own work. Bernoulli resumed writing on January 26, 1694. That letter is lost, but Spiess infers that Bernoulli had requested that the Marquis provide him an honorarium for the services he was rendering him through their correspondence (Bernoulli 1955, p. 201). We do not know how much l'Hôpital paid Bernoulli for his lessons in 1691–92, but it appears to have been enough to support him during his year or so in Paris and Oucques and perhaps for some time afterwards in Basel. In January 1694, Bernoulli was newly engaged to be married and not gainfully employed while he

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<sup>6</sup>*Journal des Sçavans*, 34, p. 401–403.

was completing his doctorate of medicine. In his response, on March 17, 1694, the Marquis wrote

I will happily to give you a *pension* of three hundred pounds, which will begin the first of January of this present year. . . . I promise you to increase this stipend shortly, which I well understand to be very modest, and it will be as soon as my affairs are somewhat straightened out . . . I am not so unreasonable as to demand all of your time for this, but I will ask you at intervals to give me a few hours of your time, to work on what I will ask you and also to communicate your discoveries to me, while asking you at the same time not to share any of them with others. I even ask you not to send here to Mr. Varignon, nor to others, any copies of the writings you have left with me; if they should become public I would not be at all pleased [p. 246].

Bernoulli's letter (20, 1) of late March is lost to posterity, but he clearly accepted the terms of what we will refer to here as "The Contract." A few words are in order. First of all, the French word *pension* here does not have the same sense as the modern English "pension," although the terms are related. L'Hôpital is offering an annual honorarium or retainer, but not necessarily in perpetuity. In fact, the duration of the Contract was a little over two years. L'Hôpital sent Bernoulli his last installment of 300  $\text{₣}$ <sup>7</sup> in June 1696, shortly after the *Analyse* appeared and some eight months after Bernoulli had taken up his position at Groningen. Bernoulli received a total of 800  $\text{₣}$ , owing to a slightly larger payment of 200  $\text{₣}$  in July of 1695. For context, 300  $\text{₣}$  has been described as half of the annual salary of a professor, but it was in fact only 21% of the salary that Bernoulli would earn at Groningen. Not a great fortune, but undoubtedly welcome to Bernoulli in his circumstances, with no other income and a new family to support (his first child was born in February 1695).

We note that it was only after agreeing to the terms of The Contract that Bernoulli began making copies of his letters to l'Hôpital. The first of these was letter 22, of April 22, 1694. From this point onwards, Bernoulli made copies of his communications to the Marquis, except in a few cases where there was no mathematical content in his letters.

We must also note that l'Hôpital was paying for more than just instruction and advice, he was essentially acquiring the rights to publish Bernoulli's discoveries. This became a bone of contention in early 1695. L'Hôpital sent Bernoulli his solution to a challenge problem that had appeared in *Acta Eruditorum*, asking him to translate it into Latin and send it to the journal. Bernoulli could not resist adding some remarks of his own, that greatly simplified the Marquis' solution. Not only did this have the effect of making the Marquis look bad in print, but it was also a violation of The Contract, to have sent his discovery concerning a matter of their discussion to a third party. In letter 41, Bernoulli reassured the Marquis that he would adhere strictly to The Contract in the future, offering as evidence that he had declined Leibniz' request that he submit material for his prospective book *de Scientiâ infiniti* (p. 272).

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<sup>7</sup>This is the symbol used for French *livres* or pounds, as transcribed in Bernoulli (1955).

The best known of the results that l'Hôpital "purchased" from Bernoulli by means of The Contract is the rule that became known as L'Hôpital's Rule, after its appearance in the *Analyse*. Bernoulli had evidently worked out the simplest case of L'Hôpital's Rule, the case of the indeterminate form  $\frac{0}{0}$  for a finite value of  $x$ , in June 1693 or perhaps earlier. In letter 11, l'Hôpital mentioned that he had learned that Varignon had been challenged by Bernoulli to evaluate

$$\frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}} = y$$

when  $x = a$  (p. 239). We note that they did not speak of limits, but rather they considered  $y$  to actually take on the value the we would call the limit as  $x$  approaches  $a$ . Modern examples involving L'Hôpital's Rule frequently involve transcendental functions, but Leibniz' calculus at this time was only a calculus of algebraic expressions. Bernoulli's  $\frac{0}{0}$  Challenge involved an algebraic function, but the problem could not be easily solved by factoring (as with  $\frac{x^2-a^2}{x-a}$ , for example), making it an excellent vehicle for demonstrating the power of L'Hôpital's Rule. In letter 11, l'Hôpital incorrectly gave the answer as  $x = 2a$ . Bernoulli apparently told him that he was wrong in (11, 1), but did not reveal his method. Between that point and The Contract, l'Hôpital asked Bernoulli for the solution on three occasions, but to no avail. Once Bernoulli had entered into The Contract, however, he had no choice but to reveal his Rule, which he did in letter 28 (p. 267). L'Hôpital presented the rule in §163 of the *Analyse* (p. 151), followed immediately by its application to Bernoulli's  $\frac{0}{0}$  Challenge.

The correspondence remained very active for the next year or so. Bernoulli clarified many issues for l'Hôpital, concerning topics that appeared in the *Analyse* and many others as well. There was also much non-mathematical discussion, including births and deaths and other family matters, and the payment of Bernoulli's stipend. In letter 47 of March 12, 1695, l'Hôpital announced that "I hope to be able to procure you a chair of mathematics in Holland" (p. 273). He was referring to the position at the University of Groningen that Bernoulli would take up later in the same year. In this letter and subsequently, l'Hôpital gives the impression that he is primarily responsible for securing the position for Bernoulli. It was Christiaan Huygens (1629–1695), in fact, who recommended Bernoulli for the position, but apparently only after asking l'Hôpital for a letter of recommendation. For the next few months, the position at Groningen and the rumored death of Huygens<sup>8</sup> dominated the non-mathematical portion of the correspondence.

In letter 56, of August 22, 1695, l'Hôpital first mentioned the *Analyse* (p.285). What he actually said was that he planned to print his treatise on conic sections very soon, a book on which he had apparently been working for some time, but as it was still unfinished upon his death in 1704, only appeared posthumously (L'Hôpital

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<sup>8</sup>Huygens died on July 8, but on June 10, l'Hôpital heard a rumor that he had died. There was some confusion over this in the letters that followed.

1707). “I will add to it,” he continued, “a small treatise on the differential calculus, in which I will give you all credit that you deserve.” Bernoulli was by this time preoccupied with his move to Groningen and did not respond until after he was settled, on November 8. He made no copy of that letter, but presumably gave his blessing for the proposed treatise. However, some time around the end of the year, l’Hôpital became very ill and after a long silence, his wife wrote to Bernoulli on February 1, 1696, to explain why the Marquis had not written for so long. L’Hôpital returned to Paris on or about the beginning of March and completed the publication process of the *Analyse*. In letter 63 of June 15, 1696, he wrote to Bernoulli that his book would “appear any day now” (p. 288), but that his illness had kept him from including his work on the conic sections, as he had originally planned. As for the *Analyse* itself, he assured Bernoulli that “you will see that I give you the credit you deserve.”

Bernoulli’s response is included in this volume for at least three reasons. Bernoulli congratulated the Marquis on the publication of the *Analyse*, which he had not yet received himself, but about which he had heard from other correspondents. He also acknowledged the receipt of 300  $\text{₣}$ , which was to be his last payment under the terms of The Contract. Also of interest is that Bernoulli described the Brachistochrone Problem to the Marquis in some detail (p. 289, 296). This challenge problem occupies a place of great significance in the history of mathematics and especially in the development of the Calculus of Variations; see Bernoulli (1988, pp. 329–334) for more on l’Hôpital’s solution of the problem and Katz (2009, pp. 586–588) for a general introduction to the Brachistochrone Problem.

It was many months before Bernoulli finally received the *Analyse*. In February 1697, he wrote

I have finally received a copy of your book and I thank you most humbly for it. You have done me too great an honor in speaking so highly of me in the preface. When I compose something in my turn I will not fail to give you the same in return. You explain things most intelligibly; I also find a beautiful order there and the propositions well organized; everything is admirably well done, and a thousand times better than I could have done. Finally, I desire nothing more than that you had put your name on the cover of the book, which would have given a much greater glory, and lent more Authority to our new method.  
[p. 220]

L’Hôpital and Bernoulli continued in frequent correspondence until the end of 1697. They wrote less frequently over the next six years and the final letter of their correspondence was written by l’Hôpital on September 15, 1702, in response to a lost letter from Bernoulli four months earlier. L’Hôpital died on February 2, 1704.

In the years immediately following the publication of the *Analyse*, Bernoulli seems to have been content with his largely unacknowledged role in its composition. However, shortly after the Marquis’ death, he became unsatisfied with the general mention he received in the preface and began making priority claims on the *Analyse*, especially with regard to L’Hôpital’s Rule. Eneström, in analyzing Bernoulli’s correspondence (Eneström 1894), found that his dissatisfaction had its first major expression in a letter he wrote to Varignon on July 18, 1705 (Bernoulli 1992, pp. 167–174). By quoting from the Marquis’ many letters involving his  $\frac{0}{0}$  Challenge, he tried to convince Varignon that the discovery of L’Hôpital’s Rule was far beyond



the Marquis' ability in the mid-1690s. Bernoulli became increasingly quarrelsome during his long career and his priority claims, made both publicly and in private correspondence, eventually became fodder for historians of mathematics. Montucla, for example, accepted Bernoulli's word and stated plainly that the Marquis did not give him due credit (Montucla 1799, p. 397), whereas Bossut argued vociferously for rejecting Bernoulli's claims of priority (Bossut 1810, pp. 49–52). With the fullness of time, both Bernoulli's *Lectiones* and his correspondence have come to light and modern scholars are now in a position to make their own assessment of the matter. We believe that readers will find that Bernoulli was indeed shortchanged in credit for the *Analyse*, but at the same time, we see the book as a very successful collaboration between a brilliant researcher and a talented expositor.

## The Contents of the *Analyse*

### *The Preface*

The *Analyse* opens with a Preface that traces the history of the “Analysis of the Infinitely Small” back to Archimedes and through the 17th century. It has been suggested that Fontenelle actually wrote the Preface, or at least this historical survey, which makes up the largest part of it. Costabel argues convincingly against this thesis (Bernoulli 1992, pp. 13–14). The suggestion that Fontenelle was the author appears to have originated with Fontenelle himself in the 1730s and then became widely known after his death through his eulogy. It seems that Fontenelle did indeed assist l'Hôpital in the process of getting his book published following his illness in early 1696, but Costabel makes a strong case against the possibility that he wrote the Preface.

### *Chapter 1: Notation and the Rules of Calculus*

In Chapter 1, l'Hôpital gives the rules for differential calculus. Modern readers should not expect anything that looks like a calculus book of our time. Leibniz' calculus concerns equations and differentials, not functions and derivatives. Even the graphs and terminology will take a little getting used to. There is no  $x$ - $y$  coordinate system. Following Descartes' analytic geometry, there is one axis, which we will refer to as the  $x$ -axis, even if the independent variable is not  $x$ . The axis is usually horizontal, but sometimes it is drawn vertically. The  $x$ -coordinates are usually referred to as *abscissas*, literally meaning that they are “cut off” on the axis at some distance  $x$  from the origin. In the place of a  $y$ -coordinate, there is an *ordinate* that is “applied” to the axis at the point corresponding to the abscissa  $x$ . This is a line segment that is usually perpendicular to the axis, although *oblique* ordinates were occasionally arranged at a different angle to the axis. Even the familiar parabola looks unfamiliar in the late 17th century. It is written as  $ax = yy$ .

As a typographical convention, squares were usually written by repeating the letter representing the variable, but higher powers were written with superscripts. Thus, the *second cubical parabola* was usually written  $axx = y^3$ .

Leibniz' calculus works by translating geometric problems into the language of algebraic expressions, performing operations on those expressions (taking differentials and integrals) and extracting solutions from the results. This is still largely true of calculus as it is practiced now, if one understands by algebraic expressions those functions that are composed of algebraic expressions involving both variables and a number of transcendental functions. Geometry has a very long history, having already reached a high degree of sophistication by the time of Euclid, Archimedes and Apollonius, whereas analytic geometry was less than half a century old when Leibniz discovered his calculus. Not surprisingly, then, the *Analyse* describes a calculus that has a much more geometrical flavor than the calculus of later centuries. For example, the parabola could be considered as a relationship between two-dimensional figures: given an abscissa  $x$  and a *parameter*  $a$  (this name for the constant goes back to Apollonius), one seeks a square with the same area as the rectangle with sides of length  $a$  and  $x$ . The equation  $ax = yy$  expresses a relation between "planar numbers" representing two-dimensional areas, so it would not have been written as  $x = cyy$ , which would seem to violate the rules of geometry by comparing a one-dimensional length to a three-dimensional volume. Descartes had taught that it was not necessary to write equations in such a *homogeneous* form, involving only terms of the same dimension, but even in the 1690s l'Hôpital, and especially Bernoulli, usually did write them in this manner.

Leibniz' differential calculus tells us how to find the relations among infinitely small increments  $dx$ ,  $dy$ , etc., among the variables  $x$ ,  $y$ , etc., in an equation. L'Hôpital gives the definition on p. 2: "The infinitely small portion by which a variable quantity continually increases or decreases is called the *Differential*." This cryptic definition might be made somewhat clearer by considering his Figure 1.  $AMB$  is a curve, presumably a parabola, with axis  $AC$  and origin  $A$ .  $AD$  is not the  $y$ -axis, but the parameter for the parabola. The abscissa is  $AP$ , denoted  $x$  and the

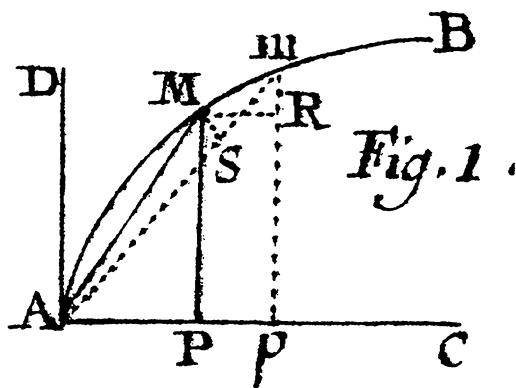


Fig. 1 Definition of the Differential

ordinate is  $PM$ , denoted  $y$ . We imagine that  $p$  is infinitely close to  $P$  and  $pm$  is the ordinate that is infinitely close to  $PM$ . Then  $Pp$  is the differential  $dx$  and  $Rm$ , where  $MR \perp pm$ , is the differential  $dy$ .

At this stage, we still don't know what "infinitely small" means, but perhaps Postulate I can clarify this. Bernoulli's first postulate is "Quantities that decrease or increase by an infinitely small quantity neither decrease nor increase." L'Hôpital is a little more comprehensible: "We suppose that two quantities that differ by an infinitely small quantity may be used interchangeably, or (what amounts to the same thing) that a quantity which is increased or decreased by another quantity that is infinitely smaller than it is, may be considered as remaining the same." Although this still does not give us much to go on, things become clearer when they begin to perform calculations. What one does is to replace the variables  $x$ ,  $y$ , etc., in an equation with  $x + dx$ ,  $y + dy$ , etc., and then cancel the original equation, much as finite differences are calculated. Then, because of Postulate I, terms involving two or more differentials multiplied together may be omitted, giving a relation among the variables and their differentials, such as  $a dx = 2y dy$  in the case of the parabola.

L'Hôpital then gives the rules for the differential calculus:

- Constant Rule: The differential of a constant is 0.
- Sum and Difference Rule: the differential of  $x \pm y$  is  $dx \pm dy$ .
- Product Rule: The differential of  $xy$  is  $y dx + x dy$ .
- Quotient Rule: The differential of  $\frac{x}{y}$  is  $\frac{y dx - x dy}{yy}$ .
- Power Rule: The differential of  $x^n$  is  $nx^{n-1}$  for any rational  $n$ .

We note that neither L'Hôpital nor Bernoulli used these names for their rules; we use them here because modern readers are familiar with these names and this will help them to see the correspondence between modern calculus and Leibniz' calculus. We also note that this is the ordering of the rules given by L'Hôpital. We find his exposition of the rules of calculus to be more elegant and satisfying than Bernoulli's and this is an example of a place where L'Hôpital has added value to Bernoulli's *Lectiões*. Finally, we note that there is no need for the Chain Rule in this calculus, because it is not restricted to functions. Variables may be freely introduced and when they are, the above rules are all that is needed. For example, if  $y = \sqrt{ax - x^2}$ , then we may let  $u = ax - x^2$ . Then the above rules give  $dy = \frac{1}{2}u^{-\frac{1}{2}} du$  and  $du = a dx - 2x dx$ , so

$$dy = \frac{a dx - 2x dx}{2\sqrt{ax - x^2}}.$$

With a little thought, modern readers can see that whenever  $y = f(x)$  for some algebraic function  $f$ , these rules will give  $dy = f'(x) dx$ , which explains the origin of our notation  $\frac{dy}{dx}$  for the derivative  $f'(x)$ . We stress that functions and derivatives are nowhere to be found in the *Analyse* or the *Lectiões*. However, the calculations on equations that take place in their pages are reminiscent of calculations in modern Related Rates and Implicit Differentiation problems.

L'Hôpital and Bernoulli both made extensive use of proportions, another legacy of classical Greek mathematics. The proportional relation  $a : b :: x : y$  means "as  $a$  is to  $b$ , so  $x$  is to  $y$ ." To modern readers, this is just the relation

$$\frac{a}{b} = \frac{x}{y},$$

which may be solved to give  $y = \frac{b}{a}x$ , but to the ancient Greeks, there was a distinction made between the number  $\frac{a}{b}$  and the proportion  $a : b$ . There was a large set of rules for manipulating proportions, which L'Hôpital's readers would have learned from Book V of Euclid's *Elements*. We note that Bernoulli wrote proportional relations in the form  $a \cdot b :: x \cdot y$  and L'Hôpital wrote them in the form  $a . b :: x . y$ ; we will write them in the form  $a : b :: x : y$  in both treatises.

### Chapter 2: Finding Tangents

Bernoulli's Postulate II is "Any Curved line consists of infinitely many straight lines, each of which is infinitely small." L'Hôpital's is much the same, explained a little more fully and with illustrations. This postulate is used for the first time to find tangents. This was not a question of calculating a slope, but rather the geometric question of how to *draw* the tangent at  $M$ . This reduces to the question of finding the point  $T$  where the tangent line intersects the axis and joining it to  $M$ . The line segment  $TP$  is called the *subtangent* so the problem is reduced to finding  $t = TP$ . By Postulate II, when  $p$  is infinitely close to  $P$ , then  $Pp$  is a straight line and by similar triangles, we have  $mR : RM :: MP : PT$  (see Fig. 3). Bernoulli expresses this in the form  $dy : dx :: y : t$ , a relation that he repeats many times in the *Lectiones*. In his Proposition I on p. 11, L'Hôpital gives it in the form

$$mR(dy) : RM(dx) :: MP(y) : PT = \frac{y dx}{dy}.$$

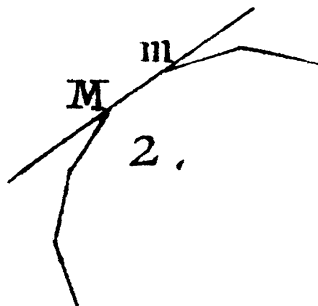


Fig. 2 Postulate II

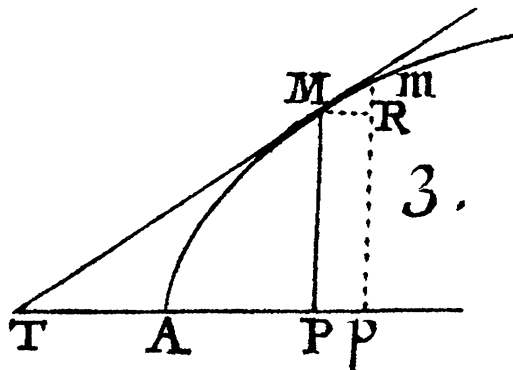


Fig. 3 The Differential Triangle

This is typical of the way that l'Hôpital uses proportional relations: The relation  $a : b :: c : d$  can always be used to solve for  $d$  when  $a$ ,  $b$ , and  $c$  are known, so l'Hôpital puts the corresponding values of the first three terms in parentheses and solves for the fourth term at the end of the expression. In this case, he has found a general formula for the length of the subtangent that can be used freely throughout the remainder of the *Analyse*. Modern readers will note that in the case where  $y = f(x)$ , this is equivalent to saying the length of the subtangent at  $(x_0, y_0)$  is  $\frac{y_0}{f'(x_0)}$ .

In both the *Analyse* and the *Lectiones*, what follows is the calculation of tangents for a large collection of curves that were known to mathematicians in the 1690s. The collection is especially extensive in the *Analyse* and is sort of a catalog of the curves that were known to mathematicians of that time. Some of these curves were already known to the ancient Greeks. L'Hôpital and Bernoulli presented almost all of them geometrically, rather than algebraically. Admittedly, in many cases, algebraic expressions are not possible without the use of transcendental functions.

### Conic Sections

The conic sections – Ellipse, Hyperbola, and Parabola – were known to the ancient Greeks and were still the object of intense study in the 17th century. In fact, l'Hôpital originally conceived of the *Analyse* as a chapter to be included as a part of a larger treatise on conic sections; see letter 56 on p. 285. L'Hôpital's treatise on conic sections did not appear until after his death (L'Hôpital 1707).

We assume that the reader is familiar with the conic sections as they are now treated in analytic geometry. Of course, they were defined by the ancient Greeks as sections of a double-napped cone, but this treatment is not given in the *Analyse*. Instead, l'Hôpital gives them through equations in  $x$  and  $y$  that may be unfamiliar to modern readers. The first mention of the ellipse is in §12 (see p. 13), where it is

given by the equation

$$\frac{ayy}{b} = ax - xx. \quad (1)$$

Rearranging terms and completing the square, this may be written as

$$\frac{\left(x - \frac{a}{2}\right)^2}{\left(\frac{a}{2}\right)^2} + \frac{y^2}{\left(\frac{\sqrt{ab}}{2}\right)^2} = 1,$$

which modern readers will recognize as an ellipse with center  $\left(\frac{a}{2}, 0\right)$ , horizontal axis  $a$  and vertical axis  $\sqrt{ab}$ . Thus, the origin is at the left end of the horizontal axis, rather than the center of the ellipse. When  $b = a$ , this is the equation of a circle of diameter  $a$ , with the origin on the circumference of the circle. Equation (1) reduces in this case to  $yy = ax - xx$ , i.e.  $y^2 = x(a - x)$ , an equation that is used freely throughout the *Analyse*.

Specifying an ellipse with a given center requires two parameters: in the modern treatment, those parameters are the horizontal and vertical semi-axes, whereas in l'Hôpital's treatment, the horizontal axis  $a$  is clearly a natural choice for one parameter. The other parameter,  $b$ , is also a natural choice for someone steeped in the classical theory of proportions. An ellipse is a curve in which the ratio of the rectangle on  $x$  and  $a - x$  to the square on  $y$  is always the same, or  $x(a - x) : y^2 :: a : b$ , for some magnitude  $b$ . Because the magnitude  $a$  is the length of a line segment, the classical theory of proportions demands that  $b$  be a magnitude of the same kind. Thus in Figure 1.4 on page 14,  $a$  and  $b$  are the lengths of the segments  $AB$  and  $AD$ . As in Figure 1 on page xvii for the case of the parabola,  $AD$  is not the  $y$ -axis, but simply a representation of the second parameter of the ellipse.

Equation (1) generalizes naturally to higher order: in the last paragraph of §12, l'Hôpital gives

$$\frac{ay^{m+n}}{b} = x^m(a - x)^n \quad (2)$$

as "the general equation of all ellipses up to infinity." L'Hôpital writes the term  $(a - x)^n$  as  $\overline{a - x}^n$ , because the overline is his way of grouping terms. In analogy to the classical case, the quantities  $x^m(a - x)^n$  and  $y^{m+n}$  here are always in the same proportion as  $a$  is to  $b$ . In §13, l'Hôpital gives

$$\frac{ay^{m+n}}{b} = x^m(a + x)^n \quad (3)$$

as the generalized equation of hyperbolas. Similarly to the ellipse, the quantities  $x^m(a + x)^n$  and  $y^{m+n}$  here are always in the same proportion as  $a$  is to  $b$ . In the

classical case, where  $m = n = 1$ , we may rearrange terms and complete the square to obtain:

$$\frac{\left(x + \frac{a}{2}\right)^2}{\left(\frac{a}{2}\right)^2} - \frac{y^2}{\left(\frac{\sqrt{ab}}{2}\right)^2} = 1,$$

which is a hyperbola with center at  $\left(-\frac{a}{2}, 0\right)$ , so that the origin is once again a point on the curve, in this case the vertex of the right-hand branch. In analogy to the ellipse,  $a$  represents the major or transverse axis of the hyperbola, while the conjugate axis (called the conjugate diameter in the last paragraph of §13) is the quantity  $\sqrt{ab}$ . The major axis of a hyperbola is the distance between its vertices. If one erects a perpendicular to the major axis at a vertex, then the conjugate axis is the segment of that line contained between the points where it intersects the asymptotes of the hyperbola. In Figure 6 on page 15, only the upper half of the right-hand branch of the hyperbola is given, but nevertheless the point  $B$  is the vertex of the left-hand branch and  $a$  is the length of the line segment  $AB$ . The second parameter,  $b$ , is the length of the line segment  $AD$ , not to be confused with  $AE$  the semi-conjugate axis.

L'Hôpital treats parabolas in §11 on page 13. The classical case is  $ax = yy$ , which may strike a modern reader as strange, since it does not express  $y$  as a function of  $x$ , but is actually quite natural in that the axis of symmetry of the parabola is the same as the  $x$ -axis. This parabola has only one parameter, a line segment of length  $a$  so that the square on  $y$  is always equal to the rectangle on  $a$  and  $x$ .

In the third part of §11, l'Hôpital gives the equation of the generalized parabola as  $y^m = x$  when  $m$  denotes a positive rational number. We note that the parameter is suppressed in this expression, but in the particular example of  $m = \frac{3}{2}$ , l'Hôpital writes the equation as  $y^3 = axx$ . Bernoulli writes this *semi-cubical* or second cubic parabola the same way in his Problem I on page 193, and gives the first cubical parabola as  $axx = y^3$ . Similarly, Bernoulli give the *biquadratic* parabolas as  $a^3x = y^4$ ,  $axx = y^4$  and  $ax^3 = y^4$ . In all of these equations, the parameter has the appropriate order so that the equations are all *homogenous*; that is, they involve terms all of the same total degree.

In §11 l'Hôpital also considers the case where  $y^m = x$  and  $m$  denotes a negative rational number. These are the hyperbolas "between the asymptotes"; i.e. hyperbolas where the asymptotes are the axis and its perpendicular. The cases  $ax = xy$  and  $a^3 = xyy$  are both mentioned in this article.

## Construction of Conic Sections

In Proposition IV on page 19, l'Hôpital gives a general construction of a new curve from two given curves that intersect. This is a generalization of a method for constructing the conic sections as the geometric means of the ordinates of two straight lines. In Figure 8, the given curves are  $AQC$  and  $BCN$ , which intersect at the point  $C$ . The ordinates  $PM$  of the new curve  $MC$ , denoted by  $y$ , are defined through





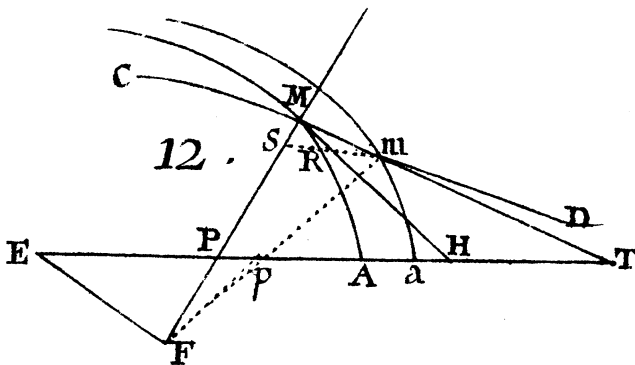


Fig. 12 Hyperbola and Related Examples

hyperbola, because

$$x = \frac{c}{d}u \quad \text{and} \quad z = \frac{c}{a+d}(a+u),$$

so that  $y^{m+n} = x^m z^n$  yields

$$\frac{d^m(a+d)^n}{c^{m+n}} y^{m+n} = u^m(a+u)^n,$$

which has the form of equation (3) with major axis  $a$  and parameter

$$b = \frac{ac^{m+n}}{d^m(a+d)^n}.$$

Finally, we consider the case where one of the straight lines is parallel to the axis. Without loss of generality, the line  $BC$  is replaced by the horizontal line passing through the point  $C$ . Thus  $x = \frac{c}{d}u$  and  $z = c$ , so that  $y^{m+n} = a^n x^m$ , which is a generalized parabola with parameter

$$a = \frac{c^{1+\frac{m}{n}}}{d^{\frac{m}{n}}}.$$

### Alternate Construction of the Hyperbola

In Proposition VII on page 23, l'Hôpital gives a general method for constructing a new curve by sliding a given curve  $MRA$  along the axis. In figure 12, the line  $FP$  rotates about a fixed point  $F$  below the axis. As the line  $FP$  rotates, the curve  $MRA$  moves rigidly in such a way as to keep the length of the segment  $PA$  fixed. The new curve  $CMD$  is then the locus of all points  $M$  where the line  $FP$  meets the curve

*MRA*. In the particular case where the curve *MRA* is actually the straight line *MH*, then *CMD* is a hyperbola. L'Hôpital mentions that his hyperbola has the axis *ET* as one of its asymptotes, but gives no further details.

We give here a derivation of the equation of this hyperbola using modern methods and notation. We suppose *ET* to be the *x*-axis and the perpendicular from the fixed point *F* to the axis to be the *y*-axis. Let *b* denote the distance from *F* to *ET*, so that *F* has coordinates  $(0, -b)$ . If we let  $(z, 0)$  be the coordinates of the point *P*, then *H* has coordinates  $(0, z + a)$  for a fixed *a*. We suppose that the line *MH* has slope *v*, where *v* may be positive, negative, or infinite (i.e., the line *MH* may be vertical). There is no need to consider the case  $v = 0$ , i.e. the case where *MH* is horizontal, because in this case the curve *CMD* coincides with the line *MH*.

Using this notation, the lines *MH* and *FP* have equations

$$y = vx - v(z + a) \quad \text{and} \quad y = \frac{b}{z}x - b,$$

respectively. The point *M* is found by solving these equations simultaneously, giving

$$x = \frac{v(z + a) - b}{vz - b}z \quad \text{and} \quad y = \frac{abv}{vz - b}.$$

We eliminate the parameter *z* by solving the second equation to get

$$z = \frac{ab}{y} + \frac{b}{v}, \quad \text{so that} \quad x = (y + b) \left( \frac{a}{y} + \frac{1}{v} \right).$$

If the line *MH* is vertical, so that  $\frac{1}{v} = 0$ , we have  $y = \frac{ab}{x - a}$ , which gives a hyperbola with asymptotes  $y = 0$  and  $x = a$ . Otherwise, we have

$$x = \frac{y^2 + (av + b)y + abv}{vy},$$

which is a second degree equation in *x* and *y*, whose general form is

$$-vxy + y^2 + (av + b)y + abv = 0.$$

The discriminant of this equation is  $v^2 > 0$ , so the curve is a hyperbola. It is a rotated, skewed hyperbola whose equation can be written as

$$-\frac{v^2}{4}x^2 + \left( -\frac{v}{2}x + \left( y + \frac{av + b}{2} \right) \right)^2 = -\frac{v(av + b)}{2}x + \frac{(av - b)^2}{4}.$$

From this form, we can determine that the asymptotes are  $y = 0$  and  $y = vx$ .

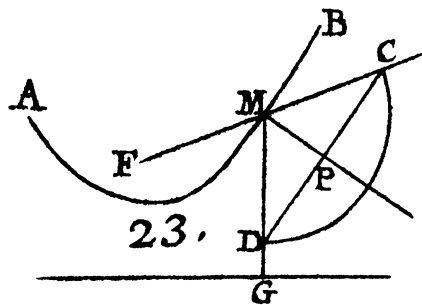


Fig. 23 Conic Sections Defined by Focus and Directrix

### Focus-Directrix Construction of the Conic Sections

In Proposition X on page 29, l'Hôpital considers curves that are defined by means of two or more foci. This is a generalization of the conic section, because ellipses and hyperbolas may be defined by means of two foci: the ellipse is the locus of all points such that the sum of their distances to two given foci is constant, whereas for the hyperbola, it is the difference of the distances that is constant. Furthermore, in this portion of the text that is more than six pages long, l'Hôpital extends to the notion of focus to include both the case of a straight line and that of a curved line. When the focus is a straight line, then we understand the distance between a point and the focus to mean the perpendicular distance from the point to the line. Such a focus is usually called a *directrix*. When the focus is a curved line, then we understand the distance to mean the length of the portion of the tangent from the focal curve to the point on the curve being constructed; see Figure 2.20 on page 33.

In §34 on page 35, l'Hôpital considers curves defined by means of two foci, one being a point and one being a directrix. Modern readers are probably familiar with the focus-directrix definition of the parabola, but l'Hôpital uses the focus and directrix to construct all of the conic sections. In Figure 23,  $F$  is the focus, and the line marked simply as  $G$  is the directrix. For a point  $M$  on the curve being defined, we require that  $MF : MG :: a : b$  for two positive constants  $a$  and  $b$ . L'Hôpital asserts that the curve is a parabola if  $a = b$ , a hyperbola if  $a > b$ , and an ellipse if  $a < b$ . To verify this, we set up a coordinate system as follows. We drop a perpendicular from  $F$  to the directrix  $G$  and make that line the  $y$ -axis. The vertex  $V$  of the curve is the point where the  $y$ -axis intersects the curve. We take  $V$  to be the origin and draw the  $x$ -axis parallel to  $G$ . Because  $V$  is one of the points  $M$  on the curve, we have  $VF : VG :: a : b$ , so without loss of generality, we may take the coordinates of  $F$  and  $G$  to be  $(0, a)$  and  $(0, -b)$ , respectively. The proportional relation  $MF : MG :: a : b$  then becomes

$$\frac{\sqrt{x^2 + (y - a)^2}}{y + b} = \frac{a}{b},$$

so that

$$b^2(x^2 + (y - a)^2) = a^2(y + b)^2.$$

Simplifying, we have

$$b^2x^2 + (b^2 - a^2)y^2 = 2ab(a + b)y.$$

If  $a = b$ , this reduces to  $x^2 = 4ay$ , the parabola with focal distance  $a$ . Otherwise, we have

$$\frac{x^2}{kc^2} + \frac{(y - c)^2}{c^2} = 1,$$

where

$$k = 1 - \frac{a^2}{b^2} \quad \text{and} \quad c = \frac{ab}{b - a}.$$

This is indeed an ellipse if  $a < b$ , so that  $k > 0$ , whereas if  $a > b$ , the coefficient of  $x^2$  is negative and the curve is hyperbola.

## Cycloid

The cycloid is given parametrically in modern textbooks by

$$\begin{aligned} x &= b\theta - a \sin \theta \\ y &= b - a \cos \theta, \end{aligned} \tag{4}$$

where  $a$  and  $b$  are positive constants. When  $a = b$ , the cycloid is called *simple*. The curve can be generated by rolling a circle of radius  $b$  along the  $x$ -axis and tracing out the path of the point on its circumference that is at the origin when  $\theta = 0$ . If  $b > a$ , the cycloid is called *curtate* and the curve is traced out by an interior point of the circle that is in the interval  $(0, b)$  on the  $y$ -axis when  $\theta = 0$ . The cycloid is *prolate* if  $b < a$ , in which case the tracing point is outside the circle, on the negative  $y$ -axis, when  $\theta = 0$ .

The cycloid is an example of a *roulette*, a very general construction of a curve that is produced by rolling one curve, called the *mobile* curve, along a different, fixed curve. In the case of the cycloid, the mobile curve is a circle and the fixed curve is a line. If one rolls a circle around inside another circle, a point on the circumference of the mobile circle traces out a *hypocycloid*. If the mobile circle rolls around the outside of the fixed circle, an *epicycloid* is traced out.

In later chapters of the *Analyse*, l'Hôpital considers cycloids, hypocycloids, and epicycloids as roulettes, which he usually calls "half-roulettes" because he is

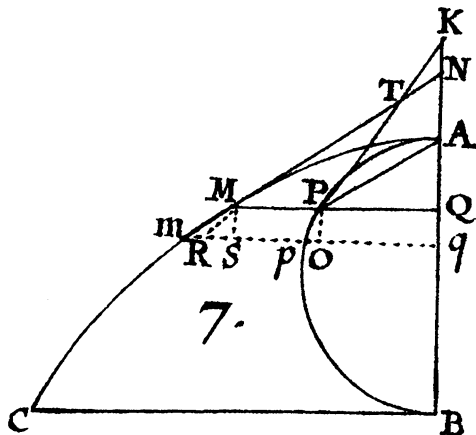


Fig. 7 The Cycloid

frequently only interested in a half-turn of the mobile circle, which in the case of the parametric definition of the cycloid means  $0 \leq \theta \leq \pi$ .

However, in Chapter 2, l'Hôpital does not define the cycloid as a roulette. Rather, in Proposition II on p. 17 he takes the axis to be a curved line  $APB$  (see fig. 7) and the abscissas  $s$  to be given by the arc length on this curve, originating at  $A$ . The ordinates  $t$ , which are perpendicular to the line segment  $AB$ , are applied to the curve  $APB$ . When  $APB$  is a semi-circle and the ordinates are determined by the equation  $s = \frac{at}{b}$  (l'Hôpital uses  $x$  and  $y$ , respectively, where we use  $s$  and  $t$ ), then the curve  $AMC$  a cycloid, simple when  $b = a$ , curtate when  $b > a$  and otherwise prolate. To see that these two definitions are equivalent, first consider the simple cycloid, the case  $a = b$ . Let  $CB$  be the  $x$ -axis, using ordinary Cartesian coordinates, and let the origin be at the point  $C$ . The diameter  $AB$  has length  $b$  and is on the vertical line  $x = \pi b$ . The parametric equations (4) can then be derived by letting  $\theta$  parameterize the angle, in radians, along the semi-circle  $APB$ , with  $\theta = 0$  corresponding to the point  $B$  and  $\theta = \pi$  to the point  $A$ . For the other cases, the semicircle has radius  $a$ , but the center of the diameter  $AB$  is still at the point  $(\pi b, b)$ . In this case, the segment  $CB$  is parallel to the  $x$ -axis, lying above the in the curtate case, or below in the prolate case.

Bernoulli also treated the cycloid (p. 198). However, he only considered the case of the simple cycloid and did not use curvilinear coordinates as l'Hôpital did. Given l'Hôpital's reported interest in the cycloid at the age of 15, it is not surprising that he devoted relatively more of his work to this curve than Bernoulli did.

The cycloid has many fascinating properties, including the fact that it gives the solution to the Brachistochrone Problem. It was also studied by Huygens in his work on the pendulum clock (Huygens 1673). For more on the cycloid, see Lockwood (1971, pp. 80–89). For roulettes, see Lockwood (1971, pp. 138–151).

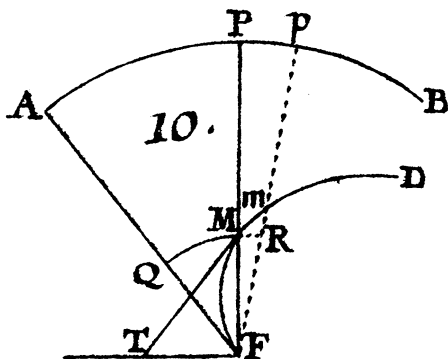


Fig. 10 The Spiral of Archimedes

### Spiral of Archimedes

In modern textbooks the Spiral of Archimedes (or Archimedean Spiral) is usually given in polar coordinates as the function  $r = \theta_0 + k\theta$ , for  $r \geq 0$ , with constants  $\theta_0$  and  $k$ . Thus, the distance from the origin to a point on this spiral is proportional to its angle, measured as the offset from the initial angle  $\theta_0$ . L'Hôpital first mentions this spiral in §23 on page 21, as an example illustrating his Proposition V.

There are no polar coordinates in the *Analyse*, although the variable  $y$ , representing the length of the line segment  $FM$  (see fig. 10) is a polar ordinate, roughly equivalent to the variable  $r$  in polar coordinates. Such polar ordinates are used extensively in Chapter 5. In place of an angular coordinate  $\theta$ , l'Hôpital uses a circular arc  $APB$  as a reference curve; it is essentially a curved axis, as the semi-circle  $APB$  was in the case of the cycloid. The arc  $APB$  is a portion of the circle of radius  $a$  and center at  $F$ , possibly a complete circle. To define the Spiral of Archimedes  $FMD$ , we let  $x$  be the length of the arc  $AP$  and  $b$  the length of the arc  $AB$ . Then the ordinate  $y$  is given by the proportion  $b : x :: a : y$ . In the *Lectiões* (see page 204), Bernoulli describes this proportional relation in words: "That curve is called the Spiral of Archimedes, which is described from a point, which is moved from the center to the circumference of a circle, the radius rotating uniformly in equal durations of time as the point is moved from the center to the circumference."

If we let  $\theta$  and  $\alpha$  be the radian measures of the arcs  $AP$  and  $AB$ , respectively, then we have  $x = a\theta$  and  $b = a\alpha$ , so that

$$y = \frac{a}{b}x = \frac{a}{\alpha}\theta.$$

Because l'Hôpital oriented the arc  $APB$  clockwise, this is essentially equivalent to  $r = \theta_0 + k\theta$ , where  $k = -\frac{a}{\alpha}$  and  $\theta_0$  is the angle between the radius  $FA$  and the positive  $x$ -axis. We note that whereas in the modern treatment, the Spiral of Archimedes is an unbounded curve with arbitrarily large radii, this construction in the *Analyse* describes a bounded portion, which terminates at the point  $B$  on the arc  $APB$ .