Christoph J. Scriba
Peter Schreiber

5000 Years of Geometry

Mathematics in History and Culture

Birkhäuser

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Mathematics in History and Culture

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Christoph J. Scriba
University Hamburg
Hamburg, Germany

Peter Schreiber
University Greifswald
Greifswald, Germany

Translated by Jana Schreiber

Editor: Project Group "History of Mathematics" of Hildesheim University
H.W. Alten, K.-H. Schlote, H. Wesemüller-Kock

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Preface of the editor of the German edition

Geometry (from the Greek word for 'measuring the Earth', the modern scientific discipline of which is now called geodesy), branch of science which deals with regular patterns, shapes and solids, was one of the first human attempts, after counting, to concern themselves with the emerging science mathematics. This is evident from the spirals on megalithic graves, incisions in stone and patterns on clay fragments.

In this book, you will learn how geometry has developed over the millennia from these earliest origins in distant times and much more. Geometry is an indispensible aid for building and surveying, and became an axiomatic science of plane and spatial shapes in Ancient Greece. It served as a basis for astronomical observations and calculations, for Islamic decorative art, and the building of medieval Christian cathedrals. Furthermore we will look at the discovery of perspective and its application in Renaissance art, at the disputes regarding the Euclidean parallel postulate, the discovery of non-Euclidean geometries in the 19^{th} century, and, finally, the theory of infinite-dimensional spaces and contemporary computer graphics.

This book is edited by the project group "History of Mathematics" at the University of Hildesheim as part of the series *Vom Zählstein zum Computer* (From Pebbles to Computers). Other titles in this series published by Springer Publishing Heidelberg are: *4000 Jahre Algebra* (4000 Years of Algebra) [Alten et al. 2003], and *6000 Jahre Mathematik* (6000 Years of Mathematics) [Wußing, in two volumes 2008/09]. To the series 'From Pebbles to Computers' two video films have been produced (University of Hildesheim): '*Mathematik in der Geschichte – Altertum*' (Mathematics in History – Antiquity) [Wesemüller-Kock/Gottwald 1998] and '*Mathematik in der Geschichte – Mittelalter*' (Mathematics in History – Middle Ages) [Wesemüller-Kock/Gottwald 2004]. Following multiple reprints and the second edition in 2004 we now present the third edition of *5000 Jahre Geometrie* including new research results on circular ditches in the Stone Age and the Nebra Sky Disk, as well as many illustrations in colour.

In this book, we will reflect on the development of geometry as part of our cultural history over the course of five millennia. Both authors have succeeded in portraying the origins and growth of this branch of mathematics, which is often thought of as dry and jejune, in a tremendously lively manner. They uncover the origins and impulses for the development of geometric notions and methods, and present how they are related to historical events and personal fates. Moreover, they describe the applications of geometrical knowledge and methods in other areas and the interdependencies that resulted from them. Finally, they emphasize their importance for other disciplines.

At the heart of this book series is portraying the history of mathematics as an integral part of the history of mankind, particularly as a fundamental part of our cultural heritage. Both authors have done justice to this task in an impeccable manner. They have depicted the genesis of geometry and its interlacing with cultural developments in other areas, such as literature, music, architecture, visual arts and religion, by a standard far higher than usual in mathematical-historical presentations. They also describe the implications of geometrical findings and methods for other areas. As such, the authors also deal far more extensively than usual with the development of geometry in other cultures, mainly in the ancient oriental cultures, in Islamic countries, as well as in India, China, Japan and the old American cultures. Tables at the beginning of each chapter give an overview of important political and cultural events of each cultural area and era dealt with. Tables at the end summarise the main geometrical contents of each chapter in note form.

Moreover, the authors compare views of ancient and medieval mathematicians with modern mathematical findings and link those to contemporary mathematics and related sciences, for example, references to computer sciences regarding the description of Euclid's "algorithmic accomplishment". Furthermore, they highlight the specifications of geometrical examinations of different eras and cultural areas and the changes in content, methods and approaches geometry has faced as a proto-physics within three-dimensional or even infinite-dimensional spaces. They discuss the relationship of geometry with other branches of mathematics, for instance with algebra, analysis, and stochastics. Refreshing asides with biographical highlights and references to unexpected relations, as well as text excerpts in the appendix, bring this book to life.

Chapters 1 through 4, with the exception of sub-chapter 2.3 (Euclid), were written by Dr. Christoph J. Scriba, professor emeritus for the history of the natural sciences in the former Institute for History of Natural Sciences, Mathematics and Engineering at the University of Hamburg. Euclid's accomplishments and the development of geometry in modern times from Chapters 5 through 8 were described by Dr. Peter Schreiber, professor for geometry and the foundations of mathematics at the University of Greifswald.

We are also grateful to the authors for numerous illustrations and the texts for the appendixes. The figures that have been added to support geometrical theorems that are not referenced were drawn by the authors themselves. They also thought of the summarising problems for every sub-chapter at the end of each chapter (cf. Introduction). They often differ from ordinary tasks in regard to type and size and also vary in level of difficulty. Thus, solving them requires very different background knowledge, as well as the use of secondary literature at times. Hence, to solve some of the problems of Chapters 1 through 4, you will mainly need knowledge gained in junior high school, while other problems will require highschool knowledge, whereas some problems to Chapters 5 through 8 demand insight into notions and methods taught at university. This is due to the nature of the subject, since mathematics has grown more and more complex and difficult over the course of the centuries and understanding modern mathematics usually assumes knowledge of the mathematics of past eras. Therefore, you will occasionally find hints to solutions within the text and also the literature. However, the solutions themselves have not been included in the appendix to avoid the following: first, we do not want you to look up the solutions too quickly; second, the solutions most often are not the result of calculations, but require the description of approaches for solving the problem at hand or retracing more or less extensive considerations.

All this has been done intentionally in order to attract as large a readership as possible. Cursory readers or those that are in a hurry should not simply skip the problems, since they include many interesting historical remarks and additions to the text, which is why reading the problems carefully will benefit everyone. The extensive bibliography and index of names invite the reader to study further.

I thank both authors sincerely for the multifaceted and intensive work in particular their dedication to setting new accents with this book integrating geometry in cultural history and composing many interesting problems.

I further express my gratitude to my colleagues Dauben, Flachsmeyer, Folkerts, Grattan-Guinness, Kahle, Lüneburg, Nádeník und Wußing for their scholarly advice and critical reviewing and thank H. Mainzer for advice on historical details and Lars-Detlef Hedde (University of Greifswald), Thomas Speck and Sylvia Voß (University of Hildesheim) for converting the manuscripts, illustrations and figures into printable electronic formats.

Moreover, I wish to thank media educator Anne Gottwald, who helped us clear the licensing for printing the illustrations, and each publisher for authorising the printing rights.

I also remain grateful to the director of the Centre for Distance Learning and Extension Studies (ZFW), Prof. Dr. Erwin Wagner, the present and former directors of the Institute for Mathematics and Applied Computer Science, Prof. Dr. Förster and Prof. Dr. Kreutzkamp, the deans Prof. Dr. Schwarzer and Prof. Dr. Ambrosi and the administration of the University of Hildesheim.

Last but not least, I wish to thank the members of the project group “History of Mathematics” of ZFW: the historian of mathematics Dr. Alireza Djafari Naini and the media expert and sociologist Heiko Wesemüller-Kock, for the great and intensive teamwork while planning and preparing this book. I express my gratitude to Springer Publishing Heidelberg for taking my requests into account and the excellent design of this book.

I hope that this volume will inspire many readers to study the history of mathematics more intensively, and to learn about the background of the origins and incredibly exciting development of geometrical notions and methods. Hopefully, this will result in the reader viewing geometry not just as a mathematical discipline or as an indispensible aid for architects, robot engineers and scientists, but also as a valuable part of our culture that we encounter everywhere and that makes the world in which we live so much richer.

On behalf of the project group

Hildesheim, August 2009 Heinz-Wilhelm Alten.

Preface of the editor of the English edition

This is the first volume of our series 'From Pebbles to Computers' that appears in English. It is a translation of the 3^{rd} edition (2010) of *5000 Jahre Geometrie*, again updated, supplemented and enriched with many illustrations by the author P. Schreiber and the editors H.-W. Alten, K.-H. Schlote, and H. Wesemüller-Kock.

Meanwhile the book *4000 Jahre Algebra* appeared 2014 in its 2^{nd} edition, in 2011 *3000 Jahre Analysis* was published by Springer Berlin Heidelberg, and we are now preparing its translation *3000 Years of Analysis* to be published by Springer Basel. Some other volumes of this series will also be published in English. Besides the film 'Mathematik in der Geschichte – Mittelalter' has now been produced in English as *History of Mathematics – Middle Ages.*

All of us have been affected by the death of our author Prof. Dr. C. J. Scriba in 2013. We are grateful for his support over many years and glad to be able to present Chapters 1 through 4 of this book with the supplements he wrote before his death as part of his scientific legacy. We shall miss his advice in future.

The translation of this book was done by Jana Schreiber, the daughter of the author P. Schreiber. We are very grateful to her because she has done this with great efficiency and commitment in a short time.

After the corrections and supplements of the editor, language copy editing by the publisher, and proof reading by the author and editor we now present this volume.

I thank the members of the project group for their intensive teamwork: A. K. Gottwald for clearing the licenses (now world-wide) for printing the illustrations, H. Wesemüller-Kock for his involvement inserting new illustrations with his comments and the index of illustrations, proof reading and preparing the graphic design and layout for the whole book, the historian of mathematics Dr. K.-H. Schlote for many comments and for transferring the index of names and the subject index, Prof. Dr. K.-J. Förster and Prof. Dr. E. Wagner for providing financial support.

We are grateful for the help of our secretaries B. David and R. Falso, the students J. Schönborn and N. Westphal for preparing the text, illustrations and indexes ready for printing.

Last, but not least we thank Springer Publishing Basel AG and its editor Dr. A. Mätzener for her kindly support and the excellent layouting of this book.

I hope that this book will please, inspire and benefit many readers all over the world.

On behalf of the project group

Hildesheim, August 2014 Heinz-Wilhelm Alten.

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Introduction

It is certainly not easy to define the content and nature of mathematics briefly. Formal explanations, which are possible nowadays due to the general notion of structure and other logical notions, neglect not only the historical development, but also the instinct and experience of a mathematician, who knows what is "substantial" and "interesting" and what is not. However, within the given understanding of mathematics, it is even more complicated to explain what geometry is and what, in turn, is part of its history. The dominant views on the subject of geometry, as well as its position and meaning within mathematics, have not only changed repeatedly over the course of time. As mathematics became increasingly sophisticated, mathematicians also took opposing positions while trying to find answers to these questions. We will look at all these aspects in this book.

Even though geometry was mainly considered as one application of a primarily arithmeticaly oriented mathematics amongst many others in the earliest cultures (such as in ancient Egypt, Mesopotamia, India, China, etc.), it became the core and main interest of mathematics in Ancient Greece. It was there and then that vague notions and procedures justified only by trial and error were transformed into a theory with definitions, axioms, theorems and proofs. The heritage stemming from this period was so powerful for over two thousand years that mathematicians were usually called geometricians. Furthermore, the axiomatic-deductive method of cognition assurance, which was based on the Greeks' methods of dealing with geometric matters, was referred to as "mos geometricus", and the implementation in other sciences, including other realms of mathematics, "more geometrico", in other words 'in geometrical fashion', became a rarely achieved scientific and theoretical program. This agenda influenced, for example, Newton in the 17^{th} century, as he re-founded mechanics, Galois at the beginning of the 19^{th} century, when criticising the contemporary situation of algebra, and Hilbert, while encouraging the scientific community to axiomatise further branches of physics in his famous speech in 1900.

As a result of the European Renaissance, geometry was flooded by an extraordinary wave of inspiration and applications in the fields of astronomy, geodesy, cartography, mechanics, optics, architecture, visual arts and, hence, leading to a wealth of new challenges. The efforts made to solve these new challenges essentially led to the development of the four pillars of the "modern" mathematics in the 17^{th} century. These pillars are: the concept of function, coordinate-systems, differential calculus and integral calculus. Geometry gave birth to these pillars, and then was superseded and lost its leading position to them in a very subtle manner. Formulae and calculus took over increasingly in the 18^{th} century and pushed visualisation and logical argumentation aside.

The 19^{th} century led to an enormous growth in the size and meaning of geometry. Projective, descriptive and n-dimensional geometry, vector calculus, non-Euclidean geometry, intrinsic differential geometry, topology, and also numerous “buds” in other areas that would only come to blossom in the 20^{th} century, such as geometrical probability and measure theory, graph theory and general polyhedral theory, began developing at first without any recognisable relationship to one another. This “explosion” of geometrical disciplines, which led to the century being named the “geometrical century” according to mathematicians, was accompanied by the disintegration of the then dominant understanding of geometry as a science of “true physical space”. We will look at how the different approaches for dealing intellectually with the new situation in geometry crucially coined the whole view of mathematics that was dominant until the invention of the computer and its rising popularity. However, we must also examine how geometry lost its central position within mathematics over the course of the first half of the 20^{th} century. This has been a development that still negatively influences the organisation of mathematics in secondary and further education nowadays, despite the fact that geometry has achieved a higher than ever level in regards to its theoretical width and depth as well as its practical significance.

At the end of the 20^{th} century, geometry was, on the one hand, a huge pool of facts on the “ordinary two and three dimensional Euclidean space” and an even bigger pool of unanswered questions on those. On the other hand, geometry was not really thought of as being part of mathematics in the ordinary sense nowadays, but rather considered a way of thinking, which is more or less useful and necessarily found in almost every realm of mathematics, depending on the scientist’s personal approach. Thus, there is a geometrical theory of numbers, a geometrical theory of functions, algebraic geometry and geometrical stochastics. There are geometrical methods within variational calculus, discrete and combinatorial geometry, as well as computer geometry. The latter is not to be confused with computational geometry, which basically refers to a “theory of complexity of geometrical algorithms”.

The dichotomy of geometry suggested here has established itself very well in the meantime. The three dimensional Euclidean space remains the appropriate model for all “ordinary” problems, even though it is only a very rough approximation of reality according to the findings of physics. Within the Euclidean plane we create “pictures” of everything we want to “look at” and understand. Their meanings are associated with the dominance of seeing amongst the human senses. Inside the n-dimensional Euclidean space, mathematics embeds functions, relations and, almost all other examined objects by using coordinates, for example. Furthermore, geometry predominates in all those areas where a number of possibly very abstract objects are viewed as a “space” by using in broader sense terms taken from geometry, such as topology, metrics, dimension and linearity, with the intention of inspiring our imagination and to use analogies. To what extent one may want to practise this is – as already pointed out – a matter of style. It is an intellectual technique, without which modern mathematics in the form described here could not have developed.

To what degree the latter can really be considered geometry and to what extent the applied branches of geometry belong to mathematics or are already part of engineering is debatable. In the following, we will also defend the concept that there is an "unconscious" unprofessional mathematics that coexists with professional, deductive mathematics. The former manifests itself in the intuitive use of notions, shapes, methods, knowledge and know-how, which is difficult to put into words, but exists as a material product of engineering, handcrafts and the arts. Hence, this book will also serve as a reflection on the historical development of geometry, which will include many, often unusual aspects. We intend to contribute to the clarification of the position and meaning of geometry within mathematics and to raise interest in it.

The critical reader, that we would like to have, may pose the question how a history of geometry fits in a series called '*From Pebbles to Computers*'. What computers have to do with geometry is investigated in detail in Chapter 8.5. With regards to 'pebbles' (accounting tokens) we refer to the Pythagoreans, who got some simple pre-numbertheoretical results from patterns of geometrically ordered stones. Thus they could realize why ab is forever equal to ba and why the distance between two square numbers n^2 and $(n+1)^2$ is always $2n+1$.

This book features problems added chapter by chapter, most of which are not historical problems strictly speaking, but problems that result from the history presented here. For instance, questions without answers when they first occurred; questions that just simply did not come to mind, but were possible; old problems that nowadays are much easier to solve given modern methods; and suggestions that result from old problems. Most of the problems are reduced to special cases, contain hints or are asked in a manner that will require only a highschool or slightly more advanced mathematical background to be solved. However, a few questions are more difficult and "open-ended". Here, the reader is invited to probe and explore.

We have avoided the use of first names and the inclusion of the dates births and deaths within the main text apart from a few, well-reasoned exceptions. As far as we could determine those data, they are available in the index of names at the end of the book.

The pictures of the people at the beginning of each chapter are of different styles. We cannot rely on authentic portraits from antiquity or the non-European Middle Ages. (One reason being that people in Islamic countries were often not portrayed due to religious reasons.) However, we must acknowledge that later eras felt the necessity to make pictures of their most important personalities. In this book, a "picture" can be an imagined portrait or a symbolic graphic representation. In this respect, stamps can also serve as a cultural document of the history of science. Multiple books have been devoted to this exact subject [Gjone 1996, Schaaf 1978, Schreiber, P. 1987, Wußing/Remane 1989]. For example, a picture of Euclid (not shown here) was taken from a manuscript of Roman field surveyors (agrimensores). Here, two things are striking. First, these agrimensores thought of Euclid, the master of the logical-axiomatic approach, as their forefather, and, second, the picture has an almost oriental ambience. Considering the mix of peoples and cultures in Alexandria at 300 BC, this may appear more realistic than some neo-classically influenced pseudo-antique art.

From the European Middle Ages onwards, portraits began to appear intentionally more similar to the individual persons, as artists started relying on themselves as models. For example, the portrait of Piero della Francesca is an alleged self-portrait. It comes from his Fresco “Resurrection” (around 1465) located in his hometown of Borgo Sansepolcro.

The picture of René Descartes presented here was painted by Frans Hals shortly before the philosopher departed for Sweden. It is not only one of the very few cases in which a genuinely famous painter portrayed a genuinely famous mathematician (a second example is the portrait of Felix Klein painted by Max Liebermann), but multiple copies of this painting were subsequently made in the 17^{th} century reflecting varying facial expressions, which since then partially even flipped horizontal have haunted encyclopaedias and the science-historical literature as images of Descartes.

Peter Schreiber

Advice for the reader

Round brackets (...) contain additional insertions, translation of original titles or information on illustrations or problems.

Square brackets [...] contain information on literature within the text, explanations or references below illustrations.

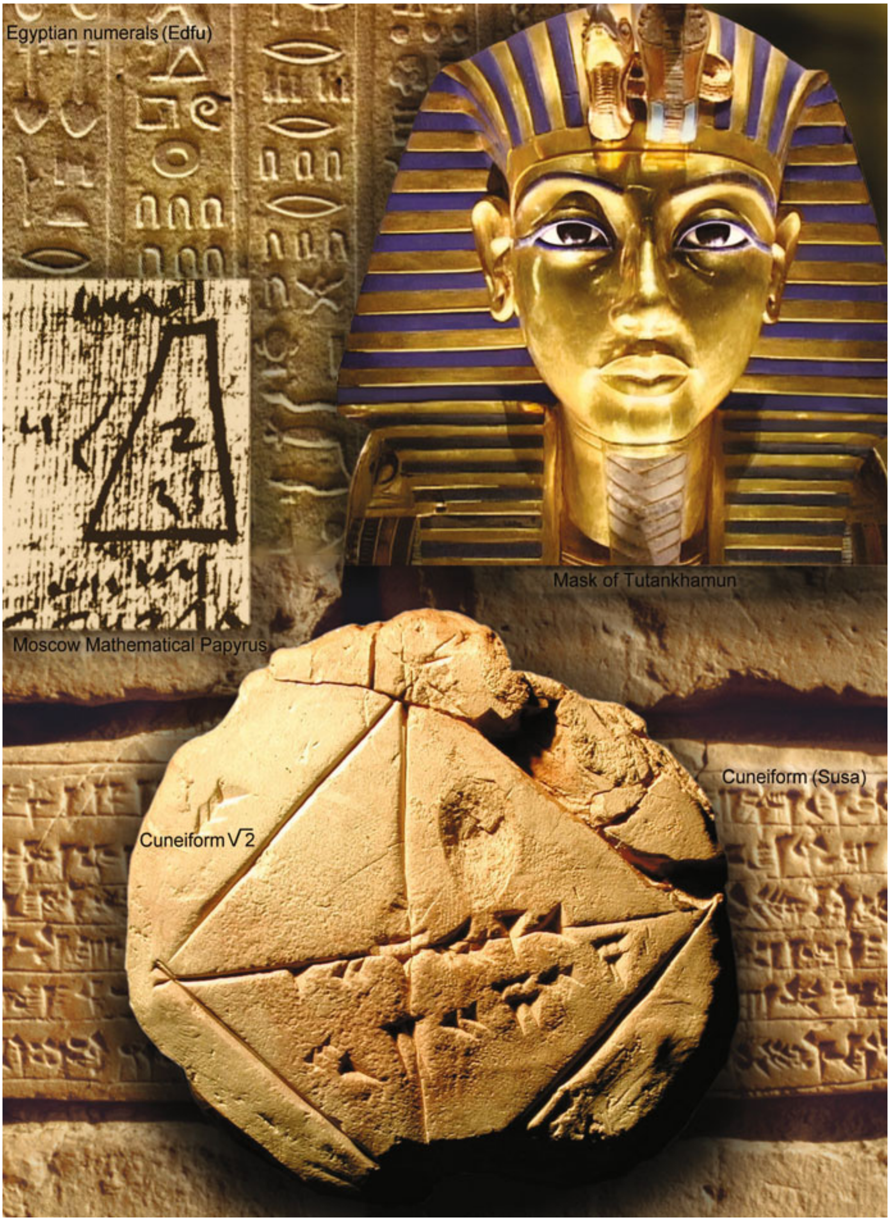
Illustrations have been numbered according to sub-chapters, e.g. illustration 7.4.3 is the third illustration of Part 4 of Chapter 7.

In order for the reader to find related texts more easily, problems have been summarised at the end of each chapter and been numbered according to sub-chapters, e.g. problem 7.3.6 is the sixth problem of Part 3 of Chapter 7.

The problems are of different sizes and vary in level of difficulty. Problems or partial problems, which the publisher believes to be especially challenging, have been marked with an*. However, we would like to point out that such a judgement is clearly subjective and depends on the reader’s individual knowledge and skills.

The kind of quotations and the references follow the style of the author P. Schreiber.

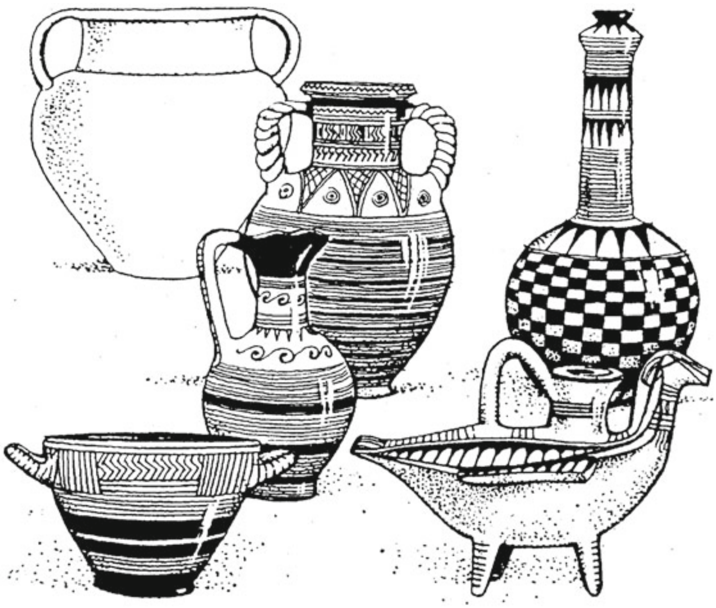
1 The beginnings of geometrical representations and calculations



C.J. Scriba, P. Schreiber, *5000 Years of Geometry*,
DOI 10.1007/978-3-0348-0898-9_1

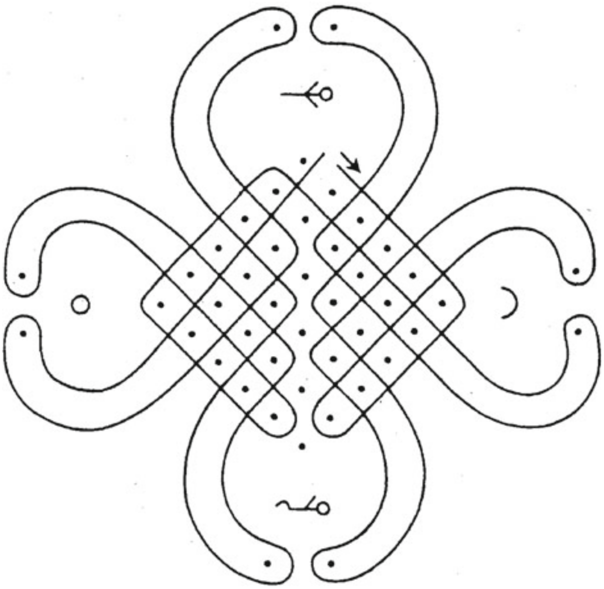
1.1 Primal Society

Long before writing was developed, mankind may have realised and systematically used geometrical structures. Nature offers the eye multiple curved lines, and a blade of grass or a tree trunk can symbolise the thought of a straight line as well as the idea of a circle (as a cross-section). When weaving or braiding we generate simple two-dimensional patterns, which then are purposely modified or also replicated as decoration on clay pots. There is evidence that such purposely geometrically shaped ornaments existed already in 40 000 BC. They can be so characteristic for such cultural societies that prehistorians can reconstruct their migrations by digging up and analyzing clay fragments. For instance, we can find in the Cretan culture patterns of folded strips on Neolithic clay pots, or six congruent circles, aligned around a central circle of the same size and touching each on two neighbouring circles. The equilateral triangle, the square (with four right-angled corners) or also the regular hexagon must have been noticed very early as special cases of plane shapes, awakening playful interest as well as first theoretical considerations (cf. e.g. [Kadeřávek 1992]).



Illus. 1.1.1 Geometrical ornaments on prehistoric ceramic
[Drawing by Hubert J. Pepper from “The Dawn of Civilization” edited by Stuart Piggott, Thames and Hudson Ltd., London]

Needs and activities of everyday life provided further inspiration: when constructing ditches, dams or houses, and land surveyingelementary geometrical ratios were required. Men probably did not realise this at first until their first logical considerations set in. Without three-dimensional solids (cuboids, cubes, pyramids, columns) building was impossible. Observing the course of the stars suggested a transition from the plane triangle to the spherical triangle. It seemed to be obvious that the diagonal bisects the square or rectangle as does the diameter the circle. All pre-Greek cultures have been aware of such immediately insightful relations and applied them in practise. Only the Greeks started probing and asking for reasons. They finally arrived at an axiomatic construction of a geometric theory that has been passed down to us by Euclid's 'Elements'. If we want to focus primarily on Egyptian and Babylonian geometry in the following, we must emphasize that there is no culture that does not reflect the versatile use of geometrical elements. Designing jewellery is often heavily influenced by religious ideas: Pots devoted to the gods would feature more abundant decorations, the altars would feature special shapes and rituals (including dances) which would be conducted in a special manner. We also must not neglect play as a source of engaging with geometrical properties. This goes beyond just board games, which are almost always sources for symmetrical patterns.



Illus. 1.1.2 Single course line concerning the cosmogonic myth of Jokwe in Angola: The course of the sun (left), moon (right) and man (below) to god (above) [Africa counts: Number and Pattern in African Culture, ©1973 by Claudia Zaslavsky. Publ. by Lawrence Hill Books, an imprint of Chicago Review Press Inc.]

Illus. 1.1.3 Goseck circle (near Halle, Germany), Nebra sky disk (State Museum of Prehistory, Halle/Saale) [Photo: H. Wesemüller-Kock]

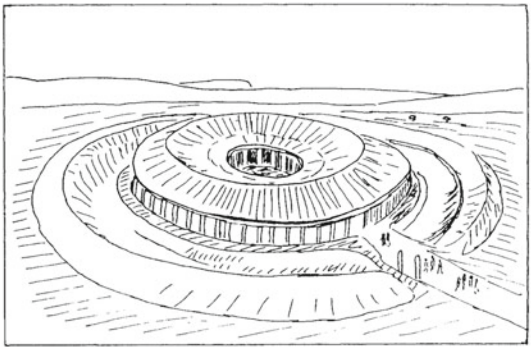
Ethnomathematics, which recently has turned towards the implicit mathematical ideas of primitive people, yields some astonishing research results. For example, there is an African tribe in Angola, whose people draw a shape freehandedly from a single curve, which interlaces elaborately, when telling their cosmogonic myth. This indicates thorough geometrical considerations, if the desired outcome with its symmetrical properties is to be achieved (Illus. 1.1.2). Since we have been aware of its changes, the starry sky has provided men with further inspiration to make basic geometrical observations. The movements of the shadow of a tree trunk or towering stone, taken over the course of a day or a year, form the basis for a simple sundial. Drawing the course of the shadow lace systematically on the ground, the result is a projection of the course of the sun on the sky in plane curves, which encourage us to think about it. In the 1990s, in Goseck (near Halle, Germany) a set of concentric circular ditches, dating back to approx. 4800 BC, was discovered, archaeologically researched and reconstructed. It is the earliest sun observatory currently known worldwide (Illus. 1.1.3). Circular ditches were constructed in Central Europe close to settlements around 4800 to 4500 BC. Goseck's circle features a dual ring of palisades with three gates, one each facing north, southeast (sunrise on 21 December) and southwest (sunset on 21 December). The distance between the palisades grows wider around 21 June. This configuration allowed farmers of 7000 years ago to determine, by means of position of the sun, the most propitious times to sow and harvest over the course of the year. However, as findings indicate, circular ditches were also used for cultural purposes. Only about 2000 years later the most famous construction of the megalithic culture (3^{rd} and 2^{nd} millennium BC), Stonehenge near Salisbury in the south of England, was erected, which has been interpreted as a sun observatory and a cult site [Gericke 1984], (Illus. 1.1.4).

Research of the last decades has shown that Stonehenge not only reflects use of astronomical knowledge but also basic geometrical ratios, e.g. Pythagoras's theorem. However, we can only assume that the Pythagorean triangle with side lengths 3, 4, 5 (for instance, it is possible to mark them with knots on a rope of length 12) was used that early to generate right angles. Researchers

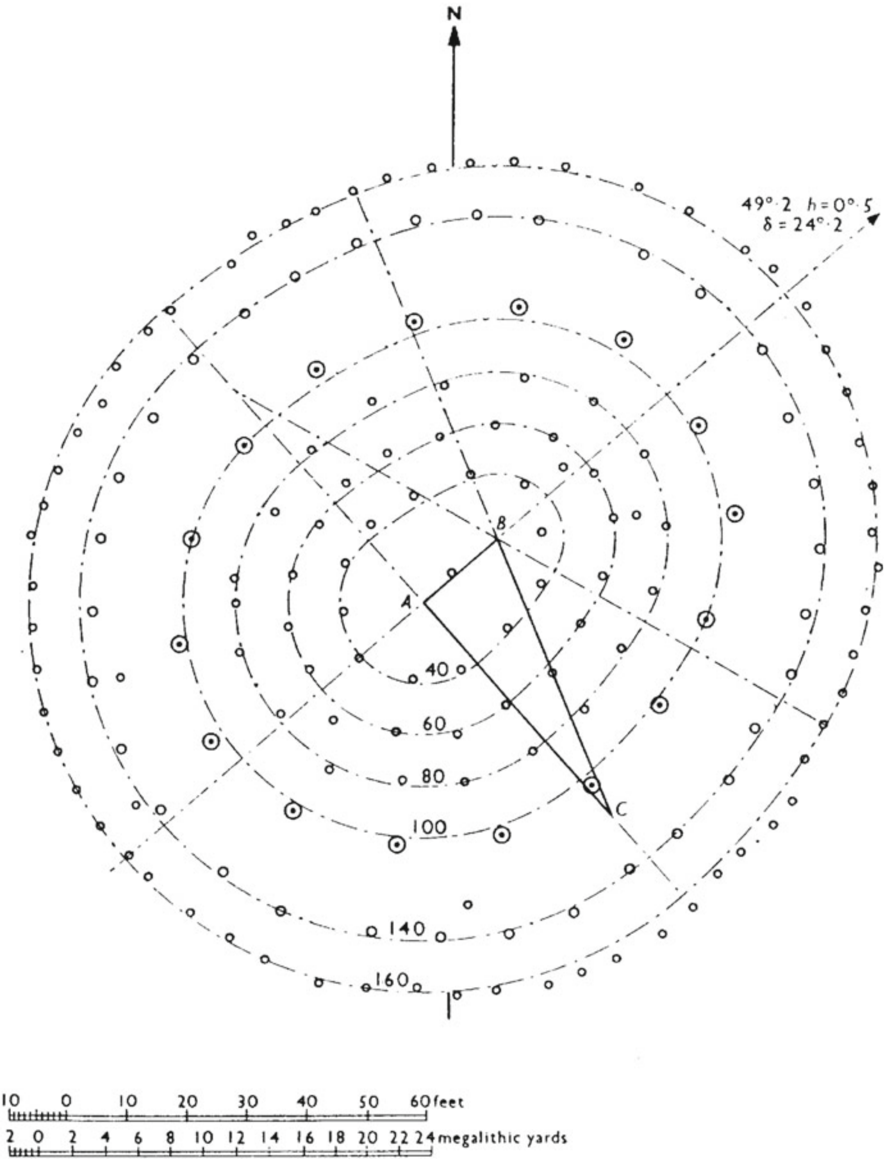
Illus. 1.1.4 Stonehenge (South England): The biggest preserved stone monument in Europe from the 3rd/2nd millennium (diameter of outer ring approx. 100 m) [Photo: H.-W. Alten]

argue that they can prove that the wood construction of Woodhenge (approx. 1800 BC) was built by applying the Pythagorean triangle 12, 35, 37 (Illus. 1.1.5, 1.1.6).

For Stonehenge see [North 1996]; for a critique on the hypothesis of the right angle view [Knorr 1985]. The bronze Nebra sky disk, found just recently near Halle (Germany), comes from approx. the same time as Woodhenge. Its constellation of the stars with the Pleiades is taken to be the first sky representation [Schlosser 2004]. This disk has been the source of lively debates in regards to theories of interpretation and meaning, whose final outcomes are expected in the near future.



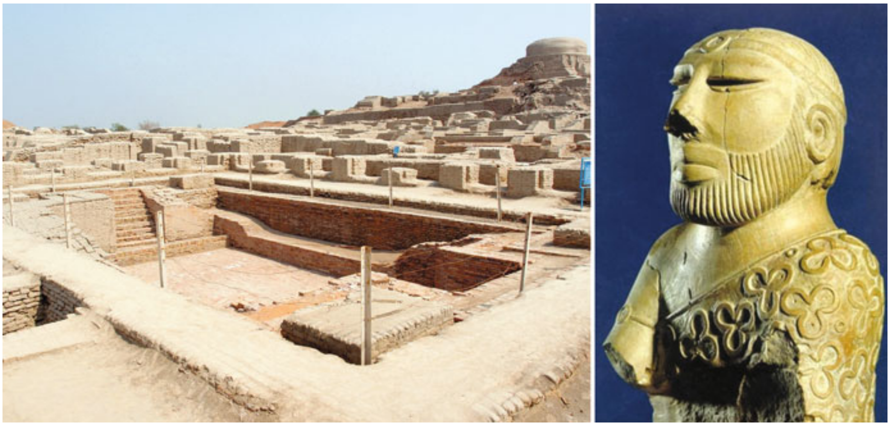
Illus. 1.1.5 Reconstruction of Woodhenge
[Ashbee, P.: The Bronze Age Round Barrow in Britain, Phoenix House Ltd, London 1960]



Illus. 1.1.6 Ground plan of Woodhenge
[Thom, A.: Megalithic Sites in Britain, Oxford, Clarendon Press 1967, Fig. 6.16 p. 74, by permission of Oxford University Press]

1.2 Old river valley civilisations

3000-2000	Town civilisations at the Indus valley: Harappa and Mohenjo Daro	Script not yet deciphered
3000-2700	Union of kingdoms at Nile	Hieroglyphs invented
3000-2700	Sumerian city states	Cuneiform on clay tablets developed
2700-2170	Old kingdom in Egypt	Pyramids built
2700-2100	Akkadian invasion and reign	Nomographs
2170-2040	First intermediate period of Egypt	
2040-1794	Middle kingdom in Egypt	Mathematical papyri
2100-1900	Several kingdoms in Mesopotamia	
1900-1600	Old Babylonian kingdom	
1728-1668	King Hammurabi in Babylon	Tablets of law
1794-1550	Second intermediate period of Egypt	
1550-1070	New kingdom in Egypt	Temple of Hatschepsut
1290-1224	Pharaoh Ramses II	Amun temple in Karnak
1285	Battle of Kadesh	Graves in Valley of the Kings
1600-625	Hittites, Kassites, Assyrians rule in Mesopotamia	Mathematical scripts in cuneiform
1070-525	Late period in Egypt: Libyans, Ethiopians, Assyrians rule at Nile	
625-539	New Babylonian kingdom	Astrology and astronomy prosper
539	Cyrus the Great conquers Babylon	
525	Persians conquer Egypt	
332	Alexander the Great conquers Egypt	
323-30	Egypt reigned by Ptolemy Dynasty	Egypt trade and cultural centre of the world Eratosthenes of Cyrene director of Library, Euclid and Apollonius in
47 BC	Library of Alexandria on fire	Alexandria
30 BC	Egypt becomes Roman province	Hero of Alexandria
391 AD	Library of Alexandria is destroyed	Pappus and Proclus work in Alexandria Mathematician Hypatia murdered by pagan persecution
395	Egypt becomes part of the Eastern Roman Empire (Byzantium) when the Roman Empire is divided	



Illus. 1.2.1 Mohenjo-Daro. Exvacated ruins of one of the largest Settlements of the ancient Indus Valley Civilisations [Photo: Saqib Qayyum, 2014]; stone statue of a 'Priest-King', found in 1927 AD in Mohenjo-Daro (National Museum, Karachi, Pakistan) [Photo: Mamoon Mangal]

1.2.1 Indus civilisations

One of the oldest advanced civilisations of mankind is the settlement Mohenjo-Daro at the Indus. The town belonging to the Harappa culture with approx. 40 000 inhabitants experienced its heyday around 2500 BC. It was almost as old as the Egyptian kingdom located along the Nile and Mesopotamia situated between the river valleys of Euphrates and Tigris. In all archaeological sites of this culture, bricks feature the same side lengths with a ratio of 1:2:4, streets follow the outline of a chessboard and weights were standardised. Since excavations and interpretations of the findings of Mohenjo-Daro (located in todays Pakistan) are still continuing, we are not able to reach final conclusions on the role of geometry in this cultural area.

1.2.2 Egyptian mathematics

We have gained better insights into the geometrical knowledge of old Egypt and Mesopotamia (also called Babylonia), since both civilisations have their origins in the Neolithic age, and have left written sources behind, which have been studied in great depth since the middle of the 19^{th} century.

Hieroglyphs had been developed since approx. 2900 BC in the strictly organized and centrally administrated Egypt. Next to the impressive constructions of the pyramids, two mathematical papyri from the time of the middle kingdom (11^{th} to 13^{th} dynasty) have served particularly well as sources for our knowledge of Egyptian geometry. Their content reflects the level of knowl-

Illus. 1.2.2 Egypt and Mesopotamia in ancient times
[Map: H. Wesemüller-Kock]

edge for approx. or shortly after 2000 BC. The two most important ones are the Rhind Mathematical Papyrus and the Moscow Mathematical Papyrus. They constitute collections of problems with relevant approaches to solving them. They seem to be texts, which have been written by teachers (writers) at schools for officials to serve as teaching handbooks. The Rhind Mathematical Papyrus was originally 5.34 m long, but only 33 cm wide. The Moscow Mathematical Papyrus was 5.44 m long, but only 8 cm wide. The latter contains 25, the former 84 problems ordered according to factual aspects, which sometimes feature visualising drawings. Thereby, geometrical solids are represented by their top or side views, since perspective drawing was unheard-of in Egypt of that time. Sometimes the same drawing even demonstrates the most important aspect in a top view and individual parts in front views, e.g. the representation of a rectangular pond with trees on the edge, the trees are folded over to the left side (Illus. 1.2.3).

Relief designs and other wall pictures provide evidence that surveying the ground of a temple was a holy act accompanied by many ceremonies, which only the pharaoh or the highest priests were allowed to carry out. The holy and mysterious aspects of the art of surveying and constructing were reflected by conserved amulets, which have the shape of simple geometric instruments. However, it does not seem likely that they drew the construction and transferred them to the building true to scale. Top and front views of columns and ledges in original size have been found on suitable plane surfaces of stone. Realisations of these can be found in surrounding buildings [Kadeřávek 1992].

Illus. 1.2.3 "Pond in a Garden" Change of perspective in the same picture, fresco from the Tomb of Nebamun, Thebes, c. 1400 BC
[British Museum London, MDID Collection]

One of the simplest geometric problems is the calculation of area A of rectangles, trapeziums and triangles. The approximation formula for any quadrilateral with sides a, b, c, d is

$$A = \frac{(a+c)}{2} \cdot \frac{(b+d)}{2}. \tag{1.2.1}$$

Hence, it entails dual averaging of the opposite sides. Interestingly, this rule has also been applied to a triangle by zeroising the fourth side (better: omitted because not existing, since the Egyptians did not know the concept of the number zero). A peculiar instruction is applied when calculating area A of a circle by means of a given diameter d: deduct 1/9 of its length and multiply the result with itself and the outcome is

$$A = (\frac{8}{9}d)^2. \tag{1.2.2}$$

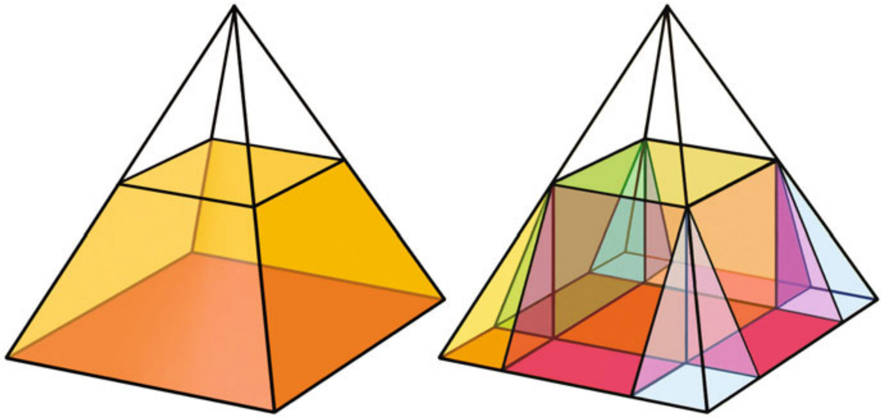
As usual, there is no reason for the astonishingly accurate method. However, problem 48 of the Rhind Mathematical Papyrus contains a drawing showing a square of side length 9, which is turned into an octagon by cutting off the edges. This can be interpreted as a circle approximation. This shape inspired Kurt Vogel in 1928 to interpret the Egyptian instruction (see Problem 1.2.1).

Apart from plane shapes, in Egyptian texts also volumes are calculated, when structurally engineered problems or calculation of the holding capacity of pots and basins are concerned. Hereby, the mention of a layer measure for volumes is remarkable. Similarly, there is a stripe measure for calculating areas. It suggests more of a calculation of the volume of a brick by multiply inserting a layer, which equals its base and whose height constitutes the unit measure, on top of one another (like when making plywood boards), rather than calculating the volume of a brick by means of filling it with unit cubes (since we use the latter method nowadays to multiply length, width and height). All problems are calculated like recipes and only with concrete numerical values. In these early times, men had neither a method to express formulae nor abstract quantities.

When calculating volumes, they mainly dealt with cuboid-shaped or cylindrical containers, whereby the mentioned formula for circular areas was used. The great pyramids suggest that the old Egyptians must have also known the capacity formula for pyramids. However, there is no proof of this. (As proven by Max Dehn in 1900, a strict derivation of this formula for any pyramid is impossible without a limit process. See also Problem 1.2.2 for special cases.).

In contrast, Problem 14 of the Moscow Mathematical Papyrus contains the correct instructions to calculate the volume of a square truncated pyramid according to the correct formula

$$V = \frac{h}{3} \cdot (a^2 + ab + b^2) \tag{1.2.3}$$



Illus. 1.2.4 Regarding the calculation of the volume of a truncated square pyramid [Design: H. Wesemüller-Kock]

(V = Volume, a = length of basis edge, b = length of top edge, h = height). You can arrive at this formula, if the one for the volume of a pyramid is known (see Problem 1.2.3). As pointed out, there is no evidence prominent in the sparsely preserved Egyptian texts that this formula was used.

Sometimes the Egyptians approximated the square truncated pyramid by calculating an average, i.e. they treated it like a cuboid, whose basis B was chosen to be the arithmetic means of the basis area and top surface area:

$$B = \frac{1}{2}(a^2 + b^2). \tag{1.2.4}$$

leads to

$$V = \frac{h}{2}(a^2 + b^2). \tag{1.2.5}$$

Historian of mathematics Kurt Vogel pointed out that the Egyptians may have realised their mistake and, as a result, have inserted a median area unit $a \cdot b$:

$$B = \frac{(a^2 + ab + b^2)}{3}. \tag{1.2.6}$$

This way, they discovered the correct calculation instruction from an incorrect formula by means of unproven generalisation. (Beyond: If we view a pyramid as a truncated pyramid with the top surface area $b^2 = 0$, the formula for the capacity of the truncated pyramid delivers the correct formula for the volume of the pyramid.)

Illus. 1.2.5 Cheops Pyramid of Giza, the tallest of all pyramids
[Photo: H.-W. Alten]

Illus. 1.2.6 Cheops Pyramid and Sphinx 1858 AD.
The Sphinx deeply covered by sand [Photo: Francis Frith 1858]