Fluid Mechanics and Its Applications

Daniel T.H. New
Simon C.M. Yu *Editors*

Vortex Rings and Jets

Recent Developments in Near-Field Dynamics

Springer

Fluid Mechanics and Its Applications

Volume 111

Aims and Scope of the Series

The purpose of this series is to focus on subjects in which fluid mechanics plays a fundamental role.

As well as the more traditional applications of aeronautics, hydraulics, heat and mass transfer etc., books will be published dealing with topics which are currently in a state of rapid development, such as turbulence, suspensions and multiphase fluids, super and hypersonic flows and numerical modeling techniques.

It is a widely held view that it is the interdisciplinary subjects that will receive intense scientific attention, bringing them to the forefront of technological advancement. Fluids have the ability to transport matter and its properties as well as to transmit force, therefore fluid mechanics is a subject that is particularly open to cross fertilization with other sciences and disciplines of engineering. The subject of fluid mechanics will be highly relevant in domains such as chemical, metallurgical, biological and ecological engineering. This series is particularly open to such new multidisciplinary domains.

The median level of presentation is the first year graduate student. Some texts are monographs defining the current state of a field; others are accessible to final year undergraduates; but essentially the emphasis is on readability and clarity.

More information about this series at http://www.springer.com/series/5980

Daniel T.H. New · Simon C.M. Yu
Editors

Vortex Rings and Jets

Recent Developments in Near-Field Dynamics

Springer

Editors
Daniel T.H. New
Division of Aerospace Engineering, School of Mechanical and Aerospace Engineering
Nanyang Technological University
Singapore
Singapore

Simon C.M. Yu
Singapore Institute of Technology
Singapore
Singapore

ISSN 0926-5112 ISSN 2215-0056 (electronic)
Fluid Mechanics and Its Applications
ISBN 978-981-287-395-8 ISBN 978-981-287-396-5 (eBook)
DOI 10.1007/978-981-287-396-5

Library of Congress Control Number: 2015933820

Springer Singapore Heidelberg New York Dordrecht London

Printed on acid-free paper

Springer Science+Business Media Singapore Pte Ltd. is part of Springer Science+Business Media (www.springer.com)

This book is dedicated to our families,
as well as past and present graduate students.

Preface

Vortex rings and jets have always fascinated and captivated researchers and engineers alike, partly due to the surprisingly rich flow phenomena underpinning such seemingly simple flow scenarios, and partly due to their immediate relevance towards a significant number of industrial applications. Without going into the details, smoke rings exhaled from the mouth, squid propulsion, industrial waste issuing out of chimneys and exhaust from jet engines are just some of the immense numbers of real-world instances that illustrate the unique characteristics of vortex rings and jets. More interestingly, from the vortical structures and behaviour standpoint, vortex rings can often be treated, albeit simplistically, as the fundamental building blocks of more complex jet flows. From the extensive pool of experimental, numerical and theoretical studies conducted up to this point in time, both phenomena share many similarities in their underlying behaviour. In fact, many of the flow characteristics and vortex flow models postulated to better explain jet flow behaviour are based on or associated with vortex rings.

Despite their long history of being at the heart of many studies performed in the past, interest in vortex rings and jets has not waned. In reality, unique demands from both well-established and new emerging engineering applications, such as flow control, renewable energy and noise emissions, just to name a few, ensure that interest in them remains as high as before, if not greater. The main discernible difference between most present and past investigations is that the present focus is now on more complex flow configurations surrounding the use of vortex rings and jets, and how to better exploit them for useful purposes in an efficient and robust manner. And coupled with significant advances made in measurement and numerical tools such as particle image velocimetry, large-eddy simulation, and not to mention data analysis in the past two decades, more exacting details can now be extracted from the flow fields of vortex rings and jets for an unprecedented level of understanding.

In view of these new developments, it will be timely to provide some fresh updates to our collective understanding on vortex ring and jet flow phenomena through the present book, while not overwhelming the readers at the same time.

There have been many excellent seminal work in this area, including *The theory of turbulent jets* by G.N. Abramovich, *Turbulent jets* by N. Rajaratnam, *Vortex dynamics* by P.G. Saffman and *Fluid vortices: Fluid mechanics and its applications* edited by S.I. Green, just to name a selected few, and the authors hope the present contribution will provide new perspectives and inspirations to readers who desire to find out some of the latest studies in related areas. In particular, this book attempts to relate vortex rings with jets in a systematic manner through several key areas, namely free vortex rings, vortex ring-structure interactions, jets formed by vortex-ring trains, jets issuing from unconventional nozzles and jet-structure interactions. The editors believe that this book will serve the readers well, either as an update to some of the emerging knowledge on vortex rings and jets, or as a guide on how to tap some of their unique flow behaviour for new engineering applications. Lastly, the editors are especially thankful to all the authors for their contributions towards this book, without which this book would not have been possible.

Singapore
October 2014

Daniel T.H. New
Simon C.M. Yu

Contents

Starting Jets and Vortex Ring Pinch-Off

L. Gao and S.C.M. Yu

Abstract Starting jet is commonly defined as the transient motion produced when a viscous incompressible fluid is forced from an initial state of rest Cantwell (Journal of Fluid Mechanics, 173, 159–189 [6]). The applied force can be time dependent in the form of an impulsive, step or ramp function acting at a point or along a line. Starting jet can be used in fundamental study of vortex ring dynamics, synthetic jets, mixing enhancement, and vortex-enhanced unsteady propulsion systems. Researches related to starting jets have been carried out broadly in two directions. The first direction is focused on the underlying mechanism for the vortex ring pinch-off, which is defined as the process whereby a forming vortex ring is no longer able to absorb vorticity flux from the jet source via the separated shear layer. Several theoretical models are proposed to predict a critical time scale for the pinch-off process, dubbed as the formation number F, for different flow conditions. The second direction is focused on its practical applications in entrainment enhancement as well as pulsed-jet propulsion systems. Specifically, due to the restricted vortex ring formation in starting jet, the propulsive efficiency can be effectively improved over the steady jet propulsion by increasing the generated thrust via the vortex over-pressure in the near-wake and by decreasing the kinetic energy loss in the wake via vortex entrainment. In this chapter, we intend to provide the readers with some basic ideas on the dynamic process of vortex ring formation in a starting jet, and its practical application in nature and engineering fields. This chapter is divided into four parts. The first part provides a brief introduction of the starting jet and the phenomenon of vortex ring pinch-off. The discussion of the underlying mechanisms and its theoretical explanation are provided in part two. In the third part, the

L. Gao (✉)
School of Aeronautics and Astronautics, Sichuan University, Chengdu 610065, People's Republic of China
e-mail: lei.gao@scu.edu.cn

S.C.M. Yu
Academic Programme Division, Singapore Institute of Technology, Singapore 179104, Singapore
e-mail: Simon.yu@singaporetech.edu.sg

D.T.H. New and S.C.M. Yu (eds.), *Vortex Rings and Jets*,
Fluid Mechanics and Its Applications 111, DOI 10.1007/978-981-287-396-5_1

practical application of starting jets in engineering systems will be explained and discussed. This chapter ends with a summary and an outlook for future study on the starting jet.

Keywords Vortex ring pinch-off · Entrainment · Unsteady-jet propulsion

1 Introduction

Before going into the details of the above-mentioned areas, the background information about a starting jet will first be introduced. As a relatively new topic of research, its introduction will be made in relation to two classical topics, i.e., the steady jets and the vortex ring dynamics.

1.1 General Description of the Starting Jet

It is well known that jet flow is referred to as an efflux of fluid being injected into the surrounding medium from a nozzle, or orifice. From a classical point of view on jet flow, a starting jet is commonly defined as a single-pulsed jet or an initial development of a continuous (steady) jet, depending on the duration of the fluid ejection. It also corresponds to the transient motion produced when a viscous fluid is forced from an initial state of rest.

If the region of interest is in the far field where the source is relatively small, the mass entrained by the flow eventually overwhelms those issued from the source, so that the jet is mostly described by their source momentum. Another feature of steady jets in the far field is that its characteristics succumb to similarity solutions with an appropriate virtual origin correction. In theoretical analysis, a jet coming out of a point source can be obtained in the limit of a practical jet when the size D of the nozzle tends to be zero while the discharge velocity U_0 increases such that the initial momentum flux $\pi U_0^2 D^2/4$ remains constant. The volume flux at the exit $\pi U_0 D^2/4$, on the other hand, will become negligible. This is a good approximation on the properties of a jet in the far field. But the idealization ignores completely the near-field flow details of the jet from a nozzle with finite size, by replacing it with a point source. Unfortunately, in the study of starting jet, the focus lies mainly on the unsteady near-field development. The finite size of the jet source can be important especially when the flow in the vicinity of the source is considered. Therefore, it requires a totally different way to investigate this kind of starting flows.

For the near field of a steady axisymmetric jet, the basic sequence of its evolution was summarized by List [31] as: "*in the immediate neighborhood of the orifice, the high-speed jet flow causes a laminar shear layer to be produced. The shear layer is unstable and grows very rapidly, forming ring-vortices that enhance* *the mixing between the irrotational ambient fluid and the ejected fluid*". The separated shear layer, its instability growth, roll-up of a series of vortices and their interactions thus determine the near field behavior of a steady jet. The motion induced in the fluid by each vortex affects other vortices in such a way that adjacent vortices pair off. The vortex motion develops a secondary circumferential instability that causes the eventual breakdown of the each vortex [32]. For steady plane jets, the basic difference is the existence of two modes of large-scale vortex formation, on alternate sides, or simultaneously on both sides. These have been called the "flapping" and "puffing" modes of oscillation. In addition, a secondary instability of vortex pairs in two-dimensional jets was found by Leweke and Williamson [28], and it is termed as, elliptic instability.

The second perspective to look at the starting jet is related to the fundamental study of vortex ring. In laboratory, vortex rings can be generated by the piston-cylinder apparatus, in which the motion of a piston pushes a column of fluid of length L through an orifice or nozzle of diameter D. The boundary layer at the edge of the orifice or nozzle separates from the orifice or nozzle edge, and subsequently rolls up into spiral, as sketched in Fig. 1.

By dimensional analysis, the starting jet flow is mainly governed by two dimensionless parameters: (1) Formation time: $t^* = U_0 t/D$ and (2) Reynolds number $Re = U_0 D/\nu$, where ν is the kinematic viscosity. It is noted that other parameters, such as piston velocity programs, geometry of the nozzle or orifice exit, may also play important roles in the dynamics of a starting jet.

The analysis of Mohseni and Gharib [34] suggested that the properties of the leading vortex ring formed in a starting jet are the final outcome of a relaxation process, dependent only on three integrals of the motion, namely the kinetic energy E, impulse I, and circulation Γ. These three quantities are invariant under the assumption of inviscid flow and no external forces. In the viscous case, total energy and circulation of the flow will be diffused with time by the viscosity. For the axisymmetric jet flow with no swirl (namely $u_\theta = 0$), the kinetic energy, axial component of the hydrodynamic impulse and circulation can be obtained by integration over the plane of symmetry as

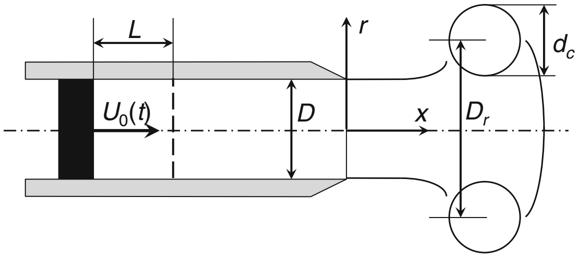

Fig. 1 Illustration of a starting jet generated by the piston-cylinder apparatus and the formation of a leading vortex ring

$$E = \pi\rho \int_0^{\infty} \int_{-\infty}^{\infty} \psi\omega dx\, dr \tag{1}$$

$$I = \pi\rho \int_0^{\infty} \int_{-\infty}^{\infty} \omega r^2 dx\, dr \tag{2}$$

$$\Gamma = \int_0^{\infty} \int_{-\infty}^{\infty} \omega dx\, dr \tag{3}$$

where ρ is the fluid density and ψ is the Stokes stream function.

For the properties of vortex rings, readers can refer to the detailed review of Shariff and Leonard [53] and Lim and Nickels [29] on the topic. Here, we only focus on the initial formation of the vortex ring from a starting jet and its kinematics. In order to understand better the following discussion in this chapter, the definitions of several kinematic concepts related to the vortex ring in starting jets are given below (which are also illustrated in Fig. 1):

1. Core center position (x_c, r_c): The vortex center can be identified through detecting the maximum of the stream function. The second method is to locate the position of minimum static pressure [19]. The time history of the vortex core center position is termed as the vortex trajectory. Vortex ring diameter D_r, defined by the distance between the spiral centers.
2. Vortex core diameter d_c: Based on the experimental results, Weigand and Gharib [56] estimated the vortex ring core size by two ways. The core diameter was determined by the distance between the minima and maxima of the axial velocity profiles along the core. Another method was to measure the core diameter in terms of the vorticity distribution. Larger core diameter by vorticity profile resulted from that the vorticity is non-zero at locations where the gradient $\partial u/\partial y = 0$. It suggested that the term $\partial v/\partial x$ contributed to the magnitude of the vorticity.

1.2 Vortex Ring Formation in a Starting Jet: Early Studies

A starting jet has been used extensively as a mechanism to study the properties of a vortex ring. Because only a single vortex ring is produced in the final flow field of a starting jet with small stroke ratio L_{max}/D, this set-up is ideal for the study of vortex rings. By using the piston-cylinder apparatus, early studies generated a large amount of results with regards to the vortex ring properties. They mainly focused on the structure, vorticity distribution, circulation and impulse of vortex rings. However, we are not going to repeat here the details of the vortex ring dynamics. Instead, we want to examine the effect of the flow parameters (e.g. geometry, initial conditions, Reynolds number, etc.) of the starting jet on the sequence of events leading to the creation of a vortex ring. We review the features of vortex ring formation only to highlight their effects on the general properties of a starting jet.

When a piston ejects fluid through a nozzle, boundary layer is generated on the inner wall of the nozzle, and separates from the sharp edge in order to satisfy the Kutta condition. Then the separated shear layer rolls-up and the vortex ring formation starts. There is strong evidence in support of the notion that the formation of vortex rings is mainly an inviscid process [38, 39]. The effect of viscosity over much of the formation process is small (except at very small times). The relaxation time toward a steady vortex ring is much shorter than the viscous diffusion time at high Reynolds number. The main effect of viscosity is to remove the singular, non-analytical behavior of the velocity the center of the spiral [36]. Immediately after the flow initiation, the dynamics of the starting jet is solely dominated by the characteristics of a large-scale vortex in the near field, and then gradually approaches to a steady jet in the far field.

It is well known that the roll-up of an inviscid plane vortex sheet, which separates at the edge of a body, is a self-similar process. The analysis stems originally from the work of Kaden [22]. It was then applied to the roll-up of a cylindrical vortex sheet by Saffman [49]. Recently, Hettel et al. [19] revisited the similarity law for the formation of free viscid vortex rings by using numerical simulation.

Kaden [22] showed that the analytical solution of the roll-up of a semi-infinite inviscid planar discontinuity sheet could be obtained using the conformal transformation of a flow along a plane wall. The velocity at position of constant angle φ in the polar coordinate system (r', φ) centered at the spiral center was found to be proportional to $1/\sqrt{r'}$. As a result of a self-similar process, the length scale l_1 and l_2 of the similar flow patterns at two points in the flow field can be related to their characteristic times t_1 and t_2 as

$$\frac{l_1}{l_2} = \left(\frac{t_1}{t_2}\right)^{\frac{2}{3}} \tag{4}$$

Similar to the planar flow around a semi-infinite flat plate, there is no length scale at the early stage of the axisymmetric flow when the spiral size d_c is much smaller than the nozzle diameter D. It implies the initial axisymmetric vortex sheet roll-up is also self-similar and the scaling behavior can be predicted from its planar counterpart. Saffman [49] applied similarity theory of the formation of a two-dimensional vortex to obtain the location of the center of the axisymmetric vortex spiral in an impulsively started jet. Therefore, the similarity laws for the time dependent axial and radial position of the vortex center, x_c and r_c, respectively, as well as the diameter of the vortex spiral d_c are given in dimensionless form:

$$x_c \sim (t^*)^{2/3}, r_c \sim (t^*)^{2/3}, d_c \sim (t^*)^{2/3} \tag{5}$$

All quantities are normalized in terms of the characteristic velocity U_c, i.e., U_0 and characteristic length scale L_c, i.e., D. These scaling laws are only valid as long as the dimension d_c of the vortex spiral is much smaller than the characteristic length scale L_c (i.e. nozzle diameter, channel height, length of plate) of the appropriate geometry. According to the experimental data of Didden [12] (see Fig. 7 therein), the self-similar development of cylindrical vortex sheet roll-up should be valid for $d_c < 0.1\ D$, or equivalently for $t^* < 0.6$.

Subsequent studies, however, have revealed contradictory results on the time-dependence of the axial position x_c of the cylindrical vortex sheet roll-up to the prediction of similarity laws. Experimental [12] and numerical [18, 21, 39] data showed $x_c \sim t^{*3/2}$. However, the similarity laws for plane vortices predict $x_c \sim t^{*2/3}$. Hettel et al. [19] suggested that the discrepancy could be explained by the fact that in each system (vortex formation behind a circular nozzle and roll-up of a semi-infinite free vortex sheet), a different frame of reference is used. The plane vortex investigated by Kaden [22] moves in the negative x-direction, while vortex ring moves in the positive x-direction during its formation process. The relative position between jet front and vortex center, i.e., $x_f - x_c$, exhibits a time dependence which can be predicted by the similarity law $(t^*)^{2/3}$.

As concluded in Hettel et al. [19], the similarity law for the self-similar roll-up of the axisymmetric vortex sheet at the early stage can be expressed as,

$$\begin{cases} \dfrac{x_c}{D} = 0.28t^{*3/2} \\ \dfrac{r_c}{D} = 0.16t^{*2/3} \\ \dfrac{d_c}{D} = 0.25t^{*2/3} \end{cases} \tag{6}$$

The development of vortex ring roll-up during this initial stage plays a key role in determining the characteristics of the vortex ring later on.

For $t^* > 0.6$, the size of the vortex core d_c increases to become comparable to the diameter of the nozzle D, leading to the failure of the similarity law. Afterwards, the self-induced velocity of the vortex ring starts to play a more important role in its downstream translation. Thus, during the self-induced translation stage, more attention should be paid to the dynamic properties of the forming vortex ring.

A simple slug flow model has been used by Didden [12], Glezer [17], and Maxworthy [33] to predict the circulation of the total starting jet with piston-cylinder apparatus. The model assumes a uniform velocity equal to the piston velocity across the exit plane of the nozzle or orifice type of generators. The increasing rate of the total starting jet circulation, i.e., the flux of vorticity is calculated as

$$\frac{d\Gamma_{jet}}{dt} = \int \omega u_x dr \approx \int -\frac{\partial u_x}{\partial r} u_x dr \approx \frac{1}{2} U_0^2 \tag{7}$$

It should be noted that the slug model for the circulation is independent of the geometry of the vortex generator (in other words, its effects are not considered). By comparing with the experimental and numerical data, the slug model was found to generally underestimate the total jet circulation. James and Madnia [21] attributed the over shoot of nozzle exit velocity profile near the edge to the higher circulation in their simulation data. Krueger [25] indicated that the under-predicted circulation by the slug model is caused by its ignorance of the effect of a rapid pressure rise at the nozzle exit during the initial vortex ring formation, which is termed as the over-pressure effect.

As a convenient tool of estimating the dynamic properties of the jet flow, the slug model can also be applied to obtain the kinetic energy and impulse of a starting jet as

$$E_{jet} = \frac{1}{8} \pi D^2 \rho L U_0^2 \tag{8}$$

$$I_{jet} = \frac{1}{4} \pi D^2 \rho L U_0 \tag{9}$$

$$\Gamma_{jet} = \frac{1}{2} L U_0 \tag{10}$$

The above estimation of the kinetic energy, impulse and circulation will be an important tool in the theoretical study of the vortex ring formation and pinch-off as discussed in the next section.

2 Vortex Pinch-Off and the Formation Number

It is obvious that at sufficiently high Reynolds number the stroke ratio *L/D* is an important controlling parameter for the development of starting jet. In the early stage of the starting jet, the size and circulation of the vortex ring will continue to grow as *L/D* increases. Gharib et al. [16] addressed the question of whether there is an upper limit in the circulation that a vortex ring can acquire by continuing to increase the stroke ratio. The investigation on this question leads to the finding of vortex ring pinch-off process, as well as the critical dimensional parameter in starting jet, i.e., the formation number.

2.1 The Process of Vortex Ring Pinch-Off

Gharib et al. [16] examined the vortex formation process for larger maximum stroke ratio, $L_{max}/D > 4$, and observed a limit on the maximum growth of vortex rings formed using a piston-cylinder apparatus. Unlike the production of a single vortex ring in starting jet with small stroke ratio, the flow field generated by larger maximum stroke ratio consists of a leading vortex ring followed by a trailing jet. The vorticity field of the leading vortex ring gradually disconnects from that of the trailing jet. This dynamical process of the vorticity field disconnection was dubbed by Gharib et al. [16] as vortex ring "pinch-off". An equivalent definition of "pinch-off" was given by Dabiri [8] as a process whereby a forming vortex ring is no longer able to entrain additional vorticity from the vortex generator. The pinch-off process is characterized by a critical dimensionless time, i.e., vortex formation number F, at which the total circulation supplied by the vortex generator is equal to the pinched-off vortex ring circulation. According to its definition, formation number should correspond to the onset of the vortex pinch-off process.

For more thorough understanding of this phenomenon, the pinch-off process is described from two perspectives. First, it is about the final state of the starting jet determined by the maximum stroke ratio L_{max}/D. Gharib et al. [16] showed that if the piston was forced to stop at $L(t)/D = 4$ (i.e., $L_{max}/D = 4$), the resulting vortex ring (in its long term development) leaves no trailing jet. Otherwise, the final state of the starting flow consists of a leading vortex ring as well a trailing jet stem. This description of the vortex ring pinch-off process can be clearly illustrated by the vorticity contours shown in Fig. 2.

Second, the dynamics of the pinch-off process can be observed as transient development of the flow field as the instantaneous stroke ratio $L(t)/D$ (or the equivalent dimensionless formation time t^*) increases. As shown in Fig. 3a, b, the leading vortex ring remains connected with the trailing shear layer near the formation number, i.e., the onset of the pinch-off process. After the critical formation number, the vorticity flux from the trailing shear layer gradually decreases, leading to the final disconnection between the vorticity fields between the leading vortex ring and the trailing jet at the end of the pinch-off process. The almost complete separation of the leading vortex ring after the pinch-off can be clearly observed in Fig. 3c. Therefore, the pinch-off process is not a sudden event and its completion will usually take up to several formation time units to finish.

According to Dabiri [8], it is should also be noted that the dimensionless formation time t^* is basically the normalized total circulation. Therefore, the normalization of the formation number in the starting jet indicates that the maximum circulation of the vortex ring is determined mainly by the length and velocity scale of the vortex-shedding flow.

In order to determine the formation number, several methods have been proposed based on its definition and physical implications. The most straightforward method to determine the formation number is by comparison of the total circulation

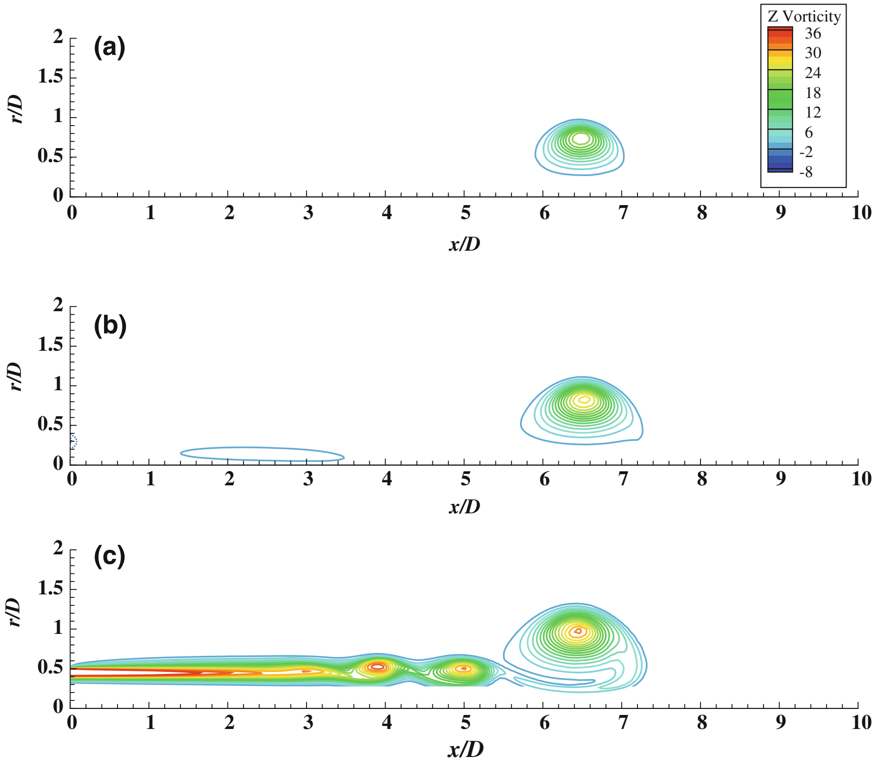

Fig. 2 Vorticity contours of flow field for the starting jets with maximum stroke ratio: **a** $L_{max}/D = 2$, **b** $L_{max}/D = 4$, and **c** $L_{max}/D = +\infty$, when vortex rings translate at downstream location $x/D \approx 6.5$. The minimum absolute vorticity value shown is 2 s^{-1}

discharged by the starting jet and the circulation of the vortex ring after pinch-off [16]. This procedure is illustrated by the dash line and the arrow in Fig. 4. This method has been extensively used in experimental and numerical studies of the vortex ring pinch-off for a variety of flow conditions using the piston-cylinder apparatus.

However, the method of Gharib et al. [16] poses several challenges for complex flow conditions, as pointed out by O'farrell and Dabiri [41]. First, determining the final circulation of the vortex ring requires the ring to be indistinguishable from its trailing jet. Second, it is essential that the vorticity field does not diffuse and the vortex does not distort by interaction with other structures in the flow. To overcome these difficulties, O'farrell and Dabiri [41] proposed an alternative method for identifying the formation number using the Lagrangian coherent structures (LCSs). It is found that the appearance of a new disconnected LCS and the termination of the original LCS is indicative of the initiation of vortex pinch-off. Using the Lagrangian criterion, the formation number is found to be consistent with the results found by the circulation criterion. It is noted that the appearance of a new LCS in

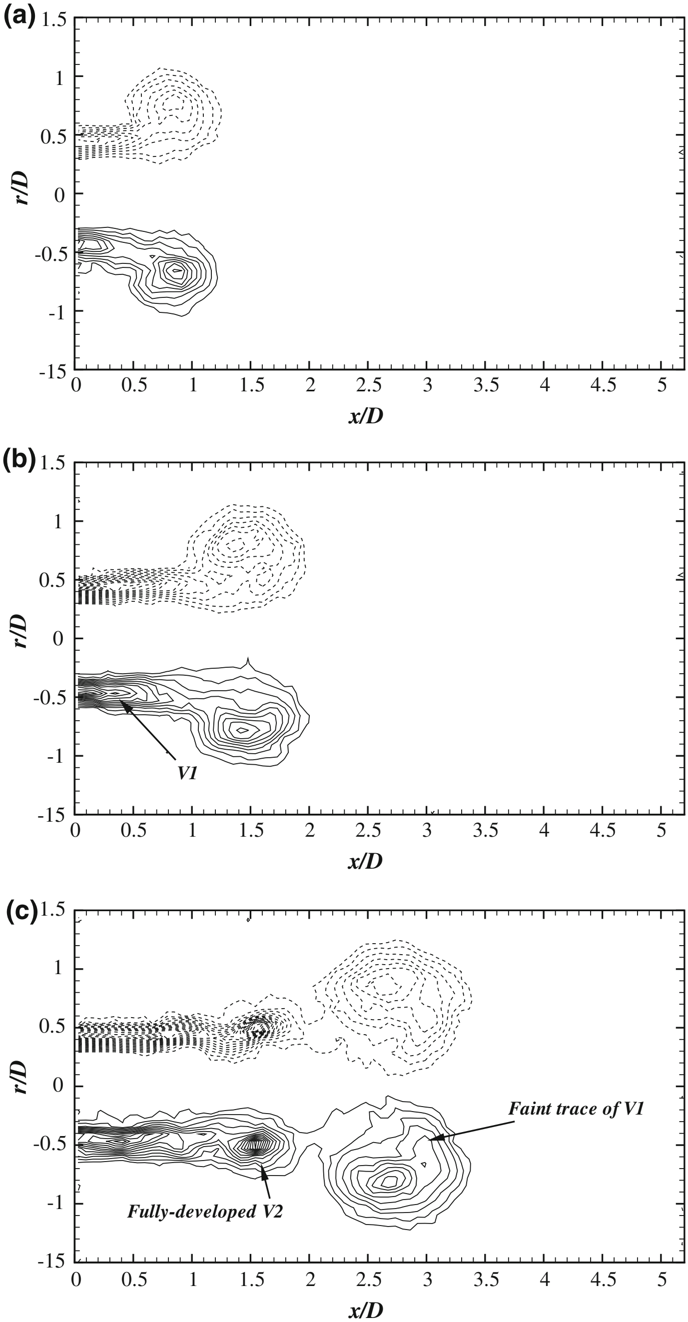

Fig. 3 The evolution of vorticity contours of a starting jet at: **a** t^* = 3.1, **b** t^* = 4.9, **c** t^* = 8.1. The formation number for this case is F = 4.5 ± 0.1. *V1* and *V2* indicate the first and second trailing vortices, respectively. Graphs are reproduced from Gao and Yu [15]

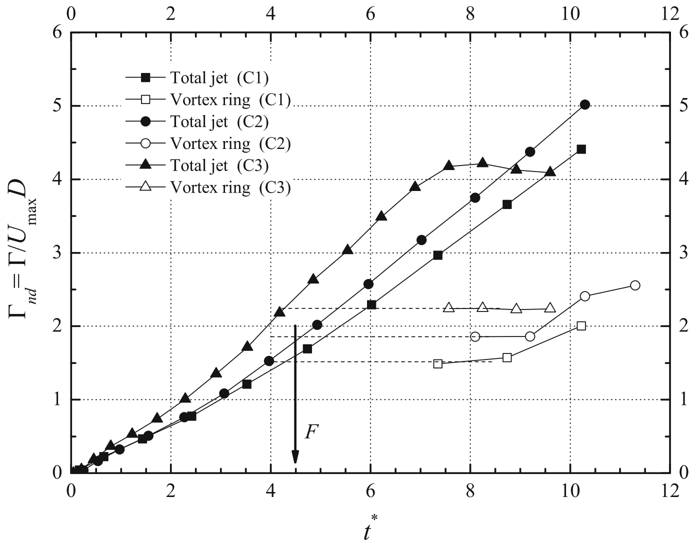

Fig. 4 The circulation method for determining the formation number *F* (reproduced from Gao and Yu [15])

the starting jet corresponding to the development of shear layer instability in the trailing jet.

In a recent experimental study of Lawson and Dawson [27], the Trailing pressure maximum (TPM) formed behind the leading vortex ring was used to identify the formation number. Their results showed that the formation number found by TPM was in agreement with those reported in previous studies.

2.2 Theoretical Analysis on the Vortex Ring Pinch-Off

The existence of the formation number was first explained by Gharib et al. [16] based on Kelvin-Benjamin variational principle for steady axis-touching vortex ring. They stated that the limiting stroke ratio $(L/D)_{\mathrm{lim}}$, i.e., the formation number, occurs when "*the apparatus is no longer able to deliver energy at a rate compatible with the requirement that a steadily translating vortex ring have maximum energy with respect to impulsive-preserving iso-vortical perturbations.*" During the vortex formation process, the ejected fluid, carrying a specific amount of kinetic energy, impulse and circulation, is fed into the leading vortex ring and serves as continuous perturbations to the "quasi-steady" vortex ring. Therefore, in that sense, the vortex ring pinch-off process was considered as a relaxation process of the leading vortex ring to an equilibrium state. To apply the Kelvin-Benjamin variational principle, a normalized energy α, which is defined as the kinetic energy normalized by impulse and circulation as,

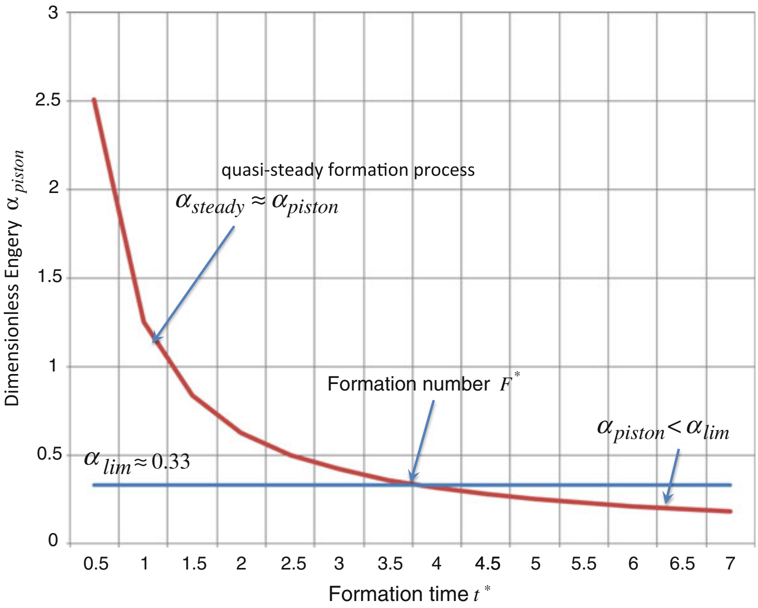

Fig. 5 Explanation of the vortex ring pinch-off process based on the Kelvin-Benjamin variational principle

$$\alpha = \frac{E}{\sqrt{\rho I \Gamma^3}} \tag{11}$$

was introduced to describe the energy restriction. The definition of the normalized energy α in terms of its impulse and circulation is consistent with the requirement of the Kelvin-Benjamin variational principle that an alternative arrangement of the vortex ring vorticity should preserve the total impulse. The theory of Gharib et al. [16] is illustrated in Fig. 5.

As shown in Fig. 5, normalized energy of the total jet α_{piston}, which is calculated by substituting Eqs. (8)–(10) into Eq. (11), decreases monotonically with the stroke ratio *L*/*D*. By assuming that all ejected fluids (with their *E*, *I* and Γ) are absorbed by the resulting vortex ring, one can deduce that $\alpha_{steady} \approx \alpha_{piston}$. Here α_{steady} is the value of α for the final steady vortex ring which is accessible from the vorticity field at stroke ratio *L*/*D* via an iso-vortical impulse-preserving rearrangement. This fact indicates that the Kelvin-Benjamin variational principle is not about the instantaneous properties of the generation process, but rather focuses on matching the quantities provided by the apparatus and those of the final vortex ring. This is actually the key point to understand the explanation proposed by Gharib et al. [16]. Due to the properties of the steady vortex rings in theoretical analysis, lower value of the α corresponds to thicker vortex ring. It means the vortex ring becomes thicker and thicker as the stroke ratio increases.

However, it is realized that the vortex ring cannot be infinitely thick. We expect α_{steady} to diminish to a limiting value, α_{lim}, as the core thickens because the existence of a α_{lim} for every family of steady vortex rings is an unproven generalization based on the Norbury-Fraenkel vortex family [13, 40]. For the Norbury-Fraenkel family of vortex rings which have uniform vorticity density ($\omega/r = const$), Hill's spherical vortex [20] is the theoretical limit for the thickest member with $\alpha = 0.16$. Nevertheless, the vortex rings generated in a piston-cylinder apparatus cannot approach this theoretical limit. It should have a slightly higher limiting normalized energy. Gharib et al. [16] suggested that, a limiting state of the resulting vortex ring, characterized in terms of the dimensionless mean core radius ε [40], is then restricted by the axis-touching state of the vortex ring. They found in their experimental results that for axis-touching vortex rings, $\alpha_{lim} = 0.33 \pm 0.01$ for a wide variety of flow conditions. They also pointed out that the specific value of α_{lim} actually depends on the shape of the vorticity profile, which in turn depends on piston history and Reynolds number. It may vary among different starting flow configurations. For example, by temporally increasing the nozzle exit diameter while keeping the volume flux constant, Dabiri and Gharib [11] found $\alpha_{lim} = 0.5$ for the final vortex ring, implying the smaller core size than that for $\alpha_{lim} = 0.33 \pm 0.01$. As a result, this critical value of α_{lim} can only be determined from experiment. In short, the total kinetic energy, impulse and circulation from the generator at a specific stroke ratio can reach the final vortex ring as long as $\alpha_{piston} > \alpha_{lim}$.

For longer stroke ratio of the starting jet, α_{piston} will become less than α_{lim}. If the assumption $\alpha_{steady} \approx \alpha_{piston}$ is still valid, the resulting steady vortex ring will become thicker than the vortex ring specified by α_{lim}. According to the above hypothesis, the α_{lim} should be the lowest value for the normalized energy of the thickest vortex ring possible in a piston-cylinder apparatus. It implies that, beyond the critical point of $\alpha_{piston} = \alpha_{lim}$, not all the ejected fluids will be entrained by the final steady vortex ring because the normalized energy provided by the jet generator cannot meet the requirement of a steady translating vortex ring with $\alpha_{steady} = \alpha_{lim}$. In other words, the kinetic energy provided by the jet generator becomes too small relative to the impulse and circulation to form a steady vortex ring with $\alpha = \alpha_{lim}$. In that situation, a trailing jet with additional vorticity (and also impulse and kinetic energy) formed behind the pinched-off leading vortex ring to maintain $\alpha = \alpha_{lim}$ for the single vortex ring. The perturbation response study of O'farrell and Dabiri [42] subsequently verified this property of the Norbury-Fraenkel family of vortex rings.

By using the slug model to quantify the total normalized energy delivered by the piston-cylinder apparatus and by using the Norbury-Fraenkel family of vortex to approximate the experimentally generated vortex ring, Gharib et al. [16] successfully predicted the observed value of the formation number, in combination of the experimental data of limiting normalized energy. To avoid the use of experimental result of α_{lim}, similar models have been developed subsequently by Mohseni and Gharib [34] to include an additional assumption on the vortex ring translational velocity (see Eqs. (9) and (10) therein), and by Linden and Turner [30] to include an additional condition on the fluid volume conservation. It is noted that Shusser et al.

[55] showed that boundary-layer growth within the hollow cylinder wall of the vortex generator was shown to lead to an increase in vorticity flux into the starting jet.

The Kelvin-Benjamin variational principle can be regarded as a before-and-after-matching of the dynamical properties between those from the total starting jet and those in the final pinched-off vortex ring. However, the dynamic process during its evolution has not been considered. To include the dynamic process of the detachment of the leading vortex ring from its trailing jet, Shusser and Gharib [54] proposed a hypothesis on the flow kinematics that "*a vortex ring completes its formation and pinches off from its generating axisymmetric jet when the translational velocity of the ring becomes equal to the jet flow near the vortex ring*." In particular, Mohseni et al. [35] proposed an explanation for the pinch-off process by considering the very detailed translational dynamics of the leading vortex ring and its trailing shear layer. At the early stage, the vortex ring grows in size and translates downstream due to its self-induced velocity. As the vortex ring becomes thicker, it pushes the shear layer toward the axis of symmetry at which the vorticity destruction occurs due to the vorticity cancellation mechanism. Since the leading vortex ring gains its strength from the shear layer, it gradually ceases to grow if the local velocity of the shear layer is less than the velocity of the vortex ring. Alternatively, the pinch-off occurs when the shear layer is unable to enlarge the leading vortex ring, as the strength of the shear layer tends to zero at the axis of symmetry.

Several models have also been proposed according to the second theory, with some assumptions. Based on the assumption of a quasi-steady formation process, the translational velocity of the leading vortex ring is estimated by the properties of the Norbury-Fraenkel family of steady vortex rings as follows:

$$E_{ring} = \frac{1}{2}\rho R \Gamma^2 \left[\ln\frac{8}{\varepsilon} - \frac{7}{4} + \frac{3}{8}\varepsilon^2 \ln\frac{8}{\varepsilon} \right] \tag{12}$$

$$I_{ring} = \rho \pi \Gamma R^2 \left(1 + \frac{3}{4}\varepsilon^2 \right) \tag{13}$$

$$U_r = \frac{\Gamma}{4\pi R} \left[\ln\frac{8}{\varepsilon} - \frac{1}{4} + \frac{3}{8}\varepsilon^2 \left(\frac{5}{4} - \ln\frac{8}{\varepsilon} \right) \right] \tag{14}$$

where U_r is the translational velocity of the vortex ring, vortex ring radius $R = D_r/2$ and $\varepsilon = d_c/D_r$ is the dimensionless mean core radius of the vortex ring. On the other hand, the local trailing jet velocity is related to the piston velocity via the conservation of mass (or volume because the fluid is incompressible) and the dynamical properties of the total jet are approximated by the slug flow model (i.e., Eqs. (8)–(10)).

As a result for the system of Eqs. (8)–(10) and (12)–(14), there are six equations but seven unknowns, i.e., E, I, Γ, L, U_r, ε and R. Thus, an extra equation is needed for the closure of the model. By comparing the vortex ring velocity U_r with the trailing jet velocity near the vortex ring and by using the conservation of mass in the trailing jet, Shusser and Gharib [54] introduced an additional equation into the model as

$$U_r = \frac{D^2}{4R^2} U_0 \tag{15}$$

They found the limiting value of the dimensionless energy to be $\alpha_{lim} \approx 0.31$, which agrees well with the experimental value of $\alpha_{lim} = 0.33$.

However, there are still some concerns about the fundamental approximation in the model of Shusser and Gharib [54] that the dynamical properties of the leading vortex ring during the formation can be estimated by the slug flow model. The approximation is regarded accurate only when all the ejected fluids are entrained completely in the leading vortex ring. It is not the real flow configuration because the trailing jet appears after the leading vortex ring travels away from the nozzle exit. Part of the ejected fluid, with its kinetic energy, impulse and circulation, resides in the trailing jet during the formation stage.

To account for the actual physical process, Gao and Yu [14] proposed a revised model by determining the properties of the leading vortex from the fluxes from the trailing jet, rather than from the slug model. The initial development of the starting jet is accounted by the similar law of vortex sheet roll-up as given in Eq. (6). In the second stage, the growth of the leading vortex is determined by the flux of those integrals of motion from the trailing jet at its rear boundary, which are estimated as

$$\frac{dE_{ring}}{dt} = \frac{1}{2}\pi b^2 \rho U^2 (U - U_r) \tag{16}$$

$$\frac{dI_{ring}}{dt} = \pi b^2 \rho U (U - U_r) \tag{17}$$

$$\frac{d\Gamma_{ring}}{dt} = \frac{1}{2}(U - U_r)^2 \tag{18}$$

where b is the radius of the trailing jet stem near the leading vortex ring and specifically $b = D/2$ for the piston-cylinder apparatus. Their analytical results revealed that the dynamic process of vortex ring pinch-off is signified by two time scales, i.e., the formation number, which indicates the onset of the pinch-off process, and the separation time, which corresponds to the time when the leading vortex ring becomes physically separated from the trailing jet and is therefore referred to as the end of the pinch-off process.

By considering the description of the pinch-off process presented in Sect. 2.1, it can be seen that the theory proposed by Gharib et al. [16], Mohseni and Gharib [34], and Linden and Turner [30] which is based on the Kelvin-Benjamin variation principle, is consistent with the first perspective to understand the pinch-off process. On the other hand, the theory proposed by Shusser and Gharib [54], Mohseni et al. [35] and Gao and Yu [14] is in accordance with the second perspective to understand the pinch-off process. As stated by Mohseni et al. [35], translational dynamics

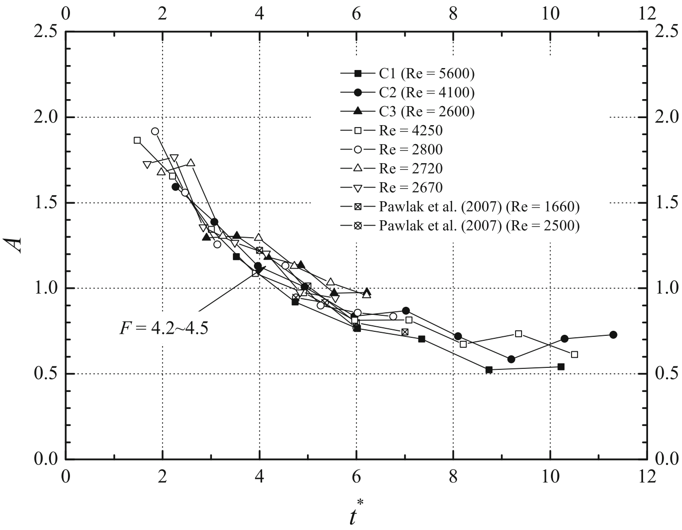

Fig. 6 The variation of the instability parameter A against the formation time t^*. The data of Pawlak et al. [45] are also included for comparison (graph reproduced from Gao and Yu [15])

can lead to insights about the formation process, but it is cumbersome for modeling. Relaxation approach appears to obviate the need for modeling the nonlinear dynamics of shear layer.

Finally, the pinch-off can also be explained in terms of the effects of the shear layer instability. For starting jets generated by large stroke ratio (typically greater than the formation number), it was observed both in experiments [15, 16, 41, 50], and simulations [35, 57] that secondary vortices (or trailing vortices) would be formed in the trailing shear layer as a result of the Kelvin-Helmholtz instability. Since the growth of the leading vortex ring is sustained by the influx of mass, vorticity, impulse and kinetic energy from the trailing shear layer at a specific rate, the development of the shear layer instability can certainly play an important role in the pinch-off process. By altering the characteristics of the shear layer instability by Reynolds number and initial shear layer thickness, Zhao et al. [57] found that the development of the trailing shear layer instability and the growth of secondary vortices actually accelerated the process of pinch-off and introduced approximately 20 % variation in the formation number and in the vortex ring circulation level.

After the pinch-off process, the interaction between the leading vortex ring and the trailing vortices can also greatly affect the evolution of the pinched-off vortex ring. One notable example is the merging of the trailing vortices into the leading vortex ring due to their interaction [15, 16, 41], as shown by the evolution of the first trailing vortex *V1* in Fig. 3b, c.

The effect of the trailing shear layer instability on the vortex ring pinch-off was investigated by Gao and Yu [15]. The experimental results showed that secondary