

Algebra and Applications

Henri Lombardi
Claude Quitté

Commutative Algebra: Constructive Methods

Finite Projective Modules

 Springer

Commutative Algebra: Constructive Methods

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Henri Lombardi · Claude Quitté

Commutative Algebra: Constructive Methods

Finite Projective Modules



Springer

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To James Brewer

Preface to the French Edition

This book is an introductory course to basic commutative algebra with a particular emphasis on finitely generated projective modules, which constitutes the algebraic version of the vector bundles in differential geometry.

We adopt the constructive point of view, with which all existence theorems have an explicit algorithmic content. In particular, when a theorem affirms the existence of an object – the solution of a problem – a construction algorithm of the object can always be extracted from the given proof.

We revisit with a new and often simplifying eye several abstract classical theories. In particular, we review theories which did not have any algorithmic content in their general natural framework, such as Galois theory, the Dedekind rings, the finitely generated projective modules, or the Krull dimension.

Constructive algebra is actually an old discipline, developed among others by Gauss and Kronecker. We are in line with the modern “bible” on the subject, which is the book by Ray Mines, Fred Richman and Wim Ruitenburg, *A Course in Constructive Algebra*, published in 1988. We will cite it in abbreviated form [MRR].

This work corresponds to an MSc graduate level, at least up to Chap. XIV, but only requires as prerequisites the basic notions concerning group theory, linear algebra over fields, determinants, modules over commutative rings, as well as the definition of quotient and localized rings. A familiarity with polynomial rings, the arithmetic properties of $\mathbb{Z}$ and Euclidian rings is also desirable.

Finally, note that we consider the exercises and problems (a little over 320 in total) as an essential part of the book.

We will try to publish the maximum amount of missing solutions, as well as additional exercises on the web page of one of the authors:

<http://hlombardi.free.fr/publis/LivresBrochures.html>.

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Finally, we could not forget to mention the Centre International de Recherches Mathématiques à Luminy and the Mathematisches Forschungsinstitut Oberwolfach, who welcomed us for research visits during the preparation of this book, offering us invaluable working conditions.

August 2011

Henri Lombardi
Claude Quitté

Preface to the English Edition

In this edition, we have corrected the errors that we either found ourselves or that were signalled to us.

We have added some exercise solutions as well as some additional content. Most of that additional content is corrections of exercises, or new exercises or problems.

The additions within the course are the following. A paragraph on the null tensors added at the end of Sect. [IV-4](#). The paragraph on the quotients of flat modules at the end of Sect. [VIII-1](#) has been fleshed out. We have added Sects. 8 and 9 in Chap. [XV](#) devoted to the local-global principles.

None of the numbering has changed, except for the local-global principle [XII-7.13](#), which has become [XII-7.14](#).

There are now 297 exercises and 42 problems.

Any useful precisions are on the site:

<http://hlombardi.free.fr/publis/LivresBrochures.html>.

Acknowledgment

We cannot thank Tania K. Roblot enough for the work achieved translating the book into English.

May 2014

Henri Lombardi
Claude Quitté

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Index of Notation

Examples

$\text{Der}_{\mathbb{R}}(\mathbf{B}, M)$	The $\mathbf{B}$ -module of the derivations of $\mathbf{B}$ in M	5
$\text{Der}(\mathbf{B})$	The $\mathbf{B}$ -module of the derivations of $\mathbf{B}$	5
$\Omega_{\mathbf{B}/\mathbb{R}}$	The $\mathbf{B}$ -module of the differentials (of Kähler) of $\mathbf{B}$, see also page 332	6

The Basic Local-Global Principle and Systems of Linear Equations

$\pi_{\mathbf{A}, \mathfrak{a}}$	The canonical homomorphism $\mathbf{A} \rightarrow \mathbf{A}/\mathfrak{a}$	15
$\mathbf{A}^{\times}$	The multiplicative group of invertible elements of $\mathbf{A}$	16
$\mathbf{A}_S$	(or $S^{-1}\mathbf{A}$) the localized ring of $\mathbf{A}$ at S	16
$j_{\mathbf{A}, S}$	The canonical homomorphism $\mathbf{A} \rightarrow \mathbf{A}_S$	16
S^{sat}	The saturated monoid of the monoid S	16
$\mathbf{A}[1/s]$	(or $\mathbf{A}_s$) the localized ring of $\mathbf{A}$ at $s^{\mathbb{N}}$	17
$(\mathfrak{b} : \mathfrak{a})_{\mathbf{A}}$	The conductor of the ideal $\mathfrak{a}$ into the ideal $\mathfrak{b}$	17
$(P : N)_{\mathbf{A}}$	The conductor of the module N into the module P	17
$\text{Ann}_{\mathbf{A}}(x)$	The annihilator of the element x	17
$\text{Ann}_{\mathbf{A}}(M)$	The annihilator of the module M	17
$(N : \mathfrak{a})_M$	$\{x \in M \mid \mathfrak{a}x \subseteq N\}$	17
$(N : \mathfrak{a}^{\infty})_M$	$\{x \in M \mid \exists n \ \mathfrak{a}^n x \subseteq N\}$	17
$\text{Reg } \mathbf{A}$	Monoid of the regular elements of $\mathbf{A}$	17
$\text{Frac } \mathbf{A}$	Total ring of fractions of $\mathbf{A}$	17
$\mathbf{A}^{m \times p}$	(or $\mathbb{M}_{m,p}(\mathbf{A})$) matrices with m rows and p columns	19
$\mathbb{M}_n(\mathbf{A})$	$\mathbb{M}_{n,n}(\mathbf{A})$	19
$\mathbb{GL}_n(\mathbf{A})$	Group of invertible matrices	19
$\mathbb{SL}_n(\mathbf{A})$	Group of matrices with determinant 1	19
$\mathbb{AG}_n(\mathbf{A})$	Projection matrices	19

$D_{\mathbf{A}}(\mathfrak{a})$	(or $\sqrt{\mathfrak{a}}$) nilradical of the ideal $\mathfrak{a}$ of $\mathbf{A}$	21
$\mathbf{A}_{\text{red}}$	$\mathbf{A}/D_{\mathbf{A}}(0)$: reduced ring associated with $\mathbf{A}$	21
$c_{\mathbf{A}, \underline{X}}(f)$	(or $c(f)$) ideal of $\mathbf{A}$, content of the polynomial $f \in \mathbf{A}[\underline{X}]$. . .	21
$\text{rk}_{\mathbf{A}}(M)$	Rank of a free module, see also the generalizations to the finitely generated projective modules pages 250, 268 and 531	36
$\text{Adj } B$	(or $\tilde{B}$) cotransposed matrix of B	36
$\mathcal{D}_k(G)$	Determinantal ideal of order k of the matrix G	38
$\mathcal{D}_k(\varphi)$	Determinantal ideal of order k of the linear map φ , see also page 576	39
$\text{rk}(\varphi) \geq k$	Notation understood with Definition II-5.7, see also notation X-6.5	39
$\text{rk}(\varphi) \leq k$	Same thing	39
$E_{ij}^{(n)}(\lambda)$	(or $E_{ij}(\lambda)$) elementary matrix	40
$\mathbb{E}_n(\mathbf{A})$	Elementary group	40
I_k	Identity matrix of order k	41
0_k	Square matrix of order k	41
$0_{k,\ell}$	Null matrix of type $k \times \ell$	41
$I_{k,q,m}$	Standard simple matrix	41
$I_{k,n}$	Standard projection matrix	41
$A_{\alpha,\beta}$	Extracted matrix	42
$\text{Adj}_{\alpha,\beta}(A)$	See notation II.5.12	42
$\mathcal{P}_{\ell}$	Set of finite subsets of $\{1, \dots, \ell\}$	42
$\mathcal{P}_{k,\ell}$	Subsets with k elements	42
$\mathbb{A}\mathbb{G}_{n,k}(\mathbf{A})$	Subsets of $\mathbb{A}\mathbb{G}_n(\mathbf{A})$: projection matrices of rank k	46
$\mathbb{G}_{n,k}(\mathbf{A})$	Projective Grassmannian over $\mathbf{A}$	46
$\mathbb{G}_n(\mathbf{A})$	Projective Grassmannian over $\mathbf{A}$	46
$\mathbb{P}^n(\mathbf{A})$	Projective space of dimension n over $\mathbf{A}$	46
$\text{Diag}(a_1, \dots, a_n)$	Diagonal square matrix	48
$\text{Tr}(\varphi)$	Trace of φ (endomorphism of $\mathbf{A}^n$), see also page 264 . . .	50
$C_{\varphi}(X)$	Characteristic polynomial of φ (idem), see also page 264	50
$[\mathbf{B} : \mathbf{A}]$	$\text{rk}_{\mathbf{A}}(\mathbf{B})$, see also page 320 and X-3.6	50
$\text{Tr}_{\mathbf{B}/\mathbf{A}}(a)$	Trace of (the multiplication by) a , see also VI-3.1	50
$\text{N}_{\mathbf{B}/\mathbf{A}}(a)$	Norm of a , see also VI-3.1	50
$\text{C}_{\mathbf{B}/\mathbf{A}}(a)$	Characteristic polynomial of (the multiplication by) a , see also VI-3.1	50
$\text{Gram}_{\mathbf{A}}(\varphi, \underline{x})$	Gram matrix of $(\underline{x})$ for φ	53
$\text{gram}_{\mathbf{A}}(\varphi, \underline{x})$	Gram determinant of $(\underline{x})$ for φ	53
$\text{disc}_{\mathbf{B}/\mathbf{A}}(\underline{x})$	Discriminant of the family $(\underline{x})$	53
$\text{Disc}_{\mathbf{B}/\mathbf{A}}$	Discriminant of a free extension	53
$\text{L}_{\mathbf{A}}(M, N)$	$\mathbf{A}$ -module of linear maps	57
$\text{End}_{\mathbf{A}}(M)$	$\text{L}_{\mathbf{A}}(M, M)$	57

$M^\star$	Dual module of M	57
$\mathbf{A}[\underline{X}]_d$	$\mathbf{A}$ -submodule of $\mathbf{A}[\underline{X}]$ of the homogeneous polynomials of degree d	60

The Method of Undetermined Coefficients

$P_f(E)$	Set of finite subsets of E	78
$P_{fe}(E)$	Set of finitely enumerated subsets of E	78
$\text{Hom}_{\mathbf{A}}(\mathbf{B}, \mathbf{B}')$	Set of homomorphisms of $\mathbf{A}$ -algebras from $\mathbf{B}$ to $\mathbf{B}'$	85
$\mu_{M,b}$	(or μ_b) $y \mapsto by, \in \text{End}_{\mathbf{B}}(M)$ ($b \in \mathbf{B}$, M a $\mathbf{B}$ -module)	86
$\mathcal{I}(f)$	Ideal of the symmetric relators	89
$\text{Adu}_{\mathbf{A},f}$	Universal splitting algebra of f over $\mathbf{A}$	89
$\text{disc}_X(f)$	Discriminant of the monic polynomial f of $\mathbf{A}[X]$	92
$\text{Tsch}_g(f)$	Tschirnhaus transform of f by g	95
$\text{Min}_{\mathbf{K},x}(T)$	or $\text{Min}_x(T)$, monic minimal polynomial of x (over the field $\mathbf{K}$)	99
$G.x$	Orbit of x under G	101
$G.x = \{x_1, \dots, x_k\}$	Orbit enumerated without repetition with $x_1 = x$. . .	101
$\text{St}_G(x)$	(or $\text{St}(x)$) stabilizer subgroup of the point x	101
$\text{Stp}_G(F)$	(or $\text{Stp}(F)$) point by point stabilizer of the subset F . . .	101
$ G : H $	Index of the subgroup H in the group G : $\#(G/H)$. . .	101
$\text{Fix}_E(H)$	(or E^H) subset of E formed from the fixed points of H . . .	101
$\sigma \in G/H$	We take a σ in each left coset of H in G	101
$C_G(x)(T)$	$= \prod_{\sigma \in G} (T - \sigma(x))$	101
$N_G(x)$	$= \prod_{\sigma \in G} \sigma(x)$	101
$\text{Tr}_G(x)$	$= \sum_{\sigma \in G} \sigma(x)$	101
$\text{Rv}_{G,x}(T)$	Resolvent of x (relative to G)	101
$\text{Aut}_{\mathbf{A}}(\mathbf{B})$	Group of $\mathbf{A}$ -automorphisms of $\mathbf{B}$	102
$\text{Gal}(\mathbf{L}/\mathbf{K})$	Idem, for a Galois extension	102
$\mathcal{G}_{\mathbf{L}/\mathbf{K}}$	Finite subgroups of $\text{Aut}_{\mathbf{K}}(\mathbf{L})$	102
$\mathcal{K}_{\mathbf{L}/\mathbf{K}}$	Strictly finite $\mathbf{K}$ -subextensions of $\mathbf{L}$	102
$\text{Gal}_{\mathbf{K}}(f)$	Galois group of the separable polynomial f	103
$\text{Syl}_X(f, p, g, q)$	Sylvester matrix of f and g in degrees p and q	109
$\text{Res}_X(f, p, g, q)$	Resultant of the polynomials f and g in degrees p and q	109
$\text{char}(\mathbf{K})$	Characteristic of a field	117
$\text{Adj}_{\mathbf{B}/\mathbf{A}}(x)$	or $\tilde{x}$: cotransposed element, see also page 319	122
$(\mathbf{A} : \mathbf{B})$	Conductor of $\mathbf{B}$ into $\mathbf{A}$	128
$\mathfrak{R}_X(f, g_1, \dots, g_r)$		130

$\text{JAC}_{\underline{X}}(\underline{f})$	Jacobian matrix of a polynomial system	137
$\text{Jac}_{\underline{X}}(\underline{f})$	Jacobian of a polynomial system	137
$ L : E _{\mathbf{A}}$	Index of a finitely generated submodule in a free module	146

Finitely Presented Modules

$R_{\underline{a}}$	Matrix of the trivial relators	178
$\langle \underline{x} \underline{z} \rangle$	$\sum_{i=1}^n x_i z_i$	179
$\mathfrak{m}_{\underline{\xi}}$	$\langle x_1 - \xi_1, \dots, x_n - \xi_n \rangle_{\mathbf{A}}$: ideal of the zero $\underline{\xi}$	182
$M \otimes_{\mathbf{A}} N$	Tensor product of two $\mathbf{A}$ -modules	187
$\bigwedge_{\mathbf{A}}^k M$	k th exterior power of M	189
$\mathbf{S}_{\mathbf{A}}^k M$	k th symmetric power of M	189
$\rho_{\star}(M)$	$\mathbf{B}$ -module obtained from the $\mathbf{A}$ -module M by the scalar extension $\rho : \mathbf{A} \rightarrow \mathbf{B}$	192
$\mathcal{F}_n(M)$	or $\mathcal{F}_{\mathbf{A},n}(M)$: n^{th} Fitting ideal of the finitely generated $\mathbf{A}$ -module M	214
$\Re s_X(\mathfrak{f})$	Resultant ideal of $\mathfrak{f}$ (with a monic polynomial in $\mathfrak{f}$) . . .	218
$\mathcal{K}_n(M)$	n^{th} Kaplansky ideal of the $\mathbf{A}$ -module M	224

Finitely Generated Projective Modules, 1

$\theta_{M,N}$	Natural $\mathbf{A}$ -linear map $M^{\star} \otimes_{\mathbf{A}} N \rightarrow L_{\mathbf{A}}(M, N)$	240
θ_M	Natural $\mathbf{A}$ -linear map $M^{\star} \otimes_{\mathbf{A}} M \rightarrow \text{End}_{\mathbf{A}}(M)$	240
$\text{Diag}(M_1, \dots, M_n)$	Block diagonal matrix	246
$\text{Bdim } \mathbf{A} < n$	Stable range (of Bass) less than or equal to n	253
$\det \varphi$	Determinant of the endomorphism φ of a finitely generated projective module	264
$C_{\varphi}(X)$	Characteristic polynomial of φ ... (idem)	264
$\tilde{\varphi}$	Cotransposed endomorphism of φ ... (idem)	264
$F_{\varphi}(X)$	Fundamental polynomial of φ , i.e., $\det(\text{Id}_P + X\varphi)$. . .	266
$\text{Tr}_P(\varphi)$	Trace of the endomorphism φ	266
$R_P(X)$	Rank polynomial of the finitely generated projective module P	266
$e_h(P)$	The idempotent associated with the integer h and with the projective module P	267
$P^{(h)}$	Component of the module P in rank h	272

Strictly Finite Algebras and Galois Algebras

$C_{\mathbf{B}/\mathbf{A}}(x)(T)$	Characteristic polynomial of (the multiplication by) $x \dots$	307
$F_{\mathbf{B}/\mathbf{A}}(x)(T)$	Fundamental polynomial of (the multiplication by) $x \dots$	307
$N_{\mathbf{B}/\mathbf{A}}(x)$	Norm of x : determinant of the multiplication by $x \dots$	307
$\mathrm{Tr}_{\mathbf{B}/\mathbf{A}}(x)$	Trace of (the multiplication by) $x \dots$	307
$a \cdot a$	$\alpha \circ \mu_a : x \mapsto \alpha(ax) \dots$	318
$\mathrm{Adj}_{\mathbf{B}/\mathbf{A}}(x)$	or $\tilde{x}$: cotransposed element $\dots$	319
$[\mathbf{B} : \mathbf{A}]$	$\mathrm{rk}_{\mathbf{A}}(\mathbf{B})$, see also pages 50 and 537 $\dots$	320
$\Phi_{\mathbf{A}/\mathbf{k}, \lambda}$	$\Phi_{\lambda}(x, y) = \lambda(xy) \dots$	321
$\phi \otimes \phi'$	Tensor product of bilinear forms $\dots$	325
$\mathbf{A}_{\mathbf{k}}^e$	$\mathbf{A} \otimes_{\mathbf{k}} \mathbf{A}$, enveloping algebra of $\mathbf{A}/\mathbf{k} \dots$	328
$J_{\mathbf{A}/\mathbf{k}}$	Ideal of $\mathbf{A}_{\mathbf{k}}^e \dots$	328
$\Delta_{\mathbf{A}/\mathbf{k}}$	$\Delta(x) = x \otimes 1 - 1 \otimes x \dots$	328
$\mu_{\mathbf{A}/\mathbf{k}}$	$\mu_{\mathbf{A}/\mathbf{k}}(\sum_i a_i \otimes b_i) = \sum_i a_i b_i \dots$	328
$\mathrm{Der}_{\mathbf{k}}(\mathbf{A}, M)$	The $\mathbf{A}$ -module of the derivations of $\mathbf{A}$ in $M \dots$	332
$\mathrm{Der}(\mathbf{A})$	The $\mathbf{A}$ -module of the derivations of $\mathbf{A} \dots$	332
$\Omega_{\mathbf{A}/\mathbf{k}}$	The $\mathbf{A}$ -module of the differentials (of Kähler) of $\mathbf{A} \dots$	332
$\varepsilon_{\mathbf{A}/\mathbf{k}}$	Idempotent that generates $\mathrm{Ann}(J_{\mathbf{A}/\mathbf{k}})$, if it exists $\dots$	336
$\mathrm{Lin}_{\mathbf{k}}(\mathbf{A}, \mathbf{A})$	$\mathbf{A}$ -module of $\mathbf{k}$ -linear maps from $\mathbf{A}$ to $\mathbf{A} \dots$	345
$\mathbb{P}\mathrm{GL}_n(\mathbf{A})$	Quotient group $\mathbb{G}\mathrm{L}_n(\mathbf{A})/\mathbf{A}^{\times} \dots$	362
A_n	Subgroup of even permutations of $S_n \dots$	359
$\mathbf{k}[G]$	Algebra of a group, or of a monoid $\dots$	359

The Dynamic Method

$\mathbb{B}(\mathbf{A})$	Boolean algebra of the idempotents of $\mathbf{A} \dots$	392
$\mathcal{B}(f)$	“Canonical” basis of the universal splitting algebra $\dots$	398

Local Rings, or Just About

$\mathrm{Rad}(\mathbf{A})$	Radical of Jacobson of $\mathbf{A} \dots$	478
$\mathbf{A}(X)$	Nagata ring of $\mathbf{A}[X] \dots$	505
$\mathrm{Suslin}(b_1, \dots, b_n)$	Suslin set of $(b_1, \dots, b_n) \dots$	509

Finitely Generated Projective Modules, 2

$\mathbf{G}_n$	$\mathbf{G}_n = \mathbb{Z}[(f_{i,j})_{i,j \in [1..n]}] / \mathcal{G}_n \dots$	530
$\mathcal{G}_n$	Relations obtained when writing $F^2 = F \dots$	530
$\mathbf{H}_0^+(\mathbf{A})$	Semi-ring of ranks of quasi-free $\mathbf{A}$ -modules $\dots$	531
$[P]_{\mathbf{H}_0^+(\mathbf{A})}$	or $[P]_{K_0}(\mathbf{A})[P]_{\mathbf{A}}$, or $[P]$: class of a quasi-free $\mathbf{A}$ -module in $\mathbf{H}_0^+(\mathbf{A}) \dots$	531

$\text{rk}_{\mathbf{A}}(M)$	(Generalized) rank of the finitely generated projective $\mathbf{A}$ -module M	531
$\mathbf{H}_0 \mathbf{A}$	Ring of the ranks over $\mathbf{A}$	532
$[\mathbf{B} : \mathbf{A}]$	$\text{rk}_{\mathbf{A}}(\mathbf{B})$, see also pages 50 and 320	537
$\mathbf{G}_n(\mathbf{A})$	$\mathbf{G}_n \otimes_{\mathbb{Z}} \mathbf{A}$	540
$\mathcal{G}_{n,k}$	$\mathcal{G}_n + \langle 1 - r_k \rangle$, with (in $\mathbf{G}_n$) $r_k = e_k(\text{Im } F)$	540
$\mathbf{G}_{n,k}$	$\mathbf{G}_{n,k} = \mathbb{Z}[(f_{i,j})_{i,j \in [1..n]}] / \mathcal{G}_{n,k}$ or $\mathbf{G}_n[1/r_k]$	540
$\mathbb{A}\mathbf{G}_{n,k}(\mathbf{A})$	“Subvariety” of $\mathbb{A}\mathbf{G}_n(\mathbf{A})$: projectors of rank k	540
$\mathbf{GK}_0 \mathbf{A}$	Semi-ring of the isomorphism classes of finitely generated projective modules over $\mathbf{A}$	557
$\mathbf{Pic} \mathbf{A}$	Group of isomorphism classes of projective modules of constant rank 1 over $\mathbf{A}$	557
$\mathbf{K}_0 \mathbf{A}$	Grothendieck ring of $\mathbf{A}$	557
$[P]_{\mathbf{K}_0(\mathbf{A})}$	or $[P]_{\mathbf{A}}$, or $[P]$: class of a finitely generated projective $\mathbf{A}$ -module in $\mathbf{K}_0(\mathbf{A})$	557
$\tilde{\mathbf{K}}_0 \mathbf{A}$	Kernel of the rank homomorphism $\text{rk} : \mathbf{K}_0 \mathbf{A} \rightarrow \mathbf{H}_0 \mathbf{A}$	558
$\text{Ifr} \mathbf{A}$	Monoid of the finitely generated fractional ideals of the ring $\mathbf{A}$	560
$\text{Gfr} \mathbf{A}$	Group of invertible elements of $\text{Ifr} \mathbf{A}$	560
$\text{Cl} \mathbf{A}$	Group of classes of invertible ideals (quotient of $\text{Gfr} \mathbf{A}$ by the subgroup of invertible principal ideals)	560

Distributive Lattices, Lattice-Groups

$\downarrow a$	$\{x \in X x \leq a\}$, see also page 612	610
$\uparrow a$	$\{x \in X x \geq a\}$, see also page 612	610
$\mathbf{T}^\circ$	Opposite lattice of the lattice $\mathbf{T}$	611
$\mathcal{I}_{\mathbf{T}}(J)$	Ideal generated by J in the distributive lattice $\mathbf{T}$	613
$\mathcal{F}_{\mathbf{T}}(S)$	Filter generated by S in the distributive lattice $\mathbf{T}$	613
$\mathbf{T}/(J = 0, U = 1)$	Particular quotient lattice	614
$\mathbb{B}\mathbf{o}(\mathbf{T})$	Boolean algebra generated by the distributive lattice $\mathbf{T}$	617
$\mathbb{Z}^{(P)}$	Orthogonal direct sum of copies of $\mathbb{Z}$, indexed by P	618
$\boxplus_{i \in I} G_i$	Orthogonal direct sum of ordered groups	618
$\mathcal{C}(a)$	Solid subgroup generated by a (in an l -group)	621
$D_{\mathbf{A}}(x_1, \dots, x_n)$	$D_{\mathbf{A}}(\langle x_1, \dots, x_n \rangle)$: an element of $\mathbf{Zar} \mathbf{A}$	634
$\mathbf{Zar} \mathbf{A}$	Zariski lattice of $\mathbf{A}$	634
$\mathbf{A}_S / \mathfrak{a}$	(or $S^{-1} \mathbf{A} / \mathfrak{a}$) we invert the elements of S and we annihilate the elements of $\mathfrak{a}$	635
$S^{\text{sat}}_{\mathbf{A}}$	or S^{sat} : the filter obtained by saturating the monoid S in $\mathbf{A}$	638
$\mathbf{A}^\bullet$	Reduced zero-dimensional ring generated by $\mathbf{A}$	644

$A \vdash B$	$\bigwedge A \leq \bigvee B$: implicative relation	647
Spec $\mathbf{T}$	Spectrum of the finite distributive lattice $\mathbf{T}$, see also page 737	649
$(b : a)_{\mathbf{T}}$	The conductor ideal of a in b (distributive lattices) . . .	650
$\mathbf{A}_{\text{pp}}$	pp-ring closure of $\mathbf{A}$	657
Min $\mathbf{A}$	Subspace of Spec $\mathbf{A}$ formed by the minimal prime ideals	654

Prüfer and Dedekind Rings

$a \div b$	$\{x \in \text{Frac } \mathbf{A} \mid xb \subseteq a\}$	676
$\mathbf{A}[\mathfrak{a}]$	Rees algebra of the ideal $\mathfrak{a}$ of $\mathbf{A}$	678
$\text{Icl}_{\mathbf{A}}(\mathfrak{a})$	Integral closure of the ideal $\mathfrak{a}$ in $\mathbf{A}$	678

Krull Dimension

Spec $\mathbf{A}$	Zariski spectrum of the ring $\mathbf{A}$	736
$\mathfrak{D}_{\mathbf{A}}(x_1, \dots, x_n)$	Compact-open set of Spec $\mathbf{A}$	736
Spec $\mathbf{T}$	Spectrum of the distributive lattice $\mathbf{T}$	737
$\mathfrak{D}_{\mathbf{T}}(u)$	Compact-open set of Spec $\mathbf{T}$	737
Oqc $(\mathbf{T})$	Distributive lattice of the compact-open sets of Spec $\mathbf{T}$	737
$\mathcal{J}_{\mathbf{A}}^{\mathbf{K}}(x)$	$\langle x \rangle + (\mathbf{D}_{\mathbf{A}}(0) : x)$: Krull boundary ideal of x in $\mathbf{A}$	739
$\mathcal{J}_{\mathbf{A}}^{\mathbf{K}}(\mathfrak{a})$	$\mathfrak{a} + (\mathbf{D}_{\mathbf{A}}(0) : \mathfrak{a})$: Krull boundary ideal of $\mathfrak{a}$ in $\mathbf{A}$. . .	739
$\mathbf{A}_{\mathbf{K}}^x$	$\mathbf{A} / \mathcal{J}_{\mathbf{A}}^{\mathbf{K}}(x)$: upper boundary ring of x in $\mathbf{A}$	739
$\mathcal{S}_{\mathbf{A}}^{\mathbf{K}}(x)$	$x^{\mathbb{N}}(1 + x\mathbf{A})$: Krull boundary monoid of x in $\mathbf{A}$	739
$\mathbf{A}_{\mathbf{K}}^{\mathbf{K}}$	$(\mathcal{S}_{\mathbf{A}}^{\mathbf{K}}(x))^{-1}\mathbf{A}$: lower boundary ring of x in $\mathbf{A}$	739
Kdim $\mathbf{A} \leq r$	The Krull dimension of the ring $\mathbf{A}$ is $\leq r$	740
Kdim $\mathbf{A} \leq \text{Kdim } \mathbf{B}$		741
$\mathcal{S}_{\mathbf{A}}^{\mathbf{K}}(x_0, \dots, x_k)$	Iterated Krull boundary monoid	742
$\mathcal{J}_{\mathbf{A}}^{\mathbf{K}}(x_0, \dots, x_k)$	Iterated Krull boundary ideal	742
$\mathcal{I}_{\mathbf{A}}^{\mathbf{K}}(x_0, \dots, x_k)$	Iterated Krull boundary ideal, variant	742
Kdim $\mathbf{T} \leq r$	The Krull dimension of the distributive lattice $\mathbf{T}$ is $\leq r$	754
$\mathcal{J}_{\mathbf{T}}^{\mathbf{K}}(x)$	$\downarrow x \vee (0 : x)_{\mathbf{T}}$: Krull boundary ideal of x in the distributive lattice $\mathbf{T}$	755
$\mathbf{T}_{\mathbf{K}}^x$	$\mathbf{T} / \mathcal{J}_{\mathbf{T}}^{\mathbf{K}}(x)$: upper boundary lattice of x	755
$\mathcal{J}_{\mathbf{T}}^{\mathbf{K}}(x_0, \dots, x_k)$	Iterated Krull boundary ideal in a distributive lattice	756
Kdim ρ	Krull dimension of the morphism ρ	757
$\mathbf{A}_{\{a\}}$	$\mathbf{A} / a^{\perp} \times \mathbf{A} / (a^{\perp})^{\perp}$	760
$\mathbf{A}_{\min}$	Minimal pp-ring closure of $\mathbf{A}$	761
Vdim $\mathbf{A}$	Valuative dimension	768