Richard H. Cushman
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Global Aspects of Classical Integrable Systems

Second Edition

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Foreword

This book gives a complete global geometric description of the motion of the two dimensional harmonic oscillator, the Kepler problem, the Euler top, the spherical pendulum and the Lagrange top. These classical integrable Hamiltonian systems one sees treated in almost every physics book on classical mechanics. So why is this book necessary? The answer is that the standard treatments are not complete. For instance in physics books one cannot see the monodromy in the spherical pendulum from its explicit solution in terms of elliptic functions nor can one read off from the explicit solution the fact that a tennis racket makes a near half twist when it is tossed so as to spin nearly about its intermediate axis. Modern mathematics books on mechanics do not use the symplectic geometric tools they develop to treat the qualitative features of these problems either. One reason for this is that their basic tool for removing symmetries of Hamiltonian systems, called regular reduction, is not general enough to handle removal of the symmetries which occur in the spherical pendulum or in the Lagrange top. For these symmetries one needs singular reduction. Another reason is that the obstructions to making local action angle coordinates global such as monodromy were not known when these works were written.

The point of view adopted in this book is to start with a somewhat unfamiliar abstract mathematical model of the physical system such as the study of the geodesic flow of a left invariant metric on the three dimensional rotation group. Using the symplectic geometric formulation of Hamiltonian mechanics we then show that the equations of motion agree with those found by more traditional methods for a well known physical system, namely, the force free rigid body or Euler top. This justifies our mathematical model. We do not try to build our model from fundamental physical principles. We have not written a book on mechanics or Hamiltonian particle dynamics. We only discuss five special integrable systems, which is a very small sample of the rich variety of general Hamiltonian systems. Moreover the behavior of the solutions of these integrable systems is much more regular than the nearly unpredictable motion of a general Hamiltonian system such as the three body problem.

Our main goal is to understand the global geometric features of our model integrable systems. The main tool we use is reduction to remove the symmetries and to obtain a system with one degree of freedom. This allows us to determine the range and the topology of every fiber of the energy momentum mapping of the system. The topology of a fiber corresponding to a singular value of the energy momentum mapping is of great interest. Physically, these motions are simpler than the general motion and therefore are easier to study experimentally. Mathematically, these fibers contain a relative equilibrium of the system, that is, a motion which is also an orbit of the symmetry group. For instance, in the spherical pendulum the relative equilibria are circular orbits on the 2-sphere which

lie in a plane parallel to and below the equator. Other examples are the regular precession and sleeping motions of the Lagrange top. Finally, to complete the qualitative picture, we describe how the fibers of the energy momentum map fit together. Sometimes this involves showing that the monodromy of certain torus bundles are nontrivial. That this phenomenon happens in the spherical pendulum and the Lagrange top was not known until the 1980s.

This book is written from a bottom up approach with examples being given prominence over theory. The examples are treated in a uniform way. First the mathematical model is described and then the equations of motion are derived. Next the symmetries and corresponding integrals are obtained and it is shown that the given problem is Liouville integrable. Finally, the geometry of the level sets of the energy momentum map, which gives a complete geometric description of the motion, are obtained by first using reduction to remove the symmetries and then reconstructing the geometry from the geometry of the reduced system. This program may seem to be excessively lengthy. There are two reasons why we have followed it. First, our procedure gives complete answers, whereas short cut ones do not. Second, in carrying out our program the reader sees enough detail in the text to be able to understand the arguments without having to look at the theory. The theory given in chapters VI through XI is what the authors feel is the minimum necessary to justify all the unproved assertions in the examples.

This book was not written to be read in a sequential fashion. We strongly encourage the reader to browse.

Introduction

The mathematical pendulum

We begin by looking at the mathematical pendulum.

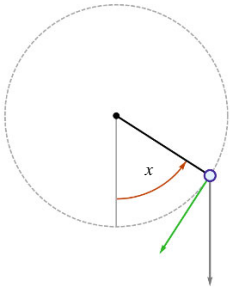

Figure 0.1. The mathematical pendulum.

Let $T^*\mathbf{R}$ be the *cotangent bundle* of $\mathbf{R}$, which we identify with $\mathbf{R}^2$ and give coordinates (x,y). The canonical symplectic form $\omega = \mathrm{d}x \wedge \mathrm{d}y$ on $T^*\mathbf{R}$ is the element of oriented area on $\mathbf{R}^2$. Consider the *Hamiltonian system* $(H, T^*\mathbf{R}, \omega)$ with *Hamiltonian*

$$H : T^*\mathbf{R} \to \mathbf{R} : (x,y) \mapsto \tfrac{1}{2}y^2 - \cos x.$$

▷ The following argument shows that the *Hamiltonian vector field* X_H on $T^*\mathbf{R}$ corresponding to the Hamiltonian H is

$$X_H(x,y) = \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} = y\frac{\partial}{\partial x} - \sin x\frac{\partial}{\partial y}. \tag{1}$$

(0.1) **Proof**: By definition of Hamiltonian vector field, see appendix A §3,

$$\mathrm{d}H(p)z_p = \omega(p)(X_H(p), z_p) \tag{2}$$

for every $z_p = (v_p, w_p)$ in the tangent space $T_p(T^*\mathbf{R})$ to $T^*\mathbf{R}$ at p. Let $X_H(p) = \big(X(p), Y(p)\big)$. Now $\mathrm{d}H(p)z_p = \frac{\partial H}{\partial x}v_p + \frac{\partial H}{\partial y}w_p$. Moreover, $\omega(p)(X_H(p), z_p)$ is the oriented area spanned by the parallelogram with sides $X_H(p)$ and z_p, that is,

$$\omega(p)(X_H(p), z_p) = \det\begin{pmatrix} X(p) & v_p \\ Y(p) & w_p \end{pmatrix} = X(p)w_p - Y(p)v_p.$$

Therefore (2) is equivalent to

$$\frac{\partial H}{\partial x}v_p + \frac{\partial H}{\partial y}w_p = -Y(p)v_p + X(p)w_p \tag{3}$$

for every $(v_p, w_p) \in \mathbf{R}^2$. In (3) choose $(v_p, w_p) = (1,0)$. Then $X(p) = \frac{\partial H}{\partial y} = y$. Next choose $(v_p, w_p) = (0,1)$. Then $Y(p) = -\frac{\partial H}{\partial x} = -\sin x$. □

Note that (1) may be written as the second order differential equation

$$\ddot{x} = -\frac{\mathrm{d}}{\mathrm{d}x}(-\cos x) = -\sin x. \tag{4}$$

By Newton's second law of motion, an integral curve of (1) describes the motion of a particle of unit mass under a force coming from the *potential* $V : \mathbf{R} \to \mathbf{R} : x \mapsto -\cos x$.

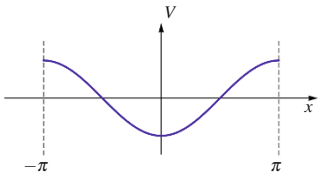

Figure 0.2. The graph of the potential $V(x) = -\cos x$.

Thus H is the total energy of the particle, namely the sum of the *kinetic* and potential ▷ energy. We now show that H is a Morse function on $T^*\mathbf{R}$.

(0.2) **Proof**: The point $p = (x,y)$ is a critical point of H if and only if $X_H(p) = 0$, that is, if and only if

$$0 = \frac{\partial H}{\partial y} = \sin x \qquad \text{and} \qquad 0 = \frac{\partial H}{\partial x} = y.$$

Thus $\{p = (n\pi, 0) \in \mathbf{R}^2 \,|\, n \in \mathbf{Z}\}$ is the set of *critical points* of H. The corresponding *critical value* of H is -1 if n is even or 1 if n is odd. Since the Hessian of H at p is

$$D^2H(p) = \left(\begin{array}{cc} \frac{\partial^2 H}{\partial x^2} & \frac{\partial^2 H}{\partial x \partial y} \\ \frac{\partial^2 H}{\partial y \partial x} & \frac{\partial^2 H}{\partial y^2} \end{array} \right) \Bigg|_p = \left(\begin{array}{cc} \cos n\pi & 0 \\ 0 & 1 \end{array} \right) = \left(\begin{array}{cc} (-1)^n & 0 \\ 0 & 1 \end{array} \right),$$

H is a Morse function, because $D^2H(p)$ is nondegenerate. □

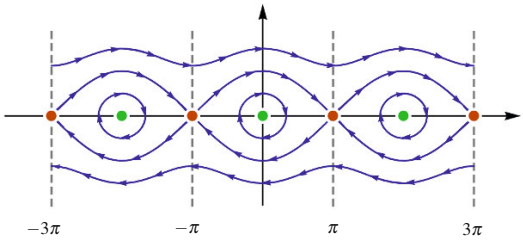

Figure 0.3. The level sets of $H(x,y) = \frac{1}{2}y^2 - \cos x$.

When $n = 2k$ the Morse index of $D^2H(n\pi, 0)$ is zero, and so the critical points $(2k\pi, 0)$ are relative minima of H; whereas when $n = 2k+1$ the *Morse index* of $D^2H(n\pi, 0)$ is

one, and so the critical points $(2k\pi,0)$ are saddle points of H, see figure .3. Using the Morse lemma, see appendix F §1, there is a neighborhood U_k of $(2k\pi,0)$ in the open strip $((2k-1)\pi,(2k+1)\pi)\times\mathbf{R}$ such that for h slightly greater than -1 the *level set* $H^{-1}(h)\cap U_k$ is diffeomorphic to a circle. Since H has no critical values in the interval $(-1,1)$, by the Morse isotopy lemma, see appendix F §3, we deduce that for every $h\in(-1,1)$ the level set $H^{-1}(h)\cap U_k$ is diffeomorphic to a circle. Thus for $h\in(-1,1)$ the whole level set $H^{-1}(h)$ is diffeomorphic to a countable disjoint union of circles. If $h\geq 1$, then $H^{-1}(h)$ is the union of the graphs of two smooth functions $y_\pm=\pm\sqrt{2(h+\cos x)}$. The graphs of $y_\pm$ are disjoint if $h>1$. On the other hand, if $h=1$, then the graphs of $y_\pm=\pm 2\cos\frac{1}{2}x$ intersect only at the points $((2k+1)\pi,0)$. There they intersect transversely as can be seen by applying the Morse lemma at the points $((2k+1)\pi,0)$. Thus we have obtained a picture of the level curves of H as given in figure .3.

To simplify the topology of the level sets of H, we make use of the fact that H is invariant under the *translation symmetry*

$$\mathbf{Z}\times T^*\mathbf{R}\to T^*\mathbf{R}:\big(n,(x,y)\big)\mapsto(x+2n\pi,y). \tag{5}$$

Thus H induces a function $\widetilde{H}$ on the space of orbits $T^*\mathbf{R}/2\pi\mathbf{Z}$. Concretely, this orbit space is identified with the cotangent bundle T^*S^1 of the circle S^1. Here S^1 is

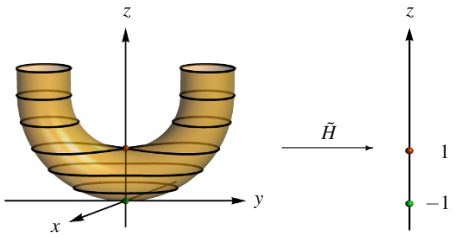

Figure 0.4. The graph of $\widetilde{H}(x,y)=\frac{1}{2}y^2-\cos x$ with $(x,y)\in T^*S^1$.

thought of as the orbit space $\mathbf{R}/2\pi\mathbf{Z}$ of the real numbers modulo 2π. Geometrically, T^*S^1 is the cylinder $S^1\times\mathbf{R}$ which is obtained from $\mathbf{R}^2$ by cutting along the vertical lines $x=0$ and $x=2\pi$ and then pasting the edges together. Applying this process to figure .3 gives figure .4 which depicts the level sets of $\widetilde{H}$ and hence the orbits of the induced Hamiltonian vector field $X_{\widetilde{H}}$. A short argument using Newton's second law shows that the second order differential equation

$$\ddot{x}=-\sin x \qquad x \bmod 2\pi$$

describes the motion of a particle of mass one on the unit circle under the influence of a constant vertical downward unit force, see figure .1.

From figure .4 we see that the topological circle, defined by the component of the level set $\widetilde{H}^{-1}(h)$ $(h>1)$ lying in the upper half cylinder, is very different from the topological circle defined by the level set $\widetilde{H}^{-1}(h)$ $(-1<h<1)$. The first circle is not contractible in TS^1 to a point whereas the second circle is. Hence it is impossible to continuously deform the first circle into the second one. This difference in the topological disposition of the

two circles corresponds to the physical fact that for small energy the particle oscillates about the bottom of the circle, while for large energy the particle loops over the top of the circle.

Exercises

1. Let (x,y) be canonical coordinates on $T^*\mathbf{R} = \mathbf{R}^2$ with symplectic form $\omega = \mathrm{d}x \wedge \mathrm{d}y$. Suppose that the Hamiltonian $H : T^*\mathbf{R} \to \mathbf{R}$ is a sum of kinetic and potential energy, that is, $H(x,y) = \frac{1}{2}y^2 + V(x)$, where $V : \mathbf{R} \to \mathbf{R}$.

   a) Find a potential function V such that the zero level set of H is connected, compact and has one singular point which is a cusp.

   b) Construct a polynomial Hamiltonian on $T^*\mathbf{R}$ whose zero level set is an n-leaf clover.

   c) Show that there is *no* smooth Hamiltonian which is the sum of kinetic and potential energy which has a 3-leaf clover as a level set.

   d) For smooth V with countable many isolated critical points give a topological characterization of the critical level sets of H.

2. Construct a Hamiltonian function on S^2 which is a Morse function with two critical points. Draw its level sets. Construct a vector field on S^2 with only one equilibrium point and sketch its orbits. Show that this vector field is not Hamiltonian.

3. a) On $\mathbf{R}^2$ consider the action $\cdot$ of $\mathbf{Z}^2$ defined by $(n,m)\cdot(x,y) = (x+n, y+m)$. The orbit space $\mathbf{R}^2/\mathbf{Z}^2$ is a two dimensional torus T^2, which may be modeled by a square with the opposite sides identified. The symplectic form $\Omega = \mathrm{d}x \wedge \mathrm{d}y$ on $\mathbf{R}^2$ induces a symplectic form $\widetilde{\Omega}$ on T^2. Show that the vector field $\widetilde{X}$ on T^2 induced by the Hamiltonian vector field $X = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ on $\mathbf{R}^2$ is *not* Hamiltonian on $(T^2, \widetilde{\Omega})$.

   b) Sketch the orbits of a Hamiltonian vector field on $(T^2, \widetilde{\Omega})$ where the Hamiltonian is a Morse function with the fewest number of critical points.

   c) Construct a vector field X on T^2 with *two* equilibrium points.

   d)* Show that a smooth function on T^2 must have at least three critical points. Find a smooth function on T^2 with exactly three critical points. Sketch its level sets.

   e) Deduce that the vector field X constructed in c) is not Hamiltonian.

4. Let M be a compact connected orientable smooth two dimensional manifold with volume form Ω. In what follows we show that every integral curve of a Hamiltonian vector field X_H of a one degree of freedom Hamiltonian system (H, M, Ω) is either an equilibrium point, a periodic orbit, or is asymptotic to an equilibrium point as $t \to \pm\infty$. For $m \in M$ let $\gamma : \mathbf{R} \to M : t \mapsto \varphi_t^H(m)$ be the integral curve of X_H through m. The *ω-limit set* $\omega(\gamma)$of γ is the closure of the set $\bigcap_{T>0} \left\{ \varphi_t^H(m) \middle| t \geq T \right\}$.

   a) Show that $\omega(\gamma)$ is nonempty.

b) If γ is an equilibrium point or a periodic orbit of X_H, show that $\omega(\gamma) = \gamma$. Is the converse true?

c)* If γ is not a periodic orbit of X_H, show that $\omega(\gamma)$ is a critical point of H, that is, an equilibrium point of X_H.

4. (Period energy relation for the mathematical pendulum.) When $|h| < 1$ show that the period of an integral curve of the mathematical pendulum which starts at $(x_+, 0)$ where $0 < x_+ = x_+(h) < \pi$ and $h + \cos x_+ = 0$, is given by

$$\tau(h) = 2\int_{-x_+}^{x_+} \frac{1}{\sqrt{2(h+\cos x)}}\, dx.$$

Show that $\tau = 4K(\sqrt{(h+1)/2})$, where K is the complete elliptic integral of the first kind, see the exercises of chapter 1. Deduce that

a) $\tau(-1) = 2\pi$, $\tau(1) = \infty$ and $\tau'(-1) = \pi/4$.

b) τ is a real analytic function on $(-1, 1)$.

c)* $\tau' > 0$ on $(-1, 1)$. Hint: show that τ satisfies a differential equation.

5. a) Suppose that a particle moves on the graph of $y = f(x)$ under the influence of gravity and that the origin is a stable equilibrium point. Determine the shape of the graph of f so that the period of oscillation of the particle about the origin is a constant independent of the energy.

b)* Show that a) is equivalent to the fact the derivative of the area enclosed by a level set of the Hamiltonian of the particle with respect to the Hamiltonian itself is a constant. Hint: see appendix D §1.

6. (Reduction of discrete symmetry of mathematical pendulum.)

a) (Discrete symmetry.) Show that

$$\zeta : S^1 \times \mathbf{R} \to S^1 \times \mathbf{R} : (x, y) \mapsto (-x, -y). \tag{6}$$

generates a $\mathbf{Z}_2$-*symmetry* of the mathematical pendulum. Show that the fundamental domain $\mathscr{D}$ of the $\mathbf{Z}_2$-action on T^*S^1 generated by ζ is the piece of the cylinder in figure .4, which lies in the half space $y \geq 0$ with the points $(\pm x, 0)$ on the circle $\partial\mathscr{D} = T^*S^1 \cap \{y = 0\}$ identified. Deduce that the orbit space $P = T^*S^1/\mathbf{Z}_2$ is homeomorphic to a cone on S^1 with vertex at the $\mathbf{Z}_2$-orbit corresponding to the point $(0, 0) \in T^*S^1$.

b) Show that the algebra of real analytic invariant functions of the abelian group $\mathbf{Z}_2$ generated by ζ is generated by

$$\tau_1 = \cos x, \qquad \tau_2 = y \sin x, \qquad \tau_3 = \tfrac{1}{2}y^2 - \cos x \tag{7}$$

subject to the relation

$$C(\tau) = \tfrac{1}{2}\tau_2^2 - (\tau_3 + \tau_1)(1 - \tau_1^2) = 0, \quad |\tau_1| \leq 1 \;\&\; \tau_3 \geq -1, \tag{8}$$

which defines the orbit space P. Draw a picture of the semialgebraic variety P.

c) (Reduced Poisson bracket.) In order to have dynamics on the $\mathbf{Z}_2$-reduced space P we first need a Poisson bracket $\{\,,\,\}_{\mathbf{R}^3}$ on $C^\infty(\mathbf{R}^3)$. A calculation shows that

$$\{\tau_1,\tau_2\} = \tau_1^2 - 1 = \frac{\partial C}{\partial \tau_3}$$
$$\{\tau_2,\tau_3\} = 2\tau_1(\tau_3+\tau_1) + \tau_1^2 - 1 = \frac{\partial C}{\partial \tau_1}$$
$$\{\tau_3,\tau_1\} = \tau_2 = \frac{\partial C}{\partial \tau_2}$$

Then for every $F, G \in C^\infty(\mathbf{R}^3)$ we have $\{F,G\} = \sum_{i,j} \frac{\partial F}{\partial \tau_i}\frac{\partial G}{\partial \tau_j}\{\tau_i,\tau_j\}$. We say that a function f on P is smooth if there is a smooth function F on $\mathbf{R}^3$ such that $f = F|P$. Let $C^\infty(P)$ be the space of smooth functions on P. Then $(P, C^\infty(P))$ is a *differential space*, which is *subcartesian* because P is a semialgebraic variety. On $C^\infty(P)$ we define a *Poisson bracket* $\{\,,\,\}_P$ as follows. Suppose that $f,g \in C^\infty(P)$. Then there are $F,G \in C^\infty(\mathbf{R}^3)$ such that $f = F|P$ and $g = G|P$. Let $\{f,g\}_P = \{F,G\}_{\mathbf{R}^3}|P$. Because the defining function C (8) of the orbit space P is a *Casimir* in the Poisson algebra $\mathscr{A} = (C^\infty(\mathbf{R}^3), \{\,,\,\}_{\mathbf{R}^3}, \cdot)$, the collection $\mathscr{I}$ of all smooth functions on $\mathbf{R}^3$, which vanish identically on P, is a *Poisson ideal* in $\mathscr{A}$. Consequently, the Poisson bracket $\{\,,\,\}_P$ is well defined. So $\mathscr{B} = \big(C^\infty(P) = C^\infty(\mathbf{R}^3/\mathscr{I}), \{\,,\,\}_P, \cdot\big)$ is a Poisson algebra.

d) (Reduced dynamics.) Consider the derivation $-\mathrm{ad}_H$ on the Poisson algebra $\mathscr{A}$. This derivation gives rise to the $\mathbf{Z}_2$-reduced Hamiltonian vector field X_H on the subcartesian differential space $(P, C^\infty(P))$ associated to the $\mathbf{Z}_2$-*reduced Hamiltonian* $H : P \to \mathbf{R} : \tau \mapsto \tau_3$. On $\mathbf{R}^3$ the integral curves of $-\mathrm{ad}_H$ satisfy

$$\dot\tau_1 = \{\tau_1,H\}_P = \{\tau_1,\tau_3\}_P = -\tau_2$$
$$\dot\tau_2 = \{\tau_2,H\}_P = \{\tau_2,\tau_3\}_P = 2\tau_1(\tau_3+\tau_1) + \tau_1^2 - 1$$
$$\dot\tau_3 = \{\tau_1,H\}_P = \{\tau_1,\tau_3\}_P = 0.$$

The equality $\dot\tau_3 = 0$ shows that H is an integral of X_H. Check that C (8) is also an integral of X_H. A calculation shows that $-\mathrm{ad}_H$ leaves $C^{-1}(0)$, $\{\tau_3+\tau_1 = 0\}$, and $\{\tau_1 = \pm 1\}$ invariant. Thus the *reduced space* P is invariant under the flow of $-\mathrm{ad}_H$. Consequently, the *reduced Hamiltonian vector field* X_H on P is $-\mathrm{ad}_H|P$. Because the Hamiltonian vector field $X_{\tilde H}$ of the mathematical pendulum is complete, the reduced vector field X_H is complete. Its flow φ_t^H is a 1-parameter group of diffeomorphisms of P. In fact, for $p \in H^{-1}(e)$ the closure of the integral curve $\{\varphi_t^H(p) \in P \,\big|\, t \in \mathbf{R}\}$ is a connected component of the level set $H^{-1}(e)$, since a level set of the reduced Hamiltonian H is compact.

Part I. Examples

Chapter I

The harmonic oscillator

1 Hamilton's equations and S^1 symmetry

Physically, the *harmonic oscillator* in the plane is described by a particle of unit mass acted upon by two linear springs of unit spring constant: one spring acts in the x_1-direction and the other in the x_2-direction. Mathematically, the *configuration space* of the harmonic oscillator is Euclidean 2-space. In other words, the space of positions of the particle is $\mathbf{R}^2$ with coordinates $x = (x_1, x_2)$ and *Euclidean inner product* $(\ ,\ )$ where $(x, x') = x_1 x_1' + x_2 x_2'$. The space of all positions and momenta of the particle is the *cotangent bundle* $T^*\mathbf{R}^2$ of $\mathbf{R}^2$. This phase space has coordinates (x, y) and a canonical *symplectic form* $\omega = \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \mathrm{d}x_2 \wedge \mathrm{d}y_2$. The *Hamiltonian* function $H : T^*\mathbf{R}^2 \to \mathbf{R}$ of the harmonic oscillator is the sum of *kinetic energy* $\frac{1}{2}(y, y)$ and *potential energy* $\frac{1}{2}(x, x)$. Letting $z = (x, y) \in \mathbf{R}^4$,

$$H(z) = \tfrac{1}{2}(y, y) + \tfrac{1}{2}(x, x) = \tfrac{1}{2}\langle z, z\rangle. \tag{1}$$

Here $\langle\ ,\ \rangle$ is the Euclidean inner product on $\mathbf{R}^4$, which we have identified with $T^*\mathbf{R}^2$.

The motion of the harmonic oscillator is described by Hamilton's equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}\begin{pmatrix} D_1 H(x,y) \\ D_2 H(x,y) \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix}.$$

Here I_2 is the 2×2 identity matrix. Since the *Hamiltonian vector field* $X_H(x, y) = (y, -x)$ is linear, the flow of the linear vector field is

$$\varphi^H : \mathbf{R} \times \mathbf{R}^4 \to \mathbf{R}^4 : (t, z) \mapsto (\exp tX_H)\, z = \begin{pmatrix} (\cos t) I_2 & (\sin t) I_2 \\ -(\sin t) I_2 & (\cos t) I_2 \end{pmatrix} z. \tag{2}$$

Given any initial condition $z \in T^*\mathbf{R}^2$, the *integral curve* of X_H through the point z is $t \mapsto \varphi_t^H(z)$. Thus from a quantitative point of view, we know everything about the vector field X_H, because we have an explicit formula (2) for every integral curve. On the other hand, from a qualitative point of view, the explicit formula is *very* unsatisfactory. For instance, we do not know if the integral curves lie on a lower dimensional invariant manifold or

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how they fit together. In the rest of this chapter we will describe the *global qualitative* features of the invariant manifolds of X_H.

Claim: The h-level set $H^{-1}(h)$ of the Hamiltonian H (1) is an invariant manifold of the vector field of the harmonic oscillator.

(1.1) **Proof**: Since

$$\dot{H} = L_{X_H} H = \langle \dot{x}, x \rangle + \langle \dot{y}, y \rangle = \langle y, x \rangle - \langle x, y \rangle = 0,$$

H is constant on the integral curves of X_H, that is, H is an integral (= conserved quantity) of X_H, see chapter VII §3. In particular the h-level set

$$H^{-1}(h) = \left\{ z \in \mathbf{R}^4 \,\middle|\, \tfrac{1}{2}\langle z, z \rangle = h \right\},$$

which is diffeomorphic to a 3-sphere S^3 when $h > 0$, a point when $h = 0$, and is empty when $h < 0$, is a smooth *invariant manifold* of X_H. In other words, every integral curve of X_H with initial condition in $H^{-1}(h)$ lies in $H^{-1}(h)$ for all time. □

The rotational symmetry of the potential energy $\frac{1}{2}(x,x)$ gives rise to another conserved quantity, namely, the *angular momentum*

$$L : T^*\mathbf{R}^2 \to \mathbf{R} : (x,y) \mapsto x_1 y_2 - x_2 y_1. \tag{3}$$

To see this, consider the $S^1 = \mathbf{R}/2\pi\mathbf{Z}$-action on $\mathbf{R}^2$

$$\psi : S^1 \times \mathbf{R}^2 \to \mathbf{R}^2 : (s,x) \mapsto R_s x = \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix} x.$$

ψ_s is a counterclockwise rotation through an angle s about the origin. The infinitesimal generator of the action ψ is the vector field

$$X(x) = \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \psi_s(x) = (-x_2, x_1).$$

ψ lifts to an S^1 action Ψ on $T^*\mathbf{R}^2$ defined by

$$\Psi : S^1 \times T^*\mathbf{R}^2 \to T^*\mathbf{R}^2 : \big(s, (x,y)\big) \mapsto (R_s x, R_s y).$$

Ψ preserves the *canonical* 1*-form* $\theta = y_1\,\mathrm{d}x_1 + y_2\,\mathrm{d}x_2 = (y, \mathrm{d}x)$ on $T^*\mathbf{R}^2$, since

$$\Psi_s^*\theta = (R_s y, \mathrm{d}R_s x) = (R_s y, R_s\,\mathrm{d}x) = (y, \mathrm{d}x) = \theta.$$

Therefore Ψ_s is a symplectic mapping, that is, $\Psi_s^*\omega = \omega$, since

$$\omega = -\,\mathrm{d}\theta = -\,\mathrm{d}\Psi_s^*\theta = -\Psi_s^*\,\mathrm{d}\theta = \Psi_s^*\omega.$$

The infinitesimal generator of the action Ψ is the vector field

$$Y(x,y) = \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \Psi_s(x,y) = (-x_2, x_1, -y_2, y_1) = \big(D_2 L(x,y), -D_1 L(x,y)\big).$$

Thus Y is a Hamiltonian vector field X_L corresponding to the angular momentum L (3). That the lifted S^1 action Ψ preserves the canonical 1-form and is the flow of the Hamiltonian vector field corresponding to the Hamiltonian $L = X \lrcorner \theta$ is a particular case of a more general set of results, see chapter VII §3. Since

$$(\Psi_s^* H)(x,y) = \tfrac{1}{2}(R_s y, R_s y) + \tfrac{1}{2}(R_s x, R_s x) = H(x,y),$$

H is constant on the integral curves of X_L. Therefore L is constant on the integral curves of X_H, that is, L is an integral of X_H, see chapter VI §4. This implies that the ℓ-level set of L,

$$L^{-1}(\ell) = \left\{(x,y) \in T^*\mathbf{R}^2 \middle| x_1 y_2 - x_2 y_1 = \ell\right\},$$

is an invariant manifold of X_H. When $\ell \neq 0$ the level set $L^{-1}(\ell)$ is diffeomorphic to $S^1 \times \mathbf{R}^2$, while when $\ell = 0$ the level set $L^{-1}(0)$ is homeomorphic but *not* diffeomorphic to $S^1 \times \mathbf{R}^2$ as it is a cone on $S^1 \times S^1$ together with its vertex at the origin.

2 S^1 energy momentum mapping

In order to organize the qualitative information about the harmonic oscillator which can be obtained from the integrals of energy and angular momentum, define the S^1 *energy momentum mapping*

$$\mathscr{EM} : T^*\mathbf{R}^2 \to \underline{\mathrm{R}}^2 : (x,y) \mapsto \big(H(x,y), L(x,y)\big) = \big(\tfrac{1}{2}(y_1^2 + y_2^2 + x_1^2 + x_2^2), x_1 y_2 - x_2 y_1\big).$$

Because H and L are polynomial integrals of X_H, the *fiber* $\mathscr{EM}^{-1}(h,\ell)$ is an invariant manifold of X_H which is a real algebraic variety. Other geometric properties of $\mathscr{EM}$ correspond to qualitative properties of X_H. To describe such global geometric properties of $\mathscr{EM}$, we shall

1. Find the critical points, critical values and range of $\mathscr{EM}$.
2. Find the topological type of *every* fiber of the energy momentum mapping. This determines the *bifurcation set* of $\mathscr{EM}$, the set of values (h,ℓ) where the *topological type of the fiber* changes.
3. Analyze how the fibers of constant angular momentum *foliate* a given energy level set.

▷ We begin by finding the critical points and corresponding critical values of the energy momentum map. A point $z = (x,y) \in T^*\mathbf{R}^2$ is a critical point of $\mathscr{EM}$ if and only if the derivative of $\mathscr{EM}$ at z is not surjective, that is,

$$D\mathscr{EM}(z) = \begin{pmatrix} DH(z) \\ DL(z) \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & y_1 & y_2 \\ y_2 & -y_1 & -x_2 & x_1 \end{pmatrix} \tag{4}$$

has rank less than two. There are two cases to be considered.

CASE I: $\operatorname{rank} D\mathscr{EM}(z) = 0$. This can only happen if $DH(z) = DL(z) = 0$. Then $z = 0$ is the critical point and $\mathscr{EM}(0) = (0,0)$ is the corresponding critical value of $\mathscr{EM}$.

CASE II. $\operatorname{rank} D\mathscr{EM}(z) = 1$. This occurs if and only if $DH(z)$ and $DL(z)$ are linearly dependent and are not both zero. From (4) it follows that $DH(z) = 0$ if and only if $DL(z) = 0$. Therefore we may suppose that $DH(z) \neq 0$, that is, $z \neq 0$. Thus for some $\mu \in \mathbf{R}$

$$0 = DL(z) - \mu\, DH(z) = (y_2, -y_1, -x_2, x_1) - \mu\,(x_1, x_2, y_1, y_2). \tag{5}$$

If $\mu = 0$, then $z = 0$, which is a contradiction. Therefore $\mu \neq 0$. Composing the linear mapping $(x_1, x_2, y_1, y_2) \mapsto \mu^{-1}(y_2, -y_1, -x_2, x_1)$ with itself gives $z = \mu^{-2} z$. Thus $\mu^2 = 1$, since $z \neq 0$. Consequently, the solutions of (5) define two punctured 2-planes

$$\Pi^*_+ = \left\{(x_1, x_2, -x_2, x_1) \in T^*\mathbf{R}^2 \middle| (x_1, x_2) \in \left(\mathbf{R}^2 \setminus \{(0,0)\}\right)\right\}$$

and

$$\Pi^*_- = \left\{(x_1, x_2, x_2, -x_1) \in T^*\mathbf{R}^2 \middle| (x_1, x_2) \in \left(\mathbf{R}^2 \setminus \{(0,0)\}\right)\right\}.$$

On Π^*_+ the corresponding set of critical values of $\mathscr{E}\mathscr{M}$ is the diagonal ray $\{(h,h) \in \mathbf{R}^2 \,|\, h > 0\}$ since $0 < \ell = L|\Pi^*_+ = x_1^2 + x_2^2 = H|\Pi^*_+ = h$; while on Π^*_- the corresponding critical values of $\mathscr{E}\mathscr{M}$ is the antidiagonal ray $\{(h,-h) \in \mathbf{R}^2 \,|\, h > 0\}$. Therefore the critical fiber $\mathscr{E}\mathscr{M}^{-1}(h,h),\ h > 0$ is the circle

$$S^1_{+,h} = H^{-1}(h) \cap \Pi^*_+ = \left\{(x_1, x_2, -x_2, x_1) \in T^*\mathbf{R}^2 \middle| x_1^2 + x_2^2 = h,\ h > 0\right\}; \tag{6}$$

while the critical fiber $\mathscr{E}\mathscr{M}^{-1}(h,-h),\ h > 0$ is the circle

$$S^1_{-,h} = H^{-1}(h) \cap \Pi^*_- = \left\{(x_1, x_2, x_2, -x_1) \in T^*\mathbf{R}^2 \middle| x_1^2 + x_2^2 = h,\ h > 0\right\}. \tag{7}$$

Note that the image of $S^1_{\pm,h}$ under the bundle projection $\pi : T^*\mathbf{R}^2 \to \mathbf{R}^2 : (x,y) \mapsto x$ is the circle $x_1^2 + x_2^2 = h$, which is positively oriented when the sign is $+$ and negatively otherwise. Another way to interpret the critical circles $S^1_{\pm,h}$ is to note that solving (5) subject to the condition that $z \in H^{-1}(h), h > 0$ is equivalent to finding the critical points
▷ of L on $H^{-1}(h)$. Thus L has two critical manifolds $S^1_{\pm,h}$ on $H^{-1}(h)$. The manifolds $S^1_{\pm,h}$ are nondegenerate of Morse index 2 for $S^1_{+,h}$ and 0 for $S^1_{-,h}$.

(2.1) **Proof**: To show that $S^1_{+,h}$ is a nondegenerate critical manifold of $L|H^{-1}(h)$ we must verify that at every $p = (x_1, x_2, -x_2, x_1) \in S^1_{+,h}$, the Hessian of $L|H^{-1}(h)$ when restricted to a 2-plane $N_p S^1_{+,h}$ normal to $S^1_{+,h}$ in $T_p H^{-1}(h)$ has Morse index 2. From the fact that p is a critical point of $L|H^{-1}(h)$ with Lagrange multiplier $\mu = 1$, it follows that the Hessian of $L|H^{-1}(h)$ at p is Q_+, which equals

$$D^2\big(L|H^{-1}(h)\big)(p) = \big(D^2L(p) - D^2H(p)\big)\,|\,T_pH^{-1}(h) = \left.\begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}\right|_{\ker DH(p)}$$

see chapter XI §2. Since $S^1_{+,h}$ is an *orbit* of X_H, we see that $T_p S^1_{+,h}$ is spanned by the vector $X_H(p) = (-x_2, x_1, -x_1, -x_2)$. As $\ker DH(p)$ is spanned by the linearly independent vectors

$$X_H = (-x_2, x_1, -x_1, -x_2),\ f_1 = (x_2, -x_1, -x_1, -x_2),\ f_2 = (x_1, x_2, x_2, -x_1),$$

a normal 2-plane $N_pS^1_{+,h}$ is spanned by the vectors $\{f_1, f_2\}$. A calculation shows that the matrix of Q_+ with respect to the basis $\{f_1, f_2\}$ is $-2I_2$. Therefore $Q_+|N_pS^1_{+,h}$ is nondegenerate with Morse index 2. A similar calculation shows that at $p \in S^1_{-,h}$ the Hessian $Q_- = D^2\big(L|H^{-1}(h)\big)(p)$ restricted to $N_pS^1_{-,h}$ is equal to $2I_2$. Thus $Q_-|N_pS^1_{-,h}$ is nondegenerate with Morse index 0. □

Consequently, on $S^1_{+,h}$ the function L assumes its maximum value h, while on $S^1_{-,h}$ the function L assumes its minimum value $-h$. Therefore, the closed wedge $\{(h,\ell) \in \mathbf{R}^2 \,\big|\, 0 \leq |\ell| \leq h\}$ is the image of the energy momentum mapping $\mathscr{EM}$.

▷ To find the topology of a fiber of $\mathscr{EM}$ corresponding to a regular value, we simultaneously diagonalize the quadratic forms defining the energy and angular momentum by a linear *symplectic coordinate change*. Consider the linear change of coordinates on $\mathbf{R}^4$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = P \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \tag{8}$$

where $A = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 0 & 0 \\ 1 & -1 \end{smallmatrix}\right)$ and $B = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Since $A^tA + B^tB = I_2$ and $A^tB = B^tA$, the matrix P is *symplectic* and orthogonal, that is, $P^*\omega = \omega$ and $\langle Pz, Pw\rangle = \langle z, w\rangle$ for every $z, w \in \mathbf{R}^4$. With respect to the (ξ, η) coordinates, the Hamiltonian H becomes

$$\widetilde{H}(\xi,\eta) = (P^*H)(\xi,\eta) = (H \circ P)(\xi,\eta) = \tfrac{1}{2}(\eta_1^2 + \eta_2^2 + \xi_1^2 + \xi_2^2).$$

Because P is symplectic, the Hamiltonian vector field $X_{\widetilde{H}}$ corresponding to $\widetilde{H}$ is $P^{-1}X_HP$, that is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \eta \\ -\xi \end{pmatrix}.$$

Moreover the angular momentum L becomes

$$\widetilde{L} = (P^*L)(\xi,\eta) = \tfrac{1}{2}(\eta_2^2 - \eta_1^2 + \xi_2^2 - \xi_1^2).$$

Since P is symplectic, $\widetilde{L}$ is an integral of $X_{\widetilde{H}}$. Therefore the fiber of $\widetilde{\mathscr{EM}} = \mathscr{EM} \circ P$ at (h,ℓ) is the set of $(\xi,\eta) \in \mathbf{R}^4$ which satisfy

$$\begin{aligned} \tfrac{1}{2}(\eta_1^2 + \eta_2^2 + \xi_1^2 + \xi_2^2) &= h = \widetilde{H}(\xi,\eta) \\ \tfrac{1}{2}(\eta_2^2 - \eta_1^2 + \xi_2^2 - \xi_1^2) &= \ell = \widetilde{L}(\xi,\eta). \end{aligned} \tag{9}$$

This implies

$$\begin{aligned} \eta_1^2 + \xi_1^2 &= h - \ell \\ \eta_2^2 + \xi_2^2 &= h + \ell. \end{aligned} \tag{10}$$

Therefore when $0 \leq |\ell| < h$, that is, when (h,ℓ) is a regular value of $\widetilde{\mathscr{EM}}$, each of the equations in (10) defines a circle. Hence $\widetilde{\mathscr{EM}}^{-1}(h,\ell)$ is a 2-torus $\widetilde{T}^2_{h,\ell}$. Since P is a diffeomorphism, we find that $\mathscr{EM}^{-1}(h,\ell)$ is a 2-torus when (h,ℓ) is a regular value. □

▷ We now describe geometrically how the orbits of $X_{\widetilde{H}}|\widetilde{T}^2_{h,\ell}$ foliate the 2-torus $\widetilde{T}^2_{h,\ell}$. Observe that the flow $\varphi_t^{\widetilde{H}}$ of $X_{\widetilde{H}}$ defines a free proper action of $S^1 = \mathbf{R}/2\pi\mathbf{Z}$ on the 2-torus $\widetilde{T}^2_{h,\ell}$.

(2.2) **Proof**: Since S^1 is compact, we need only show that at every point $p \in \widetilde{T}^2_{h,\ell}$ the isotropy group $\{t \in S^1 \,|\, \varphi_t^{\widetilde{H}}(p) = p\}$ is the identity element of S^1. For $(\xi,\eta) \neq 0$ and $t \in \mathbf{R}$

$$\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \varphi_t^{\widetilde{H}}\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}(\cos t)I_2 & (\sin t)I_2\\ -(\sin t)I_2 & (\cos t)I_2\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}$$

implies that $t = 2n\pi$ for every $n \in \mathbf{Z}$. These values of t correspond to the identity element in S^1 under the orbit mapping $\mathbf{R} \to \mathbf{R}/2\pi\mathbf{Z}$. □

This implies the orbit space $\widetilde{T}^2_{h,\ell}/S^1$ is a smooth one dimensional manifold, see chapter VII ((2.9)).

Claim: The orbit space $\widetilde{T}^2_{h,\ell}/S^1$ is diffeomorphic to S^1.

(2.3) **Proof**: Since the S^1-action defined by the flow of $X_{\widetilde{H}}$ is free and proper, we know from results proved in chapter VII ((2.12)) that $\widetilde{T}^2_{h,\ell}$ is the total space of a smooth principal bundle with base a smooth one dimensional manifold with no boundary. Because the bundle projection map ρ is smooth, compactness and connectedness of $\widetilde{T}^2_{h,\ell}$ implies that the base is a compact connected one dimensional smooth manifold with no boundary. This implies that the base is a circle C. □

▷ We now show that a fiber $\mathscr{C} = \rho^{-1}(p)$ where $p \in C$ of the principal bundle ρ is a global *cross section* for the flow of $X_{\widetilde{H}}$ on $\widetilde{T}^2_{h,\ell}$. Suppose that $(\xi,\eta) \in \mathscr{C}$. After time 2π the integral curve of $X_{\widetilde{H}}$ through (ξ,η) intersects $\mathscr{C}$ for the first positive time at $\varphi^{\widetilde{H}}_{2\pi}(\xi,\eta)$, which is in fact (ξ,η). □

Since the image under the bundle projection ρ of the integral curve through (ξ,η) of $X_{\widetilde{H}-\widetilde{L}}$ parameterizes C and crosses p for the first positive time at 2π, we find that

$$\varphi^{\widetilde{H}-\widetilde{L}}_{2\pi}(\xi,\eta) = \varphi^{\widetilde{H}}_{2\pi}(\xi,\eta).$$

In other words, the integral curve of $X_{\widetilde{H}}$ through a point on $\mathscr{C}$ winds once around C as its projection winds once around the circle $\{(0,\xi_2,\sqrt{h-\ell},\eta_2) \in \widetilde{T}^2_{h,\ell} \,\big|\, \xi_2^2+\eta_2^2 = h+\ell\}$. Therefore the rotation number of the flow of $X_{\widetilde{H}}$ on $\widetilde{T}^2_{h,\ell}$ is 1. Applying the linear symplectic coordinate change P^{-1} with P given by (8), we find that $P^{-1}\mathscr{C}$ is a global cross section for the flow of X_H on $T^2_{h,\ell}$ and that every integral curve of $X_H|T^2_{h,\ell}$ has rotation number 1.

The information we have obtained so far about the level sets of the energy momentum mapping of the harmonic oscillator is summarized in the *bifurcation diagram* figure 2.1. The set of regular values of $\mathscr{EM}$ is the open wedge $\mathscr{T}$ in the (h,ℓ) plane defined by $0 \leq |\ell| < h$, since the critical values of $\mathscr{EM}$ are the two rays $\{(h,\pm h) \in \mathbf{R}_{>0} \times \mathbf{R} \,|\, h > 0\}$. Because $\mathscr{T}$ is *simply connected*, the energy momentum mapping $\mathscr{EM}$ defines a *trivial* smooth fibration over $\mathscr{T}$ with fiber T^2, that is, $\mathscr{EM}^{-1}(\mathscr{T})$ is diffeomorphic to $\mathscr{T} \times T^2$, see chapter VIII §2.

To complete the qualitative analysis of the energy momentum mapping we need only understand how an energy surface $H^{-1}(h)$, $h > 0$, which is diffeomorphic to the 3-sphere

S^3, is built up from the fibers of the energy momentum mapping $\mathscr{EM}^{-1}(h,\ell)$ as ℓ varies over $[-h,h]$. Recall that $\mathscr{EM}^{-1}(h,\ell)$ is a circle when $|\ell| = h$ and is a 2-torus T^2 when $|\ell| < h$. This problem will be solved in the next two sections by showing the the S^1 momentum mapping $\mathscr{EM}$ has an extension to a U(2)-momentum mapping whose restriction to $H^{-1}(h)$ is the Hopf fibration.

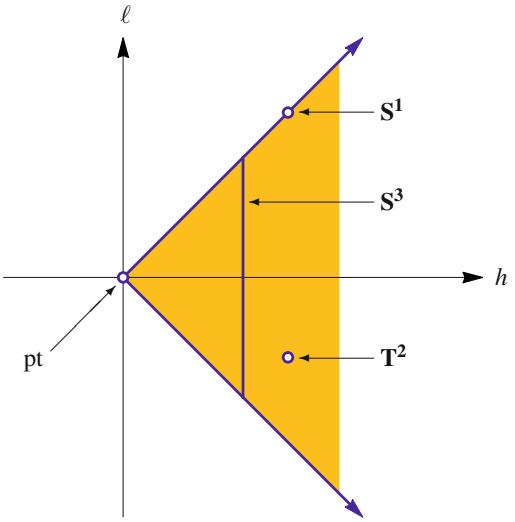

Figure 2.1. The bifurcation diagram. The image of $\mathscr{EM}$ is shaded. The topological type of its fibers or union of fibers is as indicated.

3 U(2)-momentum mapping

As with the construction of the S^1 energy momentum mapping, we begin our construction of the U(2)-momentum mapping by looking for additional integrals of the harmonic oscillator.

We start by looking for quadratic ones. Suppose that F is a homogeneous real quadratic function on $\mathbf{R}^4$. Then F is an integral of X_H if and only if $0 = L_{X_H}F = \{F,H\}$, where $\{,\}$ is the *standard Poisson bracket* on $C^\infty(\mathbf{R}^4)$, see chapter VI §4. From $[X_F,X_H] = -X_{\{F,H\}}$, it follows that $[X_F,X_H] = 0$. Conversely, if $[X_F,X_H] = 0$, then $X_{\{F,H\}} = 0$. Thus the function $\{F,H\} = (\frac{\partial F}{\partial x},\frac{\partial H}{\partial y}) - (\frac{\partial F}{\partial y},\frac{\partial H}{\partial x})$ is constant. But F is a homogeneous real quadratic function on $\mathbf{R}^4$. Hence $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0$ at the origin. Therefore $\{F,H\} = 0$, that is, F is an integral of X_H. This proves

Claim: The homogeneous real quadratic function $F : \mathbf{R}^4 \to \mathbf{R}$ is an integral of the harmonic oscillator vector field X_H if and only if $[X_F,X_H] = 0$.

Since F is a homogeneous quadratic function, the Hamiltonian vector field X_F is linear. Thus F is a homogeneous quadratic integral of X_H if and only if the matrices X_F and X_H commute. Now every homogeneous real quadratic function F on $\mathbf{R}^4$ is given by a 4×4 symmetric matrix, which can be written as $\begin{pmatrix} -B & A^t \\ A & C \end{pmatrix}$, where A is a 2×2 real matrix and B and C are 2×2 real symmetric matrices. Therefore the Hamiltonian vector field X_F corresponding to F has integral curves which satisfy

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{\partial F}{\partial y} = Ax + Cy \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -\frac{\partial F}{\partial x} = Bx - A^t y. \end{aligned}$$

So X_F is the 4×4 infinitesimally symplectic matrix $\begin{pmatrix} A & C \\ B & -A^t \end{pmatrix}$, where $B = B^t$ and $C = C^t$, that is, $X_F \in \mathrm{sp}(4,\mathbf{R})$, see chapter VII §5.1 example 2. The infinitesimally symplectic matrix corresponding to the harmonic oscillator vector field X_H is $\begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$. A calculation shows that X_F and X_H commute if and only if $X_F = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$, where $A = -A^t$ and $B = B^t$.

Claim: The set of all 4×4 infinitesimally symplectic matrices which commute with the infinitesimally symplectic matrix X_H corresponding to the harmonic oscillator vector field is isomorphic to the Lie algebra $\mathrm{u}(2)$ of the Lie group $\mathrm{U}(2)$ of 2×2 unitary matrices.

(3.1) **Proof**: Define a mapping

$$\vartheta : \mathrm{u}(2) \to \mathrm{sp}\,(4,\mathbf{R}) : A + iB \to \begin{pmatrix} A & -B \\ B & A \end{pmatrix}.$$

Since $A+iB \in \mathrm{u}(2)$, it follows that $A^t - iB^t = (\overline{A+iB})^t = -(A+iB)$. Hence $A = -A^t$ and $B = B^t$, which implies that $\mathscr{A} \in \mathrm{sp}(4,\mathbf{R})$. Thus the image of the mapping ϑ is contained in $\mathrm{sp}(4,\mathbf{R})$. Clearly ϑ is bijective on its image and linear. □

From now on we will consider $\mathrm{u}(2)$ to be a subspace of $\mathrm{sp}(4,\mathbf{R})$. Let

$$E_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},\quad E_2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},\quad E_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},\quad E_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Then $\{E_1, E_2, E_3, E_4\}$ form a basis for $\mathrm{u}(2)$. The quadratic Hamiltonian function corresponding to the linear Hamiltonian vector field E_i is $W_i(z) = \frac{1}{2}\omega(E_i z, z)$. Here $z = (x,y)$ and the matrix of the canonical symplectic form ω on $\mathbf{R}^4$ is $-E_4$. This establishes the

Claim: The functions

$$\begin{aligned} W_1(x,y) &= x_1x_2 + y_1y_2 \\ W_2(x,y) &= x_1y_2 - x_2y_1 = L(x,y) \\ W_3(x,y) &= \tfrac{1}{2}(y_1^2 + x_1^2 - y_2^2 - x_2^2) \\ W_4(x,y) &= \tfrac{1}{2}(y_1^2 + x_1^2 + y_2^2 + x_2^2) = H(x,y) \end{aligned} \tag{11}$$

are a basis for the vector space of all the quadratic integrals of the harmonic oscillator vector field.

From the claim we see that the functions W_1, W_2, W_3, W_4 are integrals of the harmonic oscillator vector field. Moreover, these integrals satisfy the relation

$$W_1^2 + W_2^2 = \widetilde{W}_3 \widetilde{W}_4 \tag{12}$$

with

$$\widetilde{W}_3 = W_4 - W_3 = y_1^2 + x_1^2 \geq 0 \text{ and } \widetilde{W}_4 = W_4 + W_3 = y_2^2 + x_2^2 \geq 0.$$

We now give a geometric interpretation of these integrals. Let $\gamma : \mathbf{R} \to \mathbf{R}^2 : t \mapsto \big(x_1(t), x_2(t)\big)$ be the projection onto configuration space of an integral curve of X_H of energy $h > 0$ starting at (x^0, y^0). From (2) we see that $\gamma(t)$ passes through x^0 at time $t = 0$ and is given by

$$\big(x_1(t), x_2(t)\big) = (x_1^0 \cos t + y_1^0 \sin t, x_2^0 \cos t + y_2^0 \sin t). \tag{13}$$

Since the integral curves of X_H of positive energy are closed, it follows that γ is a planar
▷ closed curve, called a *Lissajous figure*, which we now show is an ellipse.

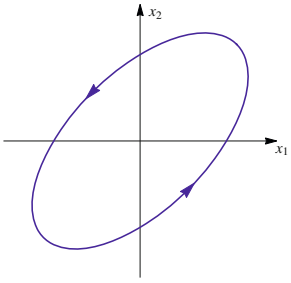

Figure 3.1. A Lissajous curve.

(3.2) **Proof**: First we note that the initial condition (x^0, y^0) determines the values of the integrals, namely,

$$W_1(x^0, y^0) = \widetilde{w}_1,\ W_2(x^0, y^0) = \widetilde{w}_2,\ \widetilde{W}_3(x^0, y^0) = \widetilde{w}_3,\ \widetilde{W}_4(x^0, y^0) = \widetilde{w}_4.$$

Therefore using the definition of $\widetilde{W}_3$ and $\widetilde{W}_4$ we get

$$\big(\widetilde{w}_3 - x_1(t)^2\big)\big(\widetilde{w}_4 - x_2(t)^2\big) = \big(y_1(t) y_2(t)\big)^2 = \big(\widetilde{w}_1 - x_1(t) x_2(t)\big)^2,$$

which upon simplification is

$$\widetilde{w}_4 x_1(t)^2 - 2\widetilde{w}_2 x_1(t) x_2(t) + \widetilde{w}_3 x_2(t)^2 = \widetilde{w}_3 \widetilde{w}_4 - \widetilde{w}_1^2 = \widetilde{w}_2^2. \tag{14}$$

Since $\widetilde{w}_3 + \widetilde{w}_4 = h > 0$ and $\widetilde{w}_3 \widetilde{w}_4 - \widetilde{w}_1^2 = \widetilde{w}_2^2 \geq 0$, if $\widetilde{w}_2 > 0$ then the quadratic form

$$Q(x_1, x_2) = \widetilde{w}_4 x_1^2 - 2\widetilde{w}_2 x_1 x_2 + \widetilde{w}_3 x_2^2 = \widetilde{w}_2^2 \tag{15}$$

is positive definite and hence the curve (13) is an ellipse $\mathscr{E}$.

We now describe the geometry of the ellipse $\mathscr{E}$ more precisely. In complex coordinates $\zeta = x_1 + i x_2$ the quadratic form $Q = \widetilde{w}_2^2$ becomes

$$4\mathscr{Q}(\zeta, \overline{\zeta}) = (\widetilde{w}_4 - \widetilde{w}_3 + 2i\widetilde{w}_1)\zeta^2 + 2(\widetilde{w}_4 + \widetilde{w}_3)\,\zeta\overline{\zeta} + (\widetilde{w}_4 - \widetilde{w}_3 - 2i\widetilde{w}_1)\overline{\zeta}^2 = 4\widetilde{w}_2^2. \tag{16}$$

Apply the rotation $\zeta = ze^{i\vartheta}$. Then (16) becomes

$$(\widetilde{w}_4 - \widetilde{w}_3 + 2i\widetilde{w}_1)e^{2i\vartheta} z^2 + 2(\widetilde{w}_4 + \widetilde{w}_3)z\bar{z} + (\widetilde{w}_4 - \widetilde{w}_3 - 2i\widetilde{w}_1)e^{-2i\vartheta}\bar{z}^2 = 4\widetilde{w}_2^2, \qquad (17)$$

which is diagonal if the angle ϑ is chosen so that $(\widetilde{w}_4 - \widetilde{w}_3 + 2i\widetilde{w}_1)e^{2i\vartheta}$ is real and, say, positive. So set

$$2\vartheta = -\tan^{-1}\frac{2\widetilde{w}_1}{\widetilde{w}_4 - \widetilde{w}_3}. \qquad (18)$$

Then the symmetry axis of $\mathscr{E}$ lies along the line in $\mathbf{R}^2$, which passes through the origin and subtends an angle 2ϑ with the positive x_1-axis. After performing the rotation $e^{i\vartheta}$, where ϑ satisfies (18), equation (17) becomes

$$Az^2 + 2Bz\bar{z} + A\bar{z}^2 = 4\widetilde{w}_2^2, \qquad (19)$$

where $A = |\widetilde{w}_4 - \widetilde{w}_3 + 2i\widetilde{w}_1| \geq 0$ and $B = \widetilde{w}_4 + \widetilde{w}_3 > 0$. In real coordinates $\binom{x_1}{x_2} = \begin{pmatrix}\cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta\end{pmatrix}\binom{\xi_1}{\xi_2}$ equation (19) becomes

$$2(A+B)\xi_1^2 + 2(B-A)\xi_2^2 = 4\widetilde{w}_2^2. \qquad (20)$$

Since

$$B^2 - A^2 = (\widetilde{w}_3 + \widetilde{w}_4)^2 - (\widetilde{w}_4 - \widetilde{w}_3)^2 - 4\widetilde{w}_1^2 = 4(\widetilde{w}_4\widetilde{w}_3 - \widetilde{w}_1^2) = 4\widetilde{w}_2^2 > 0,$$

it follows that $B - A > 0$. Thus (20) is an equation for the ellipse $\mathscr{E}$, which in standard form is $\left(\frac{\xi_1}{b}\right)^2 + \left(\frac{\xi_2}{a}\right)^2 = 1$, where $a = \frac{\sqrt{2}w_2}{\sqrt{B-A}}$ is the major semi-axis a and $b = \frac{\sqrt{2}w_2}{\sqrt{A+B}}$ is its minor semi-axis. Consequently, the eccentricity e of $\mathscr{E}$ is

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{B-A}{A+B}} = \sqrt{\frac{2A}{A+B}}.$$

Now suppose that $\widetilde{w}_2 = 0$. Then the quadratic form Q (15) factors into the product of the linear factors $\widetilde{w}_4 x_1 - \widetilde{w}_1 x_2$ and $-\widetilde{w}_1 x_1 + \widetilde{w}_3 x_2$. Hence $Q = 0$ defines two lines given by

$$\widetilde{w}_4 x_1 - \widetilde{w}_1 x_2 = 0 \quad \text{or} \quad -\widetilde{w}_1 x_1 + \widetilde{w}_3 x_2 = 0. \qquad (21)$$

Because x^0 lies on γ, it satisfies exactly one of the equations in (21). Since the energy is fixed, we have

$$x_1^2 + x_2^2 \leq x_1^2 + y_1^2 + x_2^2 + y_2^2 = 2h. \qquad (22)$$

Therefore γ is a line segment, which lies inside the disc in configuration space defined by equation (22). □

We now return to the problem of finding an extension of the S^1-momentum map of section 2. Following the construction of the S^1-momentum mapping, we look for a group acting on $T^*\mathbf{R}^2$ which properly contains S^1 and has Hamiltonian vector fields for its infinitesimal generators. In contrast to the S^1 case, the action on $T^*\mathbf{R}^2$ we find will *not* be a *lift of an action* on the configuration space $\mathbf{R}^2$. This is reflected in the observation that

some of the new integrals are not linear in the momenta, see chapter VII §5. Consider the *linear action* of the *unitary group*

$$\mathrm{U}(2) = \left\{ \mathrm{U} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in \mathrm{Gl}_4(\mathbf{R}) \;\middle|\; \begin{array}{c} a^t a + b^t b = I_2 \;\&\; a^t b = b^t a \\ a, b \in \mathrm{gl}_2(\mathbf{R}) \end{array} \right\}$$

on $\mathbf{R}^4 = T^*\mathbf{R}^2$ defined by $\Psi : \mathrm{U}(2) \times \mathbf{R}^4 \to \mathbf{R}^4 : \big(U,(x,y)\big) \mapsto U \begin{pmatrix} x \\ y \end{pmatrix}$. Since

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}^t \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix},$$

Ψ_U is a linear symplectic map, that is, $\Psi_U^* \omega = \omega$. For $u = \left(\begin{smallmatrix} A & -B \\ B & A \end{smallmatrix}\right) \in \mathrm{u}(2)$, the infinitesimal generator Y^u corresponding to u is the vector field

$$Y^u(x,y) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \Psi_{\exp su} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

with flow $\psi_t^u : \mathbf{R}^4 \to \mathbf{R}^4 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \exp tu \begin{pmatrix} x \\ y \end{pmatrix}$. Since the matrix Y^u is infinitesimally symplectic, the vector field Y^u is linear Hamiltonian with Hamiltonian function

$$J^u : \mathbf{R}^4 \to \mathbf{R} : (x,y) \mapsto \tfrac{1}{2}\omega\big(u(x,y)^t, (x,y)^t\big). \tag{23}$$

For fixed $(x,y) \in \mathbf{R}^4$, the function $u \mapsto J^u(x,y)$ is linear. Therefore by duality it makes sense to define the mapping $J : T^*\mathbf{R}^2 \to \mathrm{u}(2)^*$ by setting $J(x,y)u = J^u(x,y)$.

▷ An important property of J is that it intertwines the linear action of $\mathrm{U}(2)$ on $\mathbf{R}^4$ with the coadjoint action of $\mathrm{U}(2)$ on $\mathrm{u}(2)^*$, the dual of the Lie algebra $\mathrm{u}(2)$.

(3.3) **Proof**: This is a consequence of the calculation

$$\begin{aligned} J(U(x,y)^t)u &= J^u\big(U(x,y)^t\big) = \tfrac{1}{2}\omega\big(uU(x,y)^t, U(x,y)^t\big) \\ &= \tfrac{1}{2}\omega\big(U^{-1}uU(x,y)^t, (x,y)^t\big), \quad \text{since } U \text{ is symplectic} \\ &= J(x,y)\big(U^{-1}uU\big) = J(x,y)(\mathrm{Ad}_{U^{-1}}u) \\ &= \big(\mathrm{Ad}^t_{U^{-1}} J(x,y)\big)\, u. \end{aligned}$$

□

Therefore J is the $\mathrm{U}(2)$-*momentum mapping* corresponding to the linear action Ψ, see chapter VII §5. If we identify $\mathrm{u}(2)^*$ with $\mathrm{u}(2)$ using the *Killing metric* k defined by $k(u,v) = \frac{1}{2}\mathrm{tr}\, u\bar{v}^t$, then the momentum mapping intertwines the linear action of $\mathrm{U}(2)$ on $\mathbf{R}^4$ with the adjoint action of $\mathrm{U}(2)$ on its Lie algebra $\mathrm{u}(2)$. Identifying $\mathrm{u}(2)$ with $\mathbf{R}^4$ by choosing the basis $\{E_i\}$, the $\mathrm{U}(2)$-momentum mapping J becomes the mapping

$$J : \mathbf{R}^4 \to \mathbf{R}^4 : (x,y) \mapsto \big(W_1(x,y), W_2(x,y), W_3(x,y), W_4(x,y)\big).$$

▷ Thus the components of the $\mathrm{U}(2)$-momentum mapping J are quadratic integrals of the harmonic oscillator.

(3.4) **Proof**: This is just the content of the equations $W_i(z) = \frac{1}{2}\omega(E_i z, z)$ for $i = 1, \ldots, 4$. An alternative argument starts by observing that every element $U \in \mathrm{U}(2)$ when written as

$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ is also an element of $\mathrm{SO}(4)$, the group of all orientation preserving linear *isometries* of $(\mathbf{R}^4, \langle\,,\,\rangle)$. Consequently, Ψ_U preserves H and so $0 = L_{Y^u}H = -L_{X_H}J^u$. □

Now consider the 2-torus

$$T^2 = \left\{ \begin{pmatrix} ac & -ad & -bc & bd \\ ad & ac & -bd & -bc \\ bc & -bd & ac & -ad \\ bd & bc & ad & ac \end{pmatrix} \in \mathrm{U}(2) \;\middle|\; \begin{matrix} a^2+b^2=1 \\ c^2+d^2=1 \end{matrix} \right\},$$

which is an abelian subgroup of $\mathrm{U}(2)$. Restricting the $\mathrm{U}(2)$ action on $\mathbf{R}^4$ to a T^2-action gives rise to a momentum mapping $j : T^*\mathbf{R}^2 \to (t^2)^* = \mathbf{R}^2$ where $(t^2)^*$ is the dual of the Lie algebra t^2 of T^2 and $j(x,y) = \big(W_4(x,y), W_1(x,y)\big)$. In other words, j is the S^1 energy momentum map $\mathscr{EM}$ studied in section 2. Therefore the $\mathrm{U}(2)$-momentum mapping J is a proper extension of $\mathscr{EM}$.

4 The Hopf fibration

In this section we study the qualitative properties of the *Hopf mapping*

$$\mathscr{H} : \mathbf{R}^4 \to \mathbf{R}^4 : (x,y) \mapsto \big(w_1(x,y), w_2(x,y), w_3(x,y), w_4(x,y)\big), \tag{24}$$

where

$$\begin{aligned} w_1 &= W_1 = x_1x_2 + y_1y_2 & w_2 &= W_2 = x_1y_2 - x_2y_1 \\ w_3 &= W_3 = \tfrac{1}{2}(y_1^2 + x_1^2 - y_2^2 - x_2^2) & w_4 &= W_4 = \tfrac{1}{2}(y_1^2 + x_1^2 + y_2^2 + x_2^2). \end{aligned}$$

The *Hopf variables* w_i, $i = 1, \ldots, 4$ satisfy the relation

$$C(w_1,w_2,w_3,w_4) = w_1^2 + w_2^2 + w_3^2 - w_4^2 = 0, \quad w_4 \geq 0. \tag{25}$$

Therefore the image of the Hopf map is contained in the *semialgebraic variety* $\mathscr{C}$ defined
▷ by (25). Topologically $\mathscr{C}$ is a cone on S^2 with vertex at 0. To show that $\mathscr{C}$ is the image of the Hopf map, it suffices to verify that $D\mathscr{H}(x,y) : T_{(x,y)}\mathbf{R}^4 \to T_{\mathscr{H}(x,y)}\mathscr{C}$ is surjective for all $(x,y) \neq (0,0)$, since $\mathscr{H}(0) = 0$ and $\mathscr{C} \setminus \{0\}$ is a smooth three dimensional manifold.

(4.1) **Proof**: When $(x,y) \neq (0,0)$ the derivative

$$D\mathscr{H}(x,y) = \begin{pmatrix} x_2 & x_1 & y_2 & y_1 \\ y_2 & -y_1 & -x_2 & x_1 \\ x_1 & -x_2 & y_1 & -y_2 \\ x_1 & x_2 & y_1 & y_2 \end{pmatrix}$$

has rank ≥ 3 because its first three rows are nonzero and pairwise orthogonal. It has rank ≤ 3, since

$$\mathrm{im}\, D\mathscr{H}(x,y) \subseteq T_{\mathscr{H}(x,y)}C = \ker DC(\mathscr{H}(x,y)) = \ker(w_1,w_2,w_3,w_4)$$

and (w_1,w_2,w_3,w_4) is nonzero. □