Andrey Popov

Lobachevsky Geometry and Modern Nonlinear Problems

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Translated by Andrei Iacob

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Andrey Popov
Department of Mathematics
Lomonosov Moscow State University
Moscow, Russia

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Introduction

"... the difficulty of concepts increases as they approach the primary truths in nature ..."

N. I. Lobachevsky

Lobachevsky geometry: sources, philosophical significance, and its role in contemporary science

The aim of this book is to reveal the potential of Lobachevsky's geometry in the context of its emergence in various branches of current interest in contemporary science, first and foremost in nonlinear problems of mathematical physics. Looking "geometrically" at a wide circle of problems from the standpoint of Lobachevsky geometry allows one to apply in their study unified approaches that rest upon the methods of non-Euclidean hyperbolic geometry and its highly developed tools.

The discovery of non-Euclidean hyperbolic geometry by the great Russian mathematician Nikolai Ivanovich Lobachevsky, announced by him of the 12th of February, 1826, inaugurated an important historical stage in the development of mathematical thought as an axiomatically impeccably built new field of analytical knowledge. At the foundations of *Lobachevsky's geometry* lies a complete rethinking of the system of axioms of an intuitive geometry and the principles of its construction. Lobachevsky's geometry represented the crowning of attempts, undertaken over many centuries by thinkers of different historical periods, at establishing the correctness of Euclid's geometry that arose already at the dawn of our era.

The system of axioms of the "new" geometry proposed by Lobachevsky differs from the axioms of Euclidean geometry only through the formulation of Postulate V (the Axiom of Parallels). Let us give descriptive formulations of the corresponding variants of the *Axiom of Parallels*.

Euclid's Postulate V: *in a plane, given a line and a point not on it, at most one line parallel to the given line can be drawn through the point.*

Lobachevsky's Axiom of Parallels: *through every point that does not lie on a given straight line there pass at least two distinct straight lines which lie in the same plane as the given straight line, and which do not intersect that straight line.*

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The axioms of Euclid's geometry that were not modified (19 axioms) form the content of the so-called *Absolute Geometry*, a uniqued fundamental component of the classical geometries.

Initially, the realization of what Lobachevsky's ideas mean did run into certain difficulties, the roots of which are in all probability hidden in the primary associative psychological perception of the notions and terminology it uses. For this reason we should mention at the outset that in Lobachevsky's geometry a "straight line" must be understood as a shortest (geodesic) line, i.e., a line along which the distance between any two points on it is minimal. At the same time, the notion of "*parallelism*" of two "straight lines" presumes only that they do not *intersect* and discards the familiar Euclidean property that two parallel straight lines are equidistant (lie at the same distance from one another). Thus, in the new non-Euclidean geometry there arises, it seems, a separation, of holding out classical notions and properties, interpretable "together" in Euclid's geometry.

The conceptual result of Lobachevsky's investigations is that Postulate V (or the Axiom of Parallels) is an independent (self-standing) assertion, which is not logically connected with the other adopted axioms. The possibility of "varying" the formulation of the "Axiom of Parallels" results in the emergence of "independent" geometries (the three known classical geometries: Euclidean, hyperbolic, and spherical). Of these, the hyperbolic geometry constructed by Lobachevsky has the special promising potential demanded by the modern scientific knowledge.

The new geometry, which rests on the introduced system of axioms, was referred to as an "imaginary" geometry already by Lobachevsky himself, who regarded it as a possible "theory of spatial relations".

The subsequent historical development of this theory confirmed objectively its depth and the fundamental prospects of its potential, as well as its definite influence on the development of such domains of knowledge as geometry in general, logic, differential equations, function theory, nonlinear problems of fundamental science, and so on. The path to recognition of the new mathematical theory did run, in particular, through achievements in the geometry of surfaces of negative curvature, the theory of functions of one complex variable, and the theory of partial differential equations.

In modern mathematical physics, the nonlinear modeling of *Lobachevsky geometry* shows up in such attributes of the aforementioned fields of knowledge as *solitons, Bäcklund transformations, pseudospherical surfaces, singularities, attractors, transcendents*, and so on. As it turns out, in the investigation of many actual nonlinear problems one can find a "unifying non-Euclidean common denominator".

Lobachevsky's works on the "theory of parallels" and their influence on the development of geometry

Noting the particular significance of N. I. Lobachevsky's geometric ideas and his contribution to the development of the foundations of the axiomatic structure of mathematical systems, we list below his works that founded the axioms of non-Euclidean hyperbolic geometry. The chronology of their public appearance establishes beyond doubt the priority of N. I. Lobachevsky, and subsequently of the Russian scientific school of geometry, in the development of the concepts of non-Euclidean hyperbolic geometry, in particular, of its relationships with other promising branches of mathematics and fundamental science.

I. "*Exposition succincte des principes de la géométrie avec une demonstration rigoureuse du theorem des paralléles*". *February 12, 1826*.

("*A concise exposition of the principles of geometry with a rigorous proof of the theorem on parallels*".)

This is the first public scientific announcement on the discovery of the new non-Euclidean geometry, made by N. I. Lobachevsky on the 12th of February 1826, as a report at the session of the Physical and Mathematical Section of Kazan university. The manuscript of the report was handed to three professors for safe keeping (however, the manuscript did not survive).

II. "*On the foundations of geometry*" (Russian). *Kazanskii Vestnik, 1829–1830.*

This is a systematic exposition of Lobachevsky's theory of parallel lines, the foundations of a new "imaginary" geometry. The work was published in separate parts over the period from February 1829 to August 1830 in the Kazanskii Vestnik (The Kazan Messenger).

In this study Lobachevsky discusses first how, in his understanding, one has to first establish and then logically develop the primary notions in geometry, and subsequently obtain propositions and theorems. Further, developing these ideas, Lobachevsky provides a systematic treatment (although in compressed form) of the *foundations of the theory of parallel straight lines*, "reaching" in this way the frontiers of analytic geometry: he finds the equations of straight lines and of the most important curves. The final part of the memoir is devoted to effective applications of the imaginary geometry to the calculation of simple and multiple definite integrals. It is precisely in the possible applications of his new theory that Lobachevsky always saw an additional confirmation of its truth and objectivity.

III. "*Imaginary geometry*" (Russian). *Uchenye Zapiski Kazanskogo Universiteta, 1835.*

IV. "*Application of imaginary geometry to certain integrals*" (Russian). *Uchenye Zapiski Kazanskogo Universiteta, 1836.*

In these works Lobachevsky provides a more detailed, and accordingly more accessible exposition of the ideas and results contained in the memoir [II]. In his treatment of the subject, Lobachevsky chooses here the opposite approach: starting from the relations that connect the sides and angles in a triangle in the *imaginary geometry*, he shows that these relations cannot lead to contradictory conclusions. Based on the relations used he obtains geometric properties of triangles and parallel straight lines. He also considers applications of the new geometry to calculus.

Soon after their publications, the works [III] and [IV] were also printed, with minor changes and additions, in French, in the well-known European mathematical

Journal Crelle—"Journal für die reine und angewandte Mathematik"; this made them more accessible to mathematical circles in Europe:

IIIa. "*Géométrie imaginaire*". *Journal für die reine und angewandte Mathematik, 1836.*

IVa. "*Application de la géométrie imaginaire à quelques integrals*". *Journal für die reine und angewandte Mathematik, 1837*

These articles were studied in detail by C. F. Gauss, the most prominent mathematician of the XIXth century, who also came very close to realizing that a *non-Euclidean geometry* exists, calling it in his works *anti-Euclidean*. However, Gauss expressed his high praise of Lobachevsky's results only in his private correspondence with mathematician colleagues.

V. "*New foundations of geometry with a complete theory of parallels*" (Russian). *Uchenye Zapiski Kazanskogo Universiteta, 1835–1838.*

This is the largest work of N. I. Lobachevsky, which sums up in detail, and in the necessary cases develops, the results of his earlier works. It is from this memoir that one can draw the most completely information on the global scientific, world-outlook and philosophical views of this great mathematician.

In this work the fundamental notions of geometry are discussed in detail: adjacency, cuts and the definition of the notion of point connected with them, lines, surfaces, and also the basic theorems on perpendicular straight lines and planes, relations in triangles, linear and angular measures, measuring of areas, and others. Starting from more general fundamental premises (compared with earlier works), a theory of parallel straight lines is constructed in detail. The fundamental equations of the *imaginary geometry* are introduced. As a whole, in this work Lobachevsky establishes the precise axiomatic foundations of geometry and defines the principles of its logical development, accompanying them with the corresponding foundational results in each of the fields he considered.

VI. "*Geometrische Untersuchungen zur Theorie der Parallellinien*". *Berlin, 1840*
("Geometric investigations on the theory of parallels").

The aim of this small, but, as it turned out, rather needed brochure, published in Berlin in 1840, was to present in an intuitive and visual manner all the fundamental ideas and results that constituted Lobachevsky's new non-Euclidean geometry. This aim was achieved; indeed, it is through this publication that the wide mathematical community (and first of all, the European one) was able to become acquainted and accept the ideas of the new geometry.

VII. *"Pangeometry"* (Russian). *Uchenye Zapiski Kazanskogo Imperatorskogo Universiteta (Scientific Memoirs of Kazan Imperial University), 1855.*
(see: "*Pangeometry*", Edited and translated by Athanase Papadopoulos, Heritage of European Mathematics, European Mathematical Society Publishing House, Zürich, 2010.)

This is essentially a summarizing work on geometry, in which Lobachevsky, then already with the experience of a venerable mathematician, did collectively generalize and complete, all the result and ideas stated in his earlier works.

The name "Pangeometry" itself implies and understanding of geometry in its widest sense—as an *all-geometry*, which draws in all known (at that time) geometric representations on space structures.

In 1856 a French translation of this work appeared in a collection of scientific papers prepared for the 50th anniversary of Kazan University:

"*Pangéométrie ou précis de géométrie fondée sur une théorie générale et rigoureuse des parallèles*". *Uchenye Zapiski Kazanskogo Universiteta, 1856.*

The works [I]–[VII] constitute the geometric herritage of the prominent Russian mathematician N. I. Lobachevsky, which allowed to broaden the understanding of the very meaning of geometry as the science of the structure of space and, accordingly, of the principles of its construction and establishment. Lobachevsky's contribution to geometry became a fundament and a kind of standard that make possible the advancement of the mathematical world view as a whole.

In this connection let us mention the special role of the geometric investigations of B. Riemann. In his 1854 lectures "Über die Hypothesen welche der Geometrie zu Grunde liegen" ("On the hypotheses which lie at bases of geometry") Riemann formulated an original idea of mathematical space, the *manifold*, in his terminology. According to Riemann, geometry should be considered as a mathematical theory of continuous manifolds (different collections of homogeneous objects of, generally speaking, different nature). In his investigations Riemann develops a series of results on the intrinsic geometry of surfaces, a branch of geometry founded by C. F. Gauss in his treatise "Disquisitiones generales circa superficies curvas" (1827) ("General investigations of curved surfaces"). Intrisic geometry studies those properties of a surface that are connected with direct measurements *on* the surface. Riemann did effectively apply the notion of linear element, a metric introduced on a *manifold*.

The geometric theory treated by Riemann rests on three conceptual components, namely, the existence of the non-Euclidean *Lobachevsky geometry*, Gauss' achievements in the theory of intrinsic geometry of surfaces, and the notion of multi-dimensional space that took shape in mathematics at that time. An indisputable historical contribution of this research is the introduction of objects that are today known as Riemannian spaces—spaces that are characterized by their own curvature and which generalize our representations about Euclidean spaces, Lobachevsky's non-Euclidean hyperbolic spaces, and the spaces of elliptic geometry studied by Riemann himself. The problem, formulated in Riemann's work, *of understanding the origins of metric properties of spaces* became the harbinger of definite achievements in the general theory of relativity and, as will be shown in this book, remained of actual interest in problems of geometric interpretation of nonlinear differential equations of contemporary mathematical physics.

The fact that Lobachevsky singled out the *axiom of parallels* as an independent, "self-standing" axiomatic assertion showed that a certain revision, a renewed understanding and systematization of the axioms of geometry (axioms that lie at the foundation of absolute geometry), was needed. The solution of this fundamental mathematical problem and, thinking globally, deep philosophical question, was presented by the prominent mathematician David Hilbert at the crossroads of the XIX–XX centuries.

In his 1899 work "The foundations of geometry", Hilbert proposed a complete, separated into groups, system of axioms, which allows one, in the framework of modern geometry, to develop all ensuing "geometric constructions" and obtain the relations that connect them. At the basis of Hilbert's approach is the adoption of three primary systems of "things": "points", "straight lines", and "planes", the elements of which can be in certain relationships, ruled by terms such as "belongs", "are situated", "between", "parallel", "congruent", "continuous", and so on. The meaning of these very "things" (primary geometric "objects"), as well as of the "relations" that connect them, is completely defined by the the logic context of a complete set of stated axioms, divided into five groups: axioms of belonging (or connection, or incidence), axioms of order, axioms of congruence, the axiom of continuity, and the axiom of parallels. A detailed discussion of Hilbert's axiomatics is given in § 1.1.

It is important to note that already Hilbert himself did emphasize that as initial "things" one can take, in principle, elements of any nature, not necessarily rigidly associated with the usual stereotypes of our perception of space. For example, a "straight line" (thing) does not have to be a (Euclidean) straight line, and so on. What matters is that in the system of "things" used the full compatibility of the adopted system of axiomatic statements is preserved. This "geometric vision" of Hilbert harnesses the serious potential of the global understanding of geometry, as well as generalized principles of axiomatic construction of a mathematical theory.

Identifying the primary structural component— bricks—of the space being modeled and prescribing the types of rules that connect them is an initial problem of utmost importance in the process of creating a geometric theory. This is a primary complex of problems, each model solution of which further deepens our knowledge of the structure of real space and builds a "bridge" between Reality and the descriptive formalism that approximates it.

All these foundational problems occupied the thoughts of thinkers in all periods of history. The principles on their rigorous scientific resolution with the aim of building a geometric theory were extremely clearly formulated by the prominent Russian geometer N. V. Efimov: "*Geometry operates with notions that arise from experience as a result of a certain abstraction of the objects of real world, in which one pays attention to only certain properties of real objects*; *in rigorously logic arguments when one proves theorems one deals only with these properties of the objects—hence these properties must be mentioned in axioms and definitions*; *all the other properties, which we got used to imagine when we hear the words* "*point*", "*straight line*", "*plane*", *play no role whatsoever in logical constructions and should not be mentioned in the fundamental statements of geometry*".

Thus, Lobachehvsky's new non-Euclidean geometry became a kind of impulse to rethinking the bases and principles of the construction of modern geometry in general.

Recognition of non-Euclidean hyperbolic geometry and its philosophical significance

The unquestionable *priority* of N. I. Lobachevsky in the *discovery* of non-Euclidean hyperbolic geometry is established by his first public report *"Exposition succincte des principes de la géométrie avec une demonstration rigoureuse du theoreme des parallèles"*, made on the *12th of February, 1826*, at Kazan University. Furthermore, the fundamental contribution of this mathematical genius to the development of analytical foundations of the new geometry, the *Lobachevsky geometry*, is firmly established by a cycle of his scientific treatises, published over the subsequent 30 year period.[1] The first printed work of Lobachevsky was "On the foundations of geometry" (1829–30, [II]), which Lobachevsky himself called an extract from "Exposition".

This work has priority also over the scientific work of the prominent Hungarian mathematician János Bolyai, published in 1831 as an appendix "*Appendix, Scientam spatii absolute veram exhibens* ("Appendix Explaining the Absolutely True Science of Space independent of the truth or falsity of Euclid's axiom XI (which can never be decided a priori)"), which contains his results on the fundamental propositions of non-Euclidean geometry. However, the brilliant independent geometric ideas of János Bolyai were not destined to have significant continuation because of the following life conflict. At the beginning of 1832 Bolyai's work reached C. F. Gauss, who in a letter to his long-standing friend Farkas Bolyai (János' father) communicated that the results he did draw from "it Appendix" where a subject of his thoughts already for a long time and, essentially, were identical with the conclusions that he reached[2], concerning which now he cannot further undertake fast attempts to publication ("*To praise it would amount to praising myself. For the entire content of the work ... coincides almost exactly with my own meditations which have occupied my mind for the past thirty or thirty-five years*"). Later, in a letter to Gerling,[3] Gauss wrote about Bolyai's work: "I consider this young geometer, v. Bolyai, to be a genius of the first class ..." J. Bolyai, however took Gauss' judgement with prejudice, deciding that Gauss intended to take away the priority of his ideas. It is probably precisely J. Bolyai's prejudice to C. F. Gauss that became a kind of barrier to the further in-depth development of the geometric theory that he announced. No longer than a decade after, Bolyai was recognized as one of the prominent geometers of the first half of the XIXth century, and in 1902, with the occasion of the anniversary of 100 years from his birth, a prize bearing his name was established, laureates of which were later geometers like H. Poincaré (1905) and D. Hilbert (1910).

The advancement of the new non-Euclidean hyperbolic geometry is intimately related to the personality of Carl Friedrich Gauss, the greatest German mathematician, whose very deep mind and extraordinary mathematical insight allowed him to immediately understand and accept the objective existence and prospects of the geometry that was taking shape. This is confirmed by the aforementioned opinion-letter to Bolyai and the subsequent comments of this great

[1] Including translations into foreign languages by well-known European publishers.
[2] But not published by Gauss.
[3] V. S. Malakhovsky, "Selected Chapters on the History of Mathematics".

mathematician on the extensive work of N. I. Lobachevsky that he made in his private correspondence with colleagues in mathematics. Unfortunately, for reasons known only to Gauss himself, he did not feel that he could publicly discuss at extent this system of representations on the new non-Euclidean geometry which, beyond any doubt, emerged independently in his thinking, and, probably, was reflected in personal scientific notes, a fact witnessed not only by Gauss' own comments, but also by the the conclusions reached by the historians of mathematics of his time.

The translations of Lobachevsky's works of 1836 "Géométrie imaginaire" ([IIIa]) into French (1840) and "Geometric investigations on the theory of parallels" ([VI]) into German became accessible to Gauss. Becoming acquainted with Lobachevsky's investigations, Gauss expressed careful opinions about them, but only in private correspondence. An example is the following fragment from a letter of Gauss to his astronomer friend H. Schumacher (1846): "*You know that for 54 years now (even since 1792) I have held the same conviction (with some later enrichment, about which I don't want to comment here). I have found in Lobachevsky's work nothing that is new to me. In developing the subject, the author followed a road different from the one I took myself; Lobachevsky carried out the task in a masterly fashion and in a truly geometric spirit. I consider it a duty to call your attention to this work, since I have no doubt that it will give you a tremendous pleasure ...*"

Gauss played a special role in the geometric contributions of Lobachevsky achieving recognition, expressing (in the form of "personal communications") his authoritative opinion on the results about the new non-Euclidean geometry to a sufficiently wide circle of respected scientist of his time. It is due to Gauss' recommendation that in 1842 Lobachevsky was elected corresponding member of the Royal Society of Göttingen.

Thus, we see that at the source of the propositions of the new non-Euclidean geometry in the first half of the XIX-th century stood three giants of mathematics: N. I. Lobachevsky, J. Bolyai, and C. F. Gauss. However, the historical role Lobachevky played in this direction was special, since besides the titanical work at elaborating the new theory, he took upon himself the heavy burden of "adapting" it to the scientific and social communities, which in essence is always a main condition for the strengthening and advancement of any "revolutionary" body of knowledge.

The work of the Italian mathematician E. Beltrami "*Saggio di interpretazione della geometria non-euclidea*" (1868) represented the next stage in strengthening the position of the new non-Euclidean geometry; the results obtained therein allowed to bring Lobachevsky's geometry out of the category of "imaginary geometries" as a geometry that admits its own interpretation (though only partially) in the framework of the habitual Euclidean representations. Beltrami, studying the behavior of geodesics on the surface of the pseudosphere, established that the metric of the pseudosphere is identical in form with the metric of the Lobachevsky plane in a certain domain of it (more precisely, in a horodisc). That is, the conclusion was reached that on the pseudosphere, which is a surface in Euclidean space, there are realized all intrinsic-geometric laws of Lobachevsky's two-dimensional geometry (as applied to the indicated domain).

The final acceptance of non-Euclidean hyperbolic geometry by the scientific community came with the introduction of "virtual Euclidean representations (models)" for the *complete* Lobachevsky plane and is connected with the model interpretations of the two-dimensional Lobachevsky geometry proposed by F. Klein (in 1871) and H. Poincaré (in 1882). The Cayley-Klein model (the Klein model in the disc of the Euclidean plane which uses Cayley's projective metric) and the Poincaré model in the disc and in the half-plane (in the complex plane) are discussed in detail in § 1.2.

Speaking about the coming into life of Lobachevsky's geometry, it is necessary also to mention the works of the Russian mathematician F. Minding during the years 1838–1839 (see § 1.3) in which, in particular, he described all surfaces of revolution of constant negative curvature, namely, the *pseudosphere* and the surfaces know today as the *Minding bobbin* and *Minding top*, and obtained the form of the linear element for surfaces of this type. Interestingly, Minding himself noted the validity of the *formulas of trigonometry* on surfaces of constant negative curvature, *derivable* from the corresponding trigonometric formulas in spherical geometry by replacing the trigonometric functions involved by the "analogous" hyperbolic functions. Beltrami (in 1868) referred to these results of Minding when he analyzed the pseudosphere and emphasized that the aforementioned trigonometric relations are trigonometric formulas in Lobachevsky's geometry. Unfortunately, Minding himself did not pose the problem of connecting his results with the Lobachevsky geometry that was taking shape at that time. And Lobachevsky, by irony of fate, missed those issues of the scientific journal he regularly browsed that contained Minding's works, in which the first intuitive geometrical images of the new non-Euclidean geometry arose. What an extraordinary historical occurrence!

Historians of mathematics should also devote consideration to the personality of J. C. M. Bartels[4] and his "special mission of accompanying and supporting" the creators of contemporary non-Euclidean geometry. Already at the beging of his career of mathematician and pedagogue, Bartels became the teacher of the future king of mathematics C. F. Gauss at the Katherinenschule in Braunschweig. It is due to Bartels' efforts that the young Gauss received from the Duke of Braunschweig a scholarship to continue his education. In the 12-year period of his activity that followed (starting with 1808), Bartels served as a professor at the newly established Kazan University, and according to recollections of his contemporaries, in known situations he watched over and defended his capricious student Nikolai Ivanovich Lobachevsky. Finally, from 1820 on, Bartels taught and engaged in scientific research at the Dorpat (now Tartu) University, where he founded the Centre for Differential Geometry. Afterwards, at the end of the 30th (in the XIXth century), F. Minding, a professor also at Dorpat University, obtained important results on surfaces of revolution of constant negative curvature, on which a partial realization of non-Euclidean hyperbolic geometry takes place. It is amazing that no accounts or results are available that could shed light on Bartels' own about judgements about the new non-Euclidean geometry and on discussion with his colleagues on this theme. However, there is no doubt that under the influence exerted by this prominent mathematician his students acquired a high mathematical

[4] Johann Christian Martin Bartels was a German and Russian mathematician, a corresponding member of the St. Petersburg Academy of Science.

culture. The scale of Bartels' personality is also witnessed by the fact that for his exceptional contribution to science and education he was awarded the (practically inaccessible to scientists) high Russian government title of *secret adviser*.

An outstanding achievement of human thought can become part of the overall intellectual-spiritual heritage only when it reflects, to a certain extent, the demands of the scientific and cultural society of its time. This is equally true for Lobachevsky's geometry, as a theory that expands the boundaries of the mathematical ideas and philosophy of space. Undoubtedly, a historical factor concomitant with Lobachevsky's doctrine was the intellectual society in XIXth century Russia, divided at times by contradictions in world outlook, yet constantly preserving the need for a deep understanding of the meaning of existence, this being a characteristic trait of Russian national mentality.

Overall, in Russia the situation around *Lobachevsky's doctrine* turned out to be rather positive. This is demonstrated by fact Lobachevsky had the opportunity to regularly present parts of the theory he was developing in publications of Kazan University, who he led as rector starting in 1827, and to organize and participate in public debates. Nevertheless, there were also some negative instances, such as academician M. V. Ostrogradsky's rejection of the work "*On the foundations of geometry*", submitted to the Council of Kazan University in the Academy of Sciences. Also, in 1834, F. Bulgarin's well-known literature and general politics journal "Сын отечества" ("Son of the fatherland") published an extensive (anonymous) paper that "ridiculed" in a narrow-minded manner the doctrine of the new non-Euclidean geometry, as well as Lobachevsky himself. At the same time, though, one must speak also about the begining of the penetration of the ideas of the new geometric theory in European scientific circles and, generally, about the rise of the scientific interest in problems related to the direction of research under discussion, a confirmation of which is represented, for instance, by the construction of the "Minding surfaces", and so on.

In Russia, as time went on, something bigger than just recognition as a geometric discovery was awaiting Lobachevsky's theory: the fruits of the scientific investigations of Lobachevsky found reflection in the thinking of the most prominent Russian minds of the XIXth century and became integral part of the discussions of Russian intellectuals in their endless quest for understanding the universe. Opinions on non-Euclidean geometry can be found, for example, in the philosophical polemic of the main heroes of the novel "The Brothers Karamazov" by the great Russian writer and thinker F. M. Dostoyevsky:[5] "Yet there have been and still are geometricians and philosophers, and even some of the most distinguished, who doubt whether the whole universe, or to speak more widely, the whole of being, was only created in Euclid's geometry; they even dare to dream that two parallel lines, which according to Euclid can never meet on earth, may meet somewhere in infinity."[6]

Dostoyevsky, who had a serious basic mathematical education, sensed the subtleties of the circle of problems reached by the mathematical thought at the middle of the XIXth century, composed of questions on the foundation of mathematics, abstract problems of the geometry of space, and so on.

[5]See, e.g., "The Brothers Karamazov", Farrar, Straus and Giroux, 2002.

[6]See the translation at http://fyodordostoevsky.com/etexts/the_brothers_karamazov.txt

One can also speak of the particularly pronounced predisposition of the *Russian national culture* as a whole to comprehend, in particular, the new ideas of non-Euclidean geometry, which, it seems, were laid at the *roots of its civilization*. Important "facets", it would appear, of the unusual non-Euclidean geometry were, at the contemplative level of perception, imprinted in consciousness over a period of almost a thousand year of history of the new civilization in ancient Russia (Русь), which adopted the spiritual Christian principles of Byzantium and introduced in this inheritance the truly Slavic traditions and knowledge, expressed in *special forms* (shapes, images) that are not found in any other culture. Among such *forms*, for example, are the onion-shaped cupolas (domes) that crown the tops of innumerable Russian orthodox churches. The upper part of such an onion cupola, which extends in a harmonious way its central part (a sphere), rising towards the Sky, realizes a shape that from a contemporary analytic point of view belongs to hyperbolic geometry (it is the classical shape of a surface of revolution of negative curvature). It is natural to regard this part of the cupola as a model of a part of the upper sheet of the pseudosphere[7], a canonical surface that tends towards the point at infinity on the Absolute. Such an embodiment of the cupola shape in space can be traced through the Russian orthodox tradition starting from at least the first half of the XIIth century, and signifies the trinity of intuitive geometries accessible to human perception. To wit, starting with the indicated historical period, one can speak with confidence about the appearance of artificial forms that from the contemporary point of view belong to hyperbolic geometry or, in other words, about the results of the precise practical development of elements of non-Euclidean geometry. It is particularly remarkable that all that was mentioned above took place *more than five hundred years* before the discovery by I. Newton, G. Leibniz, and others of the differential and integral calculus, which lies at the foundations of the contemporary scientific and technological paradigm.

Side by side with the aforementioned contribution of N. I. Lobachevsky to the development of a global mathematical conception, we should address also the philosophical value of Lobachevsky's geometry as a theory that influences the development of various fields of knowledge. The general philosophical value of Lobachevsky's geometry can be described as follows. First, this geometric theory had a decisive role in the formation of the analytic conception of possible intuitive geometries (side by side with the Euclidean and spherical geometries) in the Euclidean space habitual to a human being (a passive observer). Figuratively speaking, *Lobachevsky's geometry became the third, crowning crystal in the triad of intuitive geometries*. Second, the geometric theory itself became a *tool* [8](rather than an *aim*, and, the more so, not a "scientific end in itself"), promoting the development of other fields of knowledge that lie at the foundation of contemporary philosophy and practice.

The stable growth in strength of a scientific theory over a long period of history is not possible without devoted followers, prominent scholars capable of developing its fundamental ideas. The author finds his duty to mention here a pleiad of eminent Russian scientists-mathematicians, "guardians of the space of

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[7] Concerning the pseudosphere see § 1.3 and § 2.4.

[8] Already Lobachevsky himself actively applied the theory he developed to the calculation of complicated definite integrals, regarding this as an additional argument in favor of its truth.

Lobachevsky's ideas", the names of which are connected with the advancement and popularization of Lobachevsky's geometric doctrine in Russia and abroad over the last, more than 150 years. Here we should mention, among others, A. V. Vasil′ev,[9] A. P. Kotel′nikov, P. A. Shirokov, B. L. Laptev, A. P. Norden, V. F. Kagan, Yu. Yu. Nut, A. S. Smogorzhevskii, N. V. Efimov, and È. G. Poznyak. Special contributions to the development of Lobachevsky's geometry and its applications are due to the scientific geometrical schools of Kazan and Moscow universities.

Structure and contents of the book

The exposition in the book begins with the consideration of the key elements lying at the foundation of Lobachevsky's geometry, including its interpretations (models), and is carried out in a form adapted to the methods of modern geometry, function theory, and the theory of nonlinear differential equations. The central part of the book is devoted to problems connected with various aspects of the realization of hyperbolic geometry in Euclidean space, the study of pseudospherical surfaces, and the elaboration of effective geometric approaches to the study of certain nonlinear partial differential equations of mathematical physics, in particular, in the context of physical applications. The main text is organized into five chapters, preceded by the Introduction.

The first chapter is devoted to the foundations of Lobachevsky's geometry, consisting of three basic "ingredients": axiomatics, model interpretations, and the analysis of surfaces of revolution of constant negative curvature. These sections are structured with a view to the subsequent applications of the results presented in actual problems of mathematical physics. We also consider examples of C^1-regular surfaces of revolution with different signs of the curvature, which realize a harmonious combination of the classical intuitive geometries.

The second chapter deals with general problems connected with the realization of the two-dimensional Lobachevsky geometry in the three-dimensional Euclidean space $\mathbb{E}^3$. Here it is natural to interpret the Lobachevsky geometry as geometry of a two-dimensional Riemannian manifold of constant negative curvature. In this connection we introduce the fundamental systems of equations of the theory of surfaces in $\mathbb{E}^3$ and discuss specific features of the application of the tools presented to the analysis of surfaces of constant negative Gaussian curvature. In this chapter we consider such canonical geometric objects as the Beltrami pseudosphere and Chebyshev nets. We also examine D. Hilbert's results on the impossibility of realizing the complete Lobachevsky plane in the space $\mathbb{E}^3$. We mention the fundamental connection between surfaces of pseudospherical type and the sine-Gordon equation, a geometrically universal nonlinear partial differential equation. We give a brief survey of a number of fundamental results on isometric immersions of Riemannian metrics of negative curvature in Euclidean spaces.

The third chapter is devoted to geometric aspects of the sine-Gordon equation. We study the geometric notion of Bäcklund transformation for pseudospher-

[9] Special mention is due to the scientific-biographical works of A. V. Vasil′ev [13], which provide a detailed exposition of the life path of N. I. Lobachevsky and an analysis of his scientific achievements.

ical surfaces. At the same time we remark that the application of the method of Bäcklund transformations for the construction of exact solutions of nonlinear differential equations is one of the most effective approaches in mathematical physics. Special attention is given to the class of soliton solutions of the sine-Gordon equation and their geometric interpretation on the example of classical surfaces—the pseudosphere and the Dini surface—as well as to the study of the classes of two-soliton and breather pseudospherical surfaces. We investigate the Painlevé transcendental functions of the third kind, which form a special class of self-similar solutions of the sine-Gordon equation, the geometric interpretation of which in $\mathbb{E}^3$ is provided by Amsler's pseudospherical surface. Further, we study fundamental solvability questions for certain classical problems of mathematical physics, namely, the Darboux problem and the Cauchy problem for the sine-Gordon equation; we then use the results obtained to derive important geometric generalizations and consequences. In particular, we show how to construct solutions of the sine-Gordon equation on multi-sheeted surfaces. Moreover, based on the unique solvability of the Cauchy problem for the sine-Gordon equation presented in this chapter we prove a theorem on the *unique determinacy* of pseudospherical surfaces (the fact that a pseudospherical surface is uniquely determined by the corresponding initial data on its irregular singularities). We discuss classical questions connected with the Joachimsthal-Enneper surfaces, indicating the connection between these surfaces and classes of solutions of the sine-Gordon equation obtained by the method of separation of variables. The final section of the chapter is devoted to the fundamental connection that exists between the method of the inverse scattering transform and the theory of pseudospherical surfaces. This connection is expressed by the fact that the basic relations that arise in these two different branches of mathematics are structurally identical. On the whole, all the essential questions considered in Chapter 3 point to the presence of a significant geometric component connected with Lobachevsky's geometry in a wide spectrum of problems of topical problems of mathematical physics.

In Chapter 4 we present a geometric approach to the interpretation of certain nonlinear partial differential equations which connects them with special coordinate nets on the Lobachevsky plane Λ^2. We introduce the notion of the Lobachevsky class of partial differential equations (Λ^2-class), equations that admit the aforementioned interpretation. The resulting geometric concepts for nonlinear equations allow one to apply in their study the well developed tools and methods of non-Euclidean hyperbolic geometry. Many well-known nonlinear equations, among them the sine-Gordon, Korteweg-de Vries, Burgers, and Liouville equations, etc., which compose the Λ^2 class, are generated by their own coordinate nets on the Lobachevsky plane Λ^2. This makes it possible to investigate these equations by net (intrinsic-geometrical) methods that rest on Lobachevsky's geometry. Overall, the chapter is devoted to laying the foundations of the geometric concept of Λ^2-equations; in it we also discuss the prospects of applying geometric methods of hyperbolic geometry to the constructive analysis of differential equations.

In Chapter 5 we consider applications of the geometric formalism proposed in Chapter 4 for nonlinear differential equations to problems of theoretical physics and the theory of difference methods for the numerical integration of differential equations.

In the first part of the chapter we introduce the notion of *non-Euclidean phase spaces*, which are nonlinear analogs (with non-zero curvature) of the phase spaces of classical mechanics and statistical physics, and of the Minkowski space of the special theory of relativity.

The concept of non-Euclidean phase spaces rests on the principle of identity between the metric of the phase space and the metric generated by the model equation that describes the physical process under investigation. Due to the non-triviality of the curvature of non-Euclidean phase spaces, they exhibit singularities, which acquire the physical meaning of attractors and determine the behavior of regular phase trajectories. Non-Euclidean phase spaces represent a kind of "curvilinear (non-Euclidean) projection screens" on which the evolution of the physical process under consideration is displayed in regular manner. This in turns leads to the establishment of general principles governing the evolution of the corresponding physical systems. By the nature of the approaches employed, the material discussed belongs first and foremost to the methodology of mathematical physics.

In the second part of the chapter, based on the elaborated methodology of discrete coordinate nets on the Lobachevsky plane, we propose a geometric algorithm for the numerical integration of Λ^2-equations. The realization of such an approach is connected exclusively with the planimetric analysis (in the framework of hyperbolic geometry) of piecewise-geodesic discrete nets in the plane Λ^2 which in the limit go over into the smooth coordinate net that generated the Λ^2-equation under study. The implementation of the method is demonstrated on the example of the sine-Gordon equation; to construct the geometric algorithm for its numerical integration, one needs to study discrete rhombic Chebyshev nest on the plane Λ^2.

In the framework of the general geometric approach, the present monograph covers a rather wide spectrum of problems, starting with problems on the foundation of geometry and ending with methods for the integration of nonlinear partial differential equations of mathematical physics and the formulation of a number of general principles governing the evolution of physical systems. In the author's view, making such a diverse material accessible to the reader was possible only by varying the level of rigorousity of the exposition so that it reflects in each individual problem considered the established traditions and methodology of study.

The book is addressed to a wide circle of specialists in various fields of mathematics, physics, and science in general.

Chapter 1

Foundations of Lobachevsky geometry: axiomatics, models, images in Euclidean space

This first chapter is devoted to an exposition of the foundations of Lobachevsky geometry, formed by three classical components: axiomatics, model interpretations, and investigation of surfaces of constant negative curvature. The discussion of these parts is carried out keeping in mind what is required for their application to problems of contemporary mathematical physics.

1.1 Introduction to axiomatics

Constructing the foundations of geometry amounts to establishing a complete and, at the same time, sufficiently simple and consistent system of axioms (statements, the truth of which is accepted without proof), and the derivation from them, as logical consequences, of the key theorems of geometry. The principal requirements for the system of axioms are *completeness, minimality* of the collection of assertions involved, and their *consistency*. In this section we present, following the universally recognized work of D. Hilbert [17], the axiomatics adopted in modern geometry.

Hilbert's axiomatics starts by introducing three different systems of "things", primary geometric objects. The things of the first system are called *points*, those of the second (*straight*) *lines*, and those of the third, *planes*. The *points* are the elements of *linear geometry*; the *points and lines* are the elements of *plane geometry*; and the *points, lines, and planes* are the elements of *space geometry*. It is assumed that the *points, lines, and planes* are in certain *relations*, which are referred to by the words *"lies"*, "*between*", "*congruent*" (equal), "*parallel*", *continuous"*, and so on. The precise meaning of the terms that express *relationships* is specified by the content of the corresponding (groups of) axioms of geometry.

Let us list the axioms of geometry, dividing them into five groups.

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I. Axioms of belonging (or of incidence) (8 axioms).

II. Axioms of order (4 axioms).

III. Axioms of congruence (equality) (5 axioms).

IV. Axioms of continuity (2 axioms).

V. Axiom of parallels.

The axioms of groups I–IV (19 axioms) are shared by Euclidean geometry as well as by Lobachevsky geometry, and constitute the axiomatics of *Absolute Geometry*. Adding to them the Axiom of Parallels results in the complete system of axioms of either Euclidean geometry, or of Lobachevsky geometry. Let us now formulate the axioms, remarking that usage in axioms of the plural for geometric objects presumes that these objects are distinct (e.g., "two points" means "two distinct points").

Axioms

I. Axioms of belonging (incidence)

1. *For any two points A and B there exists a straight line a that passes through each of the points A and B* (Figure 1.1.1).

Figure 1.1.1

2. *For any two points A and B there exists no more than one straight line that passes through both A and B* (Figure 1.1.1).
3. *On each straight line there exist at least two points. There exist at least three points that do not lie on the same straight line.*
4. *For any three points A, B, C that do not lie on the same straight line there exists a plane α that contains each of the points A, B, C. For every plane there always exists a point which it contains.*
5. *For any three points A, B, C that do not lie on one and the same straight line there exists no more than one plane that passes through each of these three points.*
6. *If two points A, B of a straight line a lie in a plane α, then every point of a lies in the plane α.*

   (In this case one says: "the straight line a lies in the plane α", or "the plane α passes through the straight line a".)
7. *If two planes α and β have a point A in common, then they have at least one more point B in common.*

8. *There exist at least four points that do not lie in one plane.*

Following Hilbert [17] and the later classical works of V. F. Kagan [38] and N. V. Efimov [25] on the foundations of geometry, in the formulation of axioms we, while taking care to preserve the correct statement of the axioms, used for the terms expressing the relations between "things" the corresponding notions that are more customary in modern mathematics. Incidentally, these mathematical "synonyms" were given already by Hilbert himself. Thus, for example in Axioms I.1, I.2 we used: "*the line a passes through the points A and B*" instead of the equivalent "in meaning" as well as admissible formulation "*the straight line a is incident to each of the points A and B*". Furthermore, for example, instead of the possible statement "*the point A lies on the straight line a*" one used "*the point A is incident to the straight line a*". Also, expressions "*the straight lines a and b intersect in the point A*" and "*the straight lines a and b have a common point*" are equivalent, and so on.

Commenting upon the eight axioms of group I, which Hilbert referred to as axioms of *incidence*, let us point out that their "diversity of meaning" is deep and is aimed at optimizing the approach by which one derives their consequences. As an example, consider the first two axioms I.1 and I2, which in the standard modern courses on mathematics are replaced by a single (more "content-loaded") axiom: "*through any two distinct points there always passes a unique straight line*". This last formulation is undoubtedly correct, but to derive further geometric consequences, one in fact does not employ its full "meaning capacity"; rather, it is only applied partially, in accordance with the content of axioms I.1 and I.2.

Based on just the axioms I.1–I.8 of the first group, one can now, for example, prove the following theorems [17,25]:

Theorem 1. *Two straight lines that lie in one and the same plane have no more than one common point. Two planes either have no point in common, or they have a common straight line, on which all the common points of the two planes lie. A plane and a straight line that does not lie on it either have no common point, or have only one common point.*

Theorem 2. *Through a straight line and a point that does not lie on that straight line, as well as through two distinct straight lines with a common point, there always passes one and only one plane.*

II. Axioms of order

1. *If the point B lies between the point A and the point C, then A, B and C are distinct points of one straight line, and the point B also lies between the point C and the point A* (Figure 1.1.2)*.*

2. *For any two points A and C, on the straight line AC there exists at least one point B such that the point C lies between the point A and the point B* (Figure 1.1.3)*.*

3. *Of any three points on a straight line there exists no more than one that lies between the other two.*

Figure 1.1.2

Figure 1.1.3

Definition. On a straight line a consider two points A and B; the system of two points A and B is called a *segment* and is denoted by AB or BA. The points lying between A and B are called *points of the segment* AB (or *interior* points of the segment); the points A and B themselves are called the *endpoints* of the segment AB. All the remaining points of the line a are called the *external* points of the segment AB.

4. **Pasch's Axiom.** *Let A, B, and C be three points that do not lie on a straight line, and let a be a straight line in the plane ABC that does not pass through any of the points A, B, C. If the straight line a passes through one of the points of the segment AB, then it necessarily passes also either through a point of the segment AC, or through a point of the segment BC.* (Figure 1.1.4)..

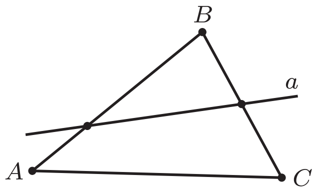

Figure 1.1.4

By their essence, the Axiom of group II define the notion of *bewteen*.

Based on this one can introduce an *order* for points on a straight line, in plane, or in space. The axioms of order were study in detail by the German mathematician M. Pasch [181].

The addition of the axioms of Group II to the axioms that we already considered allows one to obtain many important consequences [17, 25], among which we mention here the following examples:

Theorem 1. *Among any three points A, B, C lying on the same straight line there is one that lies between the two other.*

Theorem 2. *Between any two point of a straight line there exists infinitely many other points of the straight line.*

Theorem 3. *If the points C and D lie between the points A and B, then all the points of the segment CD belong to the segment AB.*

III. Axioms of congruence (equality)

The axioms of Group III will be formulated and commented upon simultaneously. This approach will allow the reader from the very beginning to follow the logic of the development of the content of the axiomatic statements of this group. The axioms in Group III define the notion of *congruence* (*equality*), and accordingly allow one to introduce the notion of *motion*.

To designate certain mutual relations that can hold between segments we use the term "*congruent*" (or "*equal*"). This kind of relation between segments is described by the axioms of Group III.

1. *If A and B are two points on the straight line a and A_1 is a point on another straight line a', then it is always possible to find a point B_1 on a given side of the straight line a' such that the segments AB and A_1B_1 are congruent* (*equal*) (Figure 1.1.5).

In particular, for the straight line a' one can also take the straight line a itself.

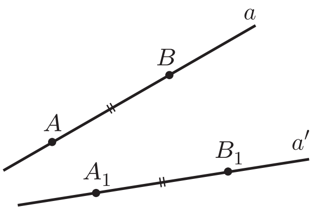

Figure 1.1.5

The congruence (equality) of the segments AB and A_1B_1 will be denoted by

$$AB = A_1B_1.$$

Axiom III.1 allows one to superpose equal segments.

We note that, according to the definition given above, a segment is given as a *system of two points A and B*, with nothing being said about the order in which they are positioned. Consequently, the following relations are equivalent in meaning:

$$AB = A_1B_1, \quad AB = B_1A_1, \quad BA = A_1B_1, \quad BA = B_1A_1.$$

2. *If both the segment A_1B_1 and the segment A_2B_2 are congruent to the segment AB, then the segments A_1B_1 and A_2B_2 are also congruent to each other.*

In other words, if two segments are congruent to a third segment, then they are congruent to one another.

3. *Let AB and BC be two segments on a straight line a that have no common interior points, and let A_1B_1 and B_1C_1 be two segments on a straight line a' that also have no common interior points. If*

$$AB = A_1B_1, \quad BC = B_1C_1,$$

then

$$AC = A_1C_1.$$

Axiom III.3 allows one to *add segments*.

To formulate the next two axioms of Group II we need to introduce the notion of an angle.

Definition A pair of half-lines (a system of two rays) ℓ and k that originate at one and the same point O and do not belong to one straight line is called an *angle*, and is denoted by $\angle(\ell, k)$ (Figure 1.1.6).

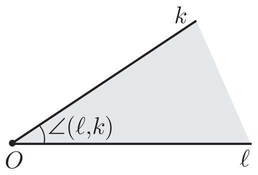

Figure 1.1.6

By *half-line* (or *ray*) with the origin at the point O one means the set of all points on a straight line that lie on the same side with respect to the point O. The rays ℓ and k are called the *sides* of the angle $\angle(\ell, k)$.

Angles can find themselves in a certain relation, termed "congruence" or equality, which is "governed" by Axioms III.4 and III.5.

4. *Suppose that on the plane α there is given an angle $\angle(\ell, k)$ and there is given a straight line a' in the same plane or some other plane α', and also that a side of the plane α' with respect to the straight line a' is chosen. Let ℓ' be a ray of the straight line a' starting from a point O'. Then in the plane α' there exists one and only one ray k' such that the angle $\angle(\ell', k')$ is congruent to the angle $\angle(\ell, k)$, and at the same time all interior points of the angle $\angle(\ell', k')$ lie on the chosen side with respect to the straight line a'.*

The congruence (equality) of angles is denoted by

$$\angle(\ell, k) = \angle(\ell', k').$$

Each angle is congruent to itself:

$$\angle(\ell, k) = \angle(k, \ell).$$

Axiom III.4 alows one to lay out angles: each angle can be placed, in a unique way, in a given plane, on a given side with respect to a given ray.

Before we formulate Axiom III.5 (the final axiom of Group III), let us clarify the notion of a triangle. By a *triangle* $\triangle ABC$ we mean a system of three segments, AB, BC, CA, which are called the *sides* of the triangle; the points A, B, C are called the *vertices* of the triangle.

5. *If for two triangles, $\triangle ABC$ and $\triangle A_1B_1C_1$, it holds that*

$$AB = A_1B_1, \quad AC = A_1C_1, \quad \angle BAC = \angle B_1A_1C_1,$$

then there also holds the equality (congruence)

$$\angle ABC = \angle A_1B_1C_1.$$

Remark. $\angle ABC$ denotes the angle with vertex B, on one side of which lies the point A, and on the other, the point C.

The first three axioms III.1–III.3 are *linear* axioms, because they concern only congruence of segments. Axiom III.4 defines the congruence of angles. Axiom III.5 connects congruence of segments as well as of angles. The last two axioms of Group III may be referred to as plane axioms, since they are assertions on geometric "objects" in the plane.

Using the axioms of Group III one introduces in geometry the notion of *motion*, as follows.

Consider two sets, Σ and Σ', between the points of which there is a one-to-one correspondence. (By set we mean a finite or infinite collection of points.) Any two points $A, B \in \Sigma$ define a segment AB, and the points $A', B' \in \Sigma'$ corresponding to them give a segment $A'B'$; we will say that the segments AB and $A'B'$ *correspond* to one another. If under the given one-to-one correspondence between Σ and Σ' any two corresponding segments are equal (congruent), then the *sets* Σ *and* Σ' and also said to be *equal* (congruent) In this case one say that the set Σ' is obtained by a *motion* of the set Σ, and conversely, Σ is obtained by a motion of Σ'.

The completion of the axiomatic system discussed by the axioms of congruence (Group III) makes it possible to obtain new wide classes of geometric consequences, which are considered in detail in, e.g., [25, 38].

IV. Axioms of continuity

1. **Archimedes' Axiom.** *Let AB and CD be two arbitrary segments; then on the straight line AB there exist a finite number of successively arranged points $A_1, A_2, A_3, \dots, A_n$ such that the segments $AA_1, A_1A_2, A_2A_3, \dots, A_{n-1}A_n$ are congruent to the segment CD and the point B lies between the points A and A_n (Figure 1.1.7).*

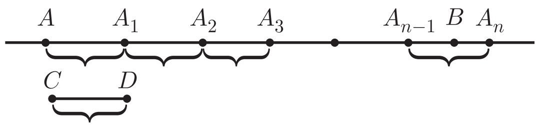

Figure 1.1.7

Axiom IV.1 is also called the *axiom of measure*. According to its meaning, the segment CD is a standard-of-length segment, a measurement *unit*, and the axiom asserts that it is possible "reach" any given point on a straight line and calculate the length of any segment.

2. **Cantor's Axiom.** *Suppose that on some straight line a there is an infinite system of segments $A_1B_1, A_2B_2, \dots, A_nB_n, \dots$, in which each successive segment is contained inside the preceding segment (Figure 1.1.8). Suppose that there is no segment that is contained inside all the segments of the given infinite system of segments. Then on the line a there exists a unique point M that lies inside all the segments $A_1B_1, A_2B_2, \dots, A_nB_n, \dots$ of the considered system.*