Systems & Control: Foundations & Applications

Alexander Poznyak
Andrey Polyakov
Vadim Azhmyakov

Attractive Ellipsoids in Robust Control

Birkhäuser

Birkhäuser

Systems & Control: Foundations & Applications

More information about this series at http://www.springer.com/series/4895

Alexander Poznyak • Andrey Polyakov
Vadim Azhmyakov

Attractive Ellipsoids in Robust Control

Birkhäuser

Alexander Poznyak
Automatic Control Department
Centro de Investigacion y Estudios Avanzados
México, Distrito Federal
Mexico

Andrey Polyakov
Non-A
INRIA-LNE
Villeneuve d'Ascq
Nord, France

Vadim Azhmyakov
Faculty of Electronic and Biomedical Engineering
University of Antonio Nariño
Neiva, HUILA
Colombia

ISSN 2324-9749 ISSN 2324-9757 (electronic)
ISBN 978-3-319-09209-6 ISBN 978-3-319-09210-2 (eBook)
DOI 10.1007/978-3-319-09210-2
Springer Cham Heidelberg New York Dordrecht London

Library of Congress Control Number: 2014946581

Mathematics Subject Classification (2010): 93D09, 93D21

© Springer International Publishing Switzerland 2014
This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed. Exempted from this legal reservation are brief excerpts in connection with reviews or scholarly analysis or material supplied specifically for the purpose of being entered and executed on a computer system, for exclusive use by the purchaser of the work. Duplication of this publication or parts thereof is permitted only under the provisions of the Copyright Law of the Publisher's location, in its current version, and permission for use must always be obtained from Springer. Permissions for use may be obtained through RightsLink at the Copyright Clearance Center. Violations are liable to prosecution under the respective Copyright Law.
The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.
While the advice and information in this book are believed to be true and accurate at the date of publication, neither the authors nor the editors nor the publisher can accept any legal responsibility for any errors or omissions that may be made. The publisher makes no warranty, express or implied, with respect to the material contained herein.

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

To Russia with love

Preface

The material discussed in this monograph is a result of our research program at the Mexican Center for Advanced Studies and Research (CINVESTAV, Mexico City, Mexico) and the Institute of Control Problems, Russian Academy of Sciences (IPU RAN, Moscow, Russia). The main purpose of this book is to provide an advanced account of a newly developed robust control design technique for a wide class of continuous-time dynamical systems. We call the approach under discussion the "Attractive Ellipsoid Method." As the general methodology of the stabilization methods develops, more and more recent results are filtering through to graduate courses. Therefore, our book also contains a coherent introduction to the proposed control design technique and related topics.

We study nonlinearly affine control systems in the presence of uncertainties and are interested in a constructive and easily implementable control strategy that guarantees in a practical sense some stability properties of the closed-loop realizations. In fact, we deal with a linear-type feedback control synthesis in the context of the above-mentioned nonlinear uncertain systems of an affine structure. Throughout, the emphasis is on understanding and the ability to apply the developed theory to examples rather that on rigorous mathematical development. Although there are theorems proved in a systematic way, the level of rigor is not that of a purely mathematical text. Nonetheless, appreciating the limitations of any method is vital, and so we have stated all results in a precise form. The choice of topics has also been influenced by a desire to cover different dynamical systems and consider possible applications. In particular, this monograph contains some illustrative examples and applications of the attractive ellipsoid method to mechanical and electromechanical systems.

We expect that this book will be useful to interested graduate students and advanced undergraduate students with sufficient knowledge of modern systems theory as well as to researchers in the fields of control engineering and applied mathematics. The book can be also considered a complementary text to graduate courses in advanced robust nonlinear control. We also assume that the reader has a rudimentary knowledge of analysis and linear algebra, while a little more is presupposed from the theory of ordinary differential equations and Lyapunov stability theory. We have attempted throughout to provide detailed and transparent proofs of the main results.

Of course, the book claims to present only an introduction to the theory and applications of stable operators in infinite-dimensional spaces. We have made an attempt to unify, simplify, and relate many scattered results in the literature. Some of the topics discussed here are new; others are not. Therefore, the book is not a collection of research papers, but is rather a monograph whose aim is to present recent developments of the theory that might constitute a foundation for further development.

The book contains this preface and twelve chapters and is organized as follows.

- **Chapter 1 (Introduction)** presents motivation and intuitive concepts.
- **Chapter 2 (Mathematical Background)** contains a short collection of necessary mathematical facts from classical analysis and related areas, namely a description of the class of nonlinear uncertain models (Quasi-Lipschitz dynamical systems and examples thereof), differential inclusions and their general solution concept, the Filippov regularization procedure, the Lyapunov approach to Quasi-Lipschitz dynamical systems, elements of linear matrix inequalities (LMIs) including the existence of solutions and some numerical approaches, the duality for LMI-constrained problems, the S-lemma, and more.
- **Chapter 3 (Robust State Feedback Control)** establishes the main concepts of linear (proportional to the current state) feedback design using an S-procedure-based approach, discusses the storage function method providing the boundedness of all possible trajectories of a controlled system from a given class, and presents a technique of minimization of the attractive ellipsoid containing all bounded trajectories. Aspects of practical stabilization are also discussed.
- **Chapter 4 (Robust Output Feedback Control)** is devoted to direct feedback control design and considers two feedback structures:
  - observer-based feedback,
  - full-order linear dynamic controllers,

  and to both of these, the attractive ellipsoid method is applied and analyzed.
- In **Chapter 5 (Control with Sample-Data Measurements),** the main problem is formulated, and some necessary mathematical concepts are discussed related to the feedback control design for nonlinear systems under sample-data output measurements. Then we present a theoretical analysis of an extended version of the invariant ellipsoid method. Then two types of feedback are analyzed:
  - a linear feedback proportional to the current state estimate obtained by a Luenberger-type estimator,
  - a full-order linear dynamic controller governed by a linear ODE with available sample data as input.

Then we construct a minimal attractive ellipsoid that guarantees stability of the system in a practical sense, varying all parameters of the suggested feedbacks. An associated numerical techniques is also presented. Implementable algorithms for the constructive treatment of the robust control design problem are proposed.

- **Chapter 6 (Sample Data and Quantifying Output Control)** considers the analysis and design of an output feedback controller for a perturbed nonlinear system in which the output is sampled and quantized. Using the invariant ellipsoid method, which is based on Lyapunov analysis techniques, together with the relaxation of a nonlinear optimization problem, sufficient conditions for the design of a robust control law are obtained. Since the original conditions result in nonlinear matrix inequalities, a numerical algorithm to obtain the solution is presented. The obtained control ensures that the trajectories of the closed-loop system will converge to a minimal (in a sense to be made specific) ellipsoidal region. Finally, numerical examples are presented in order to illustrate the applicability of the proposed design method.
- **Chapter 7 (Robust Control of Implicit Systems)** focuses on the analysis and synthesis of robust feedback for a class of implicit systems whose state derivatives cannot be expressed analytically as functions of its state coordinates. The transformation to differential–algebraic form is presented, and the attractive ellipsoid method is designed for such systems. The reduction of bilinear matrix inequalities to linear inequalities is presented in detail, and some specific numerical aspects are also discussed.
- **Chapter 8 (Attractive Ellipsoids in Sliding Mode Control)** deals with the minimization-of-unmatched-uncertainties effect in sliding mode control. In particular, LMI-based sliding mode control design is considered, and the optimal sliding surface is constructed. In addition, gain matrix tuning in dynamic actuators is analyzed, and the sliding mode control of time-delay systems with a predictive control is discussed in detail.
- **Chapter 9 (Robust Stabilization of Time-Delay Systems)** considers the class of uncertain time-delay affine-controlled systems whereby a delay is admitted to be in state variables as well as in their derivatives (neutral systems), and shows that the attractive ellipsoid method makes it possible to create feedback that provides the convergence of any state trajectory of the controlled system from a given class to an ellipsoid whose “size” depends on the parameters of the applied feedback. Finally, we present a method for numerical calculation of these parameters providing the “smallest” zone convergence for controlled trajectories.
- **Chapter 10 (Robust Control of Switched Systems)** deals with robust control problems in which a structure of the controlled dynamics may vary in time according to some fixed program strategy or in which the controlled trajectories cross some given surfaces. All nonlinearities of each structure are admitted to be uncertain but to belong a wide class of quasi-Lipschitz functions. The corresponding switching of the applied feedback is shown to be much more effective than a “solid structure” of the feedback providing a smaller convergence zone.

- **Chapter 11 (Bounded Robust Control)** describes the application of the attractive ellipsoid method to controlled systems in which the control actions are a priori bounded, so that a current control at each time is the projection of a linear function of the state or its estimate. This constraint certainly makes the convergence zone a bit larger in comparison to the unconstrained case, but our approach allows us to make this zone as small as possible.
- **Chapter 12 (Attractive Ellipsoid Method with Adaptation)** deals with designing a state estimator and adaptive controller for a class of uncertain nonlinear systems having "quasi-Lipschitz" nonlinearities as well as external perturbations. The set of stabilizing feedback matrices is given by a specific matrix inequality including the characteristic matrix of the attractive ellipsoid, which contains all possible bounded trajectories around the origin. Here we present two modifications of the attractive ellipsoid method that allow us

    – to use online information obtained during the process,
    – to adjust matrix parameters participating in some constraint that characterizes the class of adaptive stabilizing feedbacks.

  The proposed approach guarantees that under a specific persistent excitation condition, the controlled system trajectories converge to an ellipsoid of a "minimal size" having a minimal trace of the corresponding inverse ellipsoidal matrix, which turns out to be significantly smaller than one without adaptation.

Standard notation is used throughout the book. The opening chapter contains a brief collection of necessary mathematical facts that will be useful for a deeper understanding of what follows.

Many individuals have influenced the content and presentation of this book, and we are grateful to all of them. We would like to thank especially Professor A.B. Kurzhanski (Lomonosov State University, Moscow, and University of California, Berkeley) and Professor F.L. Chernousko (Institute for Problems in Mechanics, Russian Academy of Sciences) as the pioneers of this approach. The authors are grateful to all participants of the seminars at the Mexican Center for Advanced Studies and Research (CINVESTAV, Mexico City) and the Institute of Control Problems the Russian Academy of Sciences (IPU RAN, Moscow, Russia) headed by Professor B.T. Polyak. We also would like to express our gratitude to Professor A.P. Kurdyukov (IPU RAN, Moscow) and to Professor S. Mondie (CINVESTAV, Mexico City) for their critical comments and suggestions.

We wish to thank Dr. I Chairez (UPIBI-IPN, Mexico) and Dr. F. Castanos (CINVESTAV, Mexico City) for reading the manuscript and noting both substantive and typographical errors.

Finally, we wish to express our appreciation to the staff of Birkhäuser Science for their accomplished handling of the manuscript as well as their understanding and patience.

México, Mexico — Alexander Poznyak
Nord, France — Andrey Polyakov
Neiva, HUILA, Colombia — Vadim Azhmyakov
2014

Contents

List of Figures

Chapter 1
Introduction

Abstract This introductory chapter briefly reviews the evolution of optimal control design. First, it considers the classical control principles of optimal design for ideal completely known systems. Then, the case of incomplete information is studied. The ideas of robust control design and related optimization issues are discussed. The general principles of ellipsoid-based control design are introduced.

Keywords Optimal control • Robust control • Ellipsoid-based feedback control design

1.1 Complete Information Case: Classical Control Approaches

Optimal control is a rapidly expanding field developed during the last half-century to analyze optimal behavior of a constrained process that evolves in time according to prescribed laws. Its applications now embrace a variety of new disciplines such as economics and production planning. The main supposition of *classical optimal control theory* (OCT) is that the mathematical techniques especially designed for analysis and synthesis of an optimal control of dynamic models are based on the assumption that a designer (or an analyst) possesses *complete information* on the model under consideration as well as on the environment in which this controlled model will evolve.

There exist two principal approaches in solving *optimal control problems* (OCPs) in the presence of complete information on considered dynamic models:

- The first is the *maximum principle* (MP) of Pontryagin (Boltyanski, Gamkrelidze, & Pontryagin 1956).
- The second is the *dynamic programming method* (DPM) of Bellman (Bellman 1956).

Formally, a description of the notion of OCP in its classical form is as follows.

A. Poznyak et al., *Attractive Ellipsoids in Robust Control*, Systems & Control: Foundations & Applications, DOI 10.1007/978-3-319-09210-2_1

1.1.1 System Description

- *Controlled plant dynamics* are given by the following system of ordinary differential equations (ODEs):

$$\left.\begin{aligned} \dot{x}(t) &= f(x(t), u(t), t), \quad \text{a.e. } t \in [0, T], \\ x(0) &= x_{0,} \end{aligned}\right\} \tag{1.1}$$

  where $x = \left(x^1, \ldots, x^n\right)^T \in \mathbb{R}^n$ is its state vector, $u = \left(u^1, \ldots, u^r\right)^T \in \mathbb{R}^r$ is a control, which may run over a given control region $U \subset \mathbb{R}^r$.
- *Cost functional* is defined as

$$J(u(\cdot)) := h_0(x(T)) + \int_{t=0}^{T} h_1(x(t), u(t), t)\, dt. \tag{1.2}$$

  It contains an integral term as well as a terminal term, and the time process or *horizon* T is supposed to be fixed or not and may be finite or infinite.
- *The terminal set* $\mathcal{M} \subseteq \mathbb{R}^n$ is given by the inequalities

$$\mathcal{M} = \{x \in \mathbb{R}^n : g_l(x) \le 0 \quad (l = 1, \ldots, L)\}. \tag{1.3}$$

- The function (1.2) is said to be given in *Bolza form*. If in (1.2), we have

$$h_0(x) = 0,$$

  then we obtain the cost functional in the *Lagrange form*, that is,

$$J(u(\cdot)) = \int_{t=0}^{T} h_1(x(t), u(t), t)\, dt \tag{1.4}$$

  If in (1.2), we have

$$h_1(x, u, t) = 0,$$

  then we obtain the cost functional in *Mayer form*, that is,

$$J(u(\cdot)) = h_0(x(T)). \tag{1.5}$$

Usually, the following assumptions are assumed to be in force:

(A1) (U, d) is a separable metric space (with metric d) and $T > 0$.

(A2) The maps

$$\left.\begin{array}{l} f : \mathbb{R}^n \times U \times [0,T] \to \mathbb{R}^n \\ \\ h_1 : \mathbb{R}^n \times U \times [0,T] \to \mathbb{R} \\ \\ h_0 : \mathbb{R}^n \times U \times [0,T] \to \mathbb{R} \\ \\ g_l : \mathbb{R}^n \to \mathbb{R}^n \ \ (l = 1, \dots, L) \end{array}\right\} \tag{1.6}$$

are measurable, and there exist a constant L and a continuity modulus

$$\bar{\omega} : [0, \infty) \to [0, \infty)$$

such that for

$$\varphi = f(x, u, t), h_1(x, u, t), h_0(x, u, t), g_l(x) \ \ (l = 1, \dots, L),$$

the following inequalities hold:

$$\left.\begin{array}{c} \|\varphi(x, u, t) - \varphi(\hat{x}, \hat{u}, t)\| \leq L \|x - \hat{x}\| + \bar{\omega}(d(u, \hat{u})) \\ \forall t \in [0, T], \ x, \hat{x} \in \mathbb{R}^n, \ u, \hat{u} \in U \\ \\ \|\varphi(0, u, t)\| \leq L \quad \forall u, t \in U \times [0, T] \end{array}\right\} \tag{1.7}$$

(A3) The maps

$$f, h_1, h_0 \text{ and } g_l \ \ (l = 1, \dots, L)$$

are from C^1 in x, and there exists a continuity modulus

$$\bar{\omega} : [0, \infty) \to [0, \infty)$$

such that for

$$\varphi = f(x, u, t), h_1(x, u, t), h_0(x, u, t), g_l(x) \ \ (l = 1, \dots, L),$$

the following inequalities hold:

$$\left.\begin{array}{c} \left\| \frac{\partial}{\partial x}\varphi(x, u, t) - \frac{\partial}{\partial x}\varphi(\hat{x}, \hat{u}, t) \right\| \leq \bar{\omega}(\|x - \hat{x}\| + d(u, \hat{u})) \\ \forall t \in [0, T], \ x, \hat{x} \in \mathbb{R}^n, \ u, \hat{u} \in U \end{array}\right\} \tag{1.8}$$

1.1.2 Feasible and Admissible Control

A function $u(t),\ t_0 \leq t \leq T$, is said to be a *feasible control* if it is measurable and $u(t) \in U$ for all $t \in [0, T]$. Denote the set of all feasible controls by

$$\mathcal{U}[0, T] := \{u(\cdot) : [0, T] \to U \mid u(t) \text{ is measurable}\}. \tag{1.9}$$

The control $u(t),\ t_0 \leq t \leq T$ is also said to be *admissible* or to *realize the terminal condition (1.3)* if the corresponding trajectory $x(t)$ satisfies the terminal condition, that is, if it satisfies the inclusion $x(T) \in \mathcal{M}$. Denote the set of all admissible controls by

$$\mathcal{U}_{admis}[0, T] := \{u(\cdot) : u(\cdot) \in \mathcal{U}[0, T],\ x(T) \in \mathcal{M}\}. \tag{1.10}$$

In view of the theorem on the existence of the solutions to an ODE (see Coddington & Levinson 1955 or Poznyak 2008), it follows that under the assumptions **(A1)–(A2)**, for every $u(t) \in \mathcal{U}[0, T]$ the equation (1.1) admits a unique solution $x(\cdot) := x(\cdot, u(\cdot))$, and the functional (1.2) is well defined.

1.1.3 Problem Setting in the General Bolza Form

Based on the definitions given above, the classical OCP can be formulated as follows.

Problem 1.1 (OCP in Bolza Form).

$$\text{Minimize (1.2) over } \mathcal{U}_{admis}[0, T]. \tag{1.11}$$

Problem 1.2 (OCP with a Fixed Terminal Term). If in the problem (1.11),

$$\begin{gathered}\mathcal{M} = \left\{x_f \in \mathbb{R}^n\right\} = \\ \left\{x \in \mathbb{R}^n : g_1(x) = x - x_f \leq 0,\ g_2(x) = -\left(x - x_f\right) \leq 0\right\} \\ \text{or equivalently, } x = x_f,\end{gathered} \tag{1.12}$$

then it is called an ***OCP*** *with fixed terminal term* x_f.

Every control $u^*(\cdot) \in \mathcal{U}_{admis}[0, T]$ satisfying

$$J\left(u^*(\cdot)\right) = \min_{u(\cdot) \in \mathcal{U}_{admis}[0,T]} J(u(\cdot)) \tag{1.13}$$

is called an *optimal control*; the corresponding state trajectories $x^*(\cdot) := x^*(\cdot, u^*(\cdot))$ and $(x^*(\cdot), u^*(\cdot))$ are called an *optimal state trajectory* and an *optimal pair*.

1.1.4 Specific Features of Classical Optimal Control

The main specific features of both the MP and DPM approaches are as follows:

1. The cost functional defining the quality of the applied control is given in the Bolza form (1.2) containing terminal term as well as the integral term characterizing the cost functional of a designer during all time of the control process.
2. The function f in (1.1) is known a priori and may be used in the control-designing process. In other words, the right-hand side of the dynamic equation (1.1) does not contain any uncertainty or disturbances that are unavailable during the control process.
3. The state vector $x(t)$ is assumed to be available for control designing.

The solution of the classical control problem can be found, for example, in Boltyanski and Poznyak (2012). If one of this three features does not hold, then the classical Optimal Control approach is not applicable.

1.2 Case of Incomplete Information

When we have complete information on a dynamic model to be controlled, the main problem consists in designing an acceptable control that remains "close to the optimal or desired one" (having small sensitivity with respect to every unknown (unpredictable) factor from a given set of possibilities). In other words, the desired control should be *robust* with respect to unknown factors. In the presence of any sort of uncertainty (e.g., parametric type, unmodeled dynamics, external perturbations), the main methodology applied in this book for obtaining a solution suitable for a class of given models is to formulate a corresponding *tracking control* problem, where we are interested in the "best approximation" to a desired trajectory. In other words, we are interested in a zone stabilization or in the practical stability of the deviation of the trajectories of the given system from the desired one. The *robust stabilization problem* considered for different classes of nonlinear systems has been a hot topic of research over the past two decades (Ioannou & Sun 1996; Narendra & Annaswamy 2005; Schweppe 1973; Utkin 1992).

1.2.1 Robust Tracking Problem Formulation

Formally, the robust tracking problem can be described as follows:

- *The controlled plant dynamics* are given by

$$\left.\begin{aligned}&\dot{\bar{x}}(t) = \bar{f}(\bar{x}(t)) + B\bar{u}(t) + \bar{\xi}_x(t), \quad \text{a.e. } t \in [0,T]\\&\bar{x}(0) = \bar{x}_0\\&y(t) = \bar{h}(\bar{x}(t)) + \xi_y(t)\end{aligned}\right\} \tag{1.14}$$

where

$\bar{x} = \left(\bar{x}^1, \ldots, \bar{x}^n\right)^T \in \mathbb{R}^n$ is its state vector.
$\bar{u} = \left(\bar{u}^1, \ldots, \bar{u}^r\right)^T \in \mathbb{R}^r$ is the control to be designed.
$y = \left(y^1, \ldots, y^m\right)^T \in \mathbb{R}^m$ is the measurable output of the system available for a designer at any time $t \geq 0$.

The functions $\bar{\xi}_x(t)$ and $\xi_y(t)$ represent external perturbations that are not measurable (unavailable) for a designer.

- The *desired dynamics* $x^*(t)$ are governed by the following *reference model*:

$$\dot{x}^*(t) = \varphi\left(x^*(t), t\right), \tag{1.15}$$

where $x^* = \left(x^{*1}, \ldots, x^{*n}\right)^T \in \mathbb{R}^n$, and $x^*(t)$ is supposed to be measurable (available) at every time $t \geq 0$. The matrix $B \in \mathbb{R}^{n \times r}$ characterizing the actuator properties is also assumed to be known.

- The tracking error $x(t)$ is defined as

$$x(t) = \bar{x}(t) - x^*(t). \tag{1.16}$$

So the ODE describing the dynamics of the tracking error is

$$\left.\begin{gathered}\dot{x}(t) = f(x(t), t) + B\bar{u}(t) - \varphi(x^*(t), t) + \bar{\xi}_x(t)\\ \text{a.e. } t \in [0,T]\\ y(t) = h(x(t), t) + \xi_y(t)\end{gathered}\right\} \tag{1.17}$$

where

$$\begin{gathered}f(x(t), t) := f(x(t) + x^*(t)), \; x(0) = \bar{x}_0 + x^*(0)\\ h(x(t), t) := \bar{h}(x(t) + x^*(t))\end{gathered} \tag{1.18}$$

- The control action $\bar{u}$ consists of two terms:

$$\bar{u}(t) := u(t) + u_{comp}(t), \tag{1.19}$$

where the *compensating control* $u_{comp}(t)$ is selected in such a way that the effect of the dynamics $\varphi(x^*(t), t)$ of the desired trajectory will be compensated or minimized, namely,

$$\begin{aligned} u_{comp}(t) &= \underset{u_{comp}}{\operatorname{argmin}} \left\| Bu_{comp} - \varphi\left(x^*(t), t\right) \right\|^2 \\ &= B^+ \varphi\left(x^*(t), t\right) \end{aligned} \tag{1.20}$$

where

$$B^+ := \left(B^\intercal B\right)^{-1} B^\intercal$$

if we assume that

$$B^\intercal B > 0.$$

In view of this, the dynamics of the tracking error can be represented as

$$\left.\begin{aligned} \dot{x}(t) &= f\left(x(t), t\right) + Bu(t) + \xi_x(t), \text{ a.e. } t \in [0, T] \\ y(t) &= h\left(x(t), t\right) + \xi_y(t) \end{aligned}\right\} \tag{1.21}$$

where

$$\xi_x(t) := \bar{\xi}_x(t) + \left(BB^+ - I\right)\varphi\left(x^*(t), t\right). \tag{1.22}$$

For the tracking error dynamics (1.21), the following assumptions usually are supposed:

- **(B1)** The dynamic plant (1.21) is *controllable* and *observable* (see, for example, Isidori 1995).
- **(B2)** The functions f and h may be unknown, but they belong to the *given classes* $\mathcal{C}_f$ and $\mathcal{C}_h$ of nonlinear functions, respectively. In this book, both classes are the classes of the *quasi-Lipschitz functions*, whose exact definition is given in the next chapter.
- **(B3)** The unmeasured functions $\xi_x(t)$ and $\xi_y(t)$ are *bounded.*
- **(B4)** The control $u(t)$ is designed as a feedback (static or dynamic) of a given structure containing the set of parameters $\mathcal{P}$, that is,

$$u(t) = u\left(y(\tau)\,|_{0 \le \tau \le t}, t, \mathcal{P}\right), \tag{1.23}$$

so that $u(t)$ depends on all measurable data $y(\tau)$ in the time interval $[0, t]$.

1.2.2 *What Is the Effectiveness of a Designed Control in the Case of Incomplete Information?*

If we have the nonzero terms $\xi_x(t)$ (1.22) and $\xi_y(t)$ (1.21), which are unmeasurable during the control process, then obviously, the application of the classical optimal control approach (as described above) is impossible. The situation looks much more difficult if the functions $f \in \mathcal{C}_f$ and $h \in \mathcal{C}_h$ describing the dynamic process are unknown a priori. In this case, the following questions seem to be important:

> *How can one formulate the problem of control in the uncertain model case, and based on what performances we can estimate the effectiveness of an applied control strategy*?

Several approaches may be considered in this situation.

One suggests that we formulate the corresponding control problem as a *min–max optimal control*, where the *maximum* is taken over all existing uncertainties, and the *minimum* is realized within an admissible control set (see, for example, the H^∞ approach Zhou, Doyle, & Glover 1996 and robust maximum principle Boltyanski & Poznyak 2012). Such min–max consideration work successfully if the set of uncertain terms has a sufficiently simple structure (external perturbations are quadratically integrable, parametric uncertainties are from finite sets or belong to a measurable compact set of a simple nature).

Here we present another approach, referred to below as the *attractive ellipsoid method (AEM)*, which turns out to be workable for a significantly wide spectrum of uncertainties participating in a model description.

1.3 Ellipsoid-Based Feedback Control Design

The main features of AEM are as follows:

- Since in the uncertain case, the optimization of the cost functional [such as (1.2)] cannot be realized exactly because of uncertain factor participation, the control problem is formulated as a *tracking problem*, which equivalently is reduced to the minimization of the vector trajectory $x(t)$ (1.21) by an adequate selection of control strategies $u(t)$.
- The set of considered control strategies is suggested to belong to a *parameterized class of nonlinear (perhaps nonstationary) feedbacks* (1.23)

$$u(t) = u\left(y(\tau)\,|_{0 \le \tau \le t}, t, \mathcal{P}\right),$$

  whose parameters $\mathcal{P}$ are selected in such a way that all possible trajectories $x(t)$ of the closed controlled systems remain bounded and closed to the origin.
- Taking into account that every set of bounded trajectories may be imposed within a convex bounded set, and particularly within an ellipsoid, the AEM suggests that we select the feedback parameters $\mathcal{P} = \mathcal{P}^*$ providing a minimal "size" of this ellipsoid containing all possible bounded trajectories of every dynamical system from the considered class of dynamics containing uncertain elements. In this case, we talk about *zone convergence* or "practical stability" (with a prescribed convex convergence zone) if the size of the convergence zone is of a predetermined value, so that *the effectiveness of such robust control strategies is associated with the "size" of the corresponding attractive ellipsoid set.*