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Mihir K. Chakraborty
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Facets of Uncertainties and Applications

ICFUA, Kolkata, India, December 2013

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Preface

The international conference on "Facets of Uncertainties and Applications" (ICFUA 2013) was organized under the joint collaboration of the Operational Research Society of India (the Calcutta chapter) and Department of Applied Mathematics, University of Calcutta.

The conference aimed at contributing to better understanding between practitioners (both the theoreticians and researchers involved in applications) dealing with uncertainties, mainly of nonprobabilistic category. These papers, but one, presented at the conference, focus on various types of uncertainties which are essentially nonprobabilistic in nature. These types include vagueness, roughness, incompleteness, ambiguity, and such other features. Various mathematical formalisms have emerged during the past few decades to deal with such uncertainties, for example, fuzzy set theory, rough set theory, soft set theory, uncertainty theory. Papers compiled here are of two categories: invited articles presented at the plenary sessions and contributed articles read at regular sessions of the said conference. Invited articles are from experts of high standing in the field, while contributed articles are by senior and young researchers. The papers deal with the state of the art of the theories as well as their applications.

The scope of the conference included the following topics:

- Modeling different types of uncertainty (nonprobabilistic)
- Logic of uncertainty (fuzzy logic and rough logic)
- Rough sets and fuzzy sets in approximate reasoning
- Rough fuzzy hybridization and applications
- Analysis of complex systems and complex network
- Applications of fuzzy sets and rough sets in optimization and decision-making problems
- Image and speech signal processing, prediction, and control
- Robotics

- Expert systems
- Biology and medicine
- Business and management
- Noncomputational mathematics
- Complex system analysis
- Risk management
- Environment engineering
- Data mining
- Other applications

The program of the conference was organized mainly along four tracks:

- Uncertainty modeling
- Logic of uncertainty
- Hybridization of uncertainties
- Role of uncertainties in real problems

Each track contained a plenary session followed by three concurrent parallel sessions. Both the plenary and parallel sessions provided participants ample opportunity to exchange ideas on further research, research collaboration, and training.

The conference was highly interactive and intensive in nature and attracted budding researchers and young faculties working in related disciplines. The conference attracted more than 80 participants from India and abroad. The exchange among these participants has provided them with a comprehensive overview of the techniques and approaches being applied to uncertainty theory and applications.

The program committee for this conference consisted of:

- Didier Dubois, University Paul Sabatier, Toulouse
- Baoding Liu, Tsinghua University, China
- Andrzej Skowron, Warsaw University, Poland
- Roman Slowinski, Poznan University of Technology, Poland
- Dominik Slezak, University of Warsaw, Poland
- Piero Pagliani, Research group on Knowledge and Communication, Italy
- Davide Ciucci, Italy
- Manoranjan Maiti, Vidyasagar University, India
- Amit Konar, Jadavpur University, India
- Mohua Banerjee, IIT Kanpur, India

The conference was supported by the Department of Applied Mathematics, University of Calcutta; Board of Research in Nuclear Science (BRNS), Department of Atomic Energy (DAE), India; Department of Science and Technology (DST), West Bengal; Department of Higher Education, West Bengal; and Indian Statistical Institute (ISI), Kolkata. We are grateful to these organizations for their very generous support.

We thank all the authors for kindly submitting their articles to the conference proceedings. We are very thankful to all the reviewers for their constructive comments and suggestions for the finalization of the papers and to the editorial board of Springer for supporting the publication of the present volume.

Mihir K. Chakraborty
Andrzej Skowron
Manoranjan Maiti
Samarjit Kar

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About the Editors

Mihir Kumar Chakraborty, Ph.D., is visiting professor at the School of Cognitive Sciences, Jadavpur University, and director of Sivatosh Mookerjee Centre of Sciences, Kolkata. Earlier, he was professor of Pure Mathematics at the University of Calcutta. Professor Chakraborty had also been visiting professor at the Centre for Soft Computing Research, Indian Statistical Institute, Kolkata; Institute for Logic Language and Cognition, Sun Yat-Sen University, Guangzhou, China; Institut de Recherche en Informatique de Toulouse (IRIT), University Paul Sabataire, Toulouse, France; University of Paris VIII, France; University of Wollongong, Australia; University of Regina, Canada; NIAS, Bangalore, India; and VisvaBharati, Santiniketan, India. A recipient of Deutscher Akademischer Austuasch Dienst (DAAD) Fellowship, IISc Fellowship, and Fellowship of West Bengal Academy of Sciences, Professor Chakraborty is member of the Council and Research Project Committee (RPC) of Indian Council for Philosophical Research (ICPR), and guest professor at South West University of Chongqing, China. He has about 150 research papers to his credit in several international journals and edited volumes, co-authored one book *A Geometry of Approximation* (Springer), authored three books in vernacular Bengali on philosophy of mathematics, and coedited several scientific publications. A member of the editorial board of several international journals and a book series *Logic in Asia: Studia Logic Library* (Springer), Professor Chakraborty's area of research include nonstandard logics, rough set theory, fuzzy set theory, reasoning in uncertainty and vagueness, logic of diagrams, topology/functional analysis, and philosophy of mathematics. He has also supervised 16 Ph.D. students. Professor Chakraborty is founder of Calcutta Logic Circle, Association for Logic in India, Indian Society for Fuzzy Mathematics and Information Processing (ISFUMIP), and member of the advisory board of the International Rough Set Society and Indian Rough Set Society.

Andrzej Skowron, European Coordination Committee for Artificial Intelligence (ECCAI) Fellow, received his Ph.D. and D.Sc. (Habilitation) from the University of Warsaw, Poland. In 1991, he received the Scientific Title of Professor. He is full professor in the Faculty of Mathematics, Computer Science and Mechanics at the

University of Warsaw, and honorary professor of Chongqing University of Posts and Telecommunications, China. He is the author of more than 400 scientific publications and editor of many books. His areas of expertise include reasoning with incomplete information, approximate reasoning, soft computing methods and applications, rough sets, rough mereology, granular computing, intelligent systems, knowledge discovery and data mining, decision support systems, adaptive and autonomous systems, perception-based computing, and interactive computational systems. He has supervised more than 20 Ph.D. theses and was the editor-in-chief of the journal *Fundamenta Informaticae* during 1995–2009. Professor Skowronis also on the editorial boards of many international journals. During 1996–2000, he was the president of the International Rough Set Society. He delivered numerous invited talks at international conferences including plenary talk at the 16th IFIP World Computer Congress (Beijing, 2000) and served as program chair at more than 200 international conferences. He was involved in numerous research and commercial projects.

Manoranjan Maiti, Ph.D., has earlier worked at Indian Institute of Tropical Meteorology, Poona; Structural Engineering Division, Vikram Sarabhai Space Centre, ISRO, Trivandrum; Department of Mathematics, Calcutta University Post Graduate Centre (presently, Tripura University), Agartala; Department of Applied Mathematics, Vidyasagar University, West Bengal. He was also dean, Faculty Council of Science, for a period of 10 years and vice-chancellor (pro-tempore) of Vidyasagar University, for a short period. Twenty six students have been awarded Ph.D. degree in mathematics under his guidance at Vidyasagar University and NIT Durgapur, West Bengal, as well as several students are pursuing Ph.D. under him. He has published more than 250 research papers in several international journals. He was associate editor of Applied Mathematical Modelling. His fields of interest include inventory control system, supply chain, fuzzy optimization, transportation, etc.

Samarjit Kar completed his Ph.D., in mathematics from Vidyasagar University, West Bengal. He is associate professor at the Department of Mathematics, National Institute of Technology, Durgapur, India. With over 15 years of experience in teaching, Professor Kar is also a visiting professor at the Department of Mathematical Sciences, Tsinghua University, China. Moreover he has visited several universities and institutes in India and abroad. He has co-authored more than 120 technical articles in international journals, contributed volumes and conference proceedings. An author of 2 textbooks and edited 5 contributed books, Professor Kar has guided 10 Ph.D. students. He is the associate editor of Journal of *Uncertainty Analysis and Application* (Springer) and is presently associated with an ongoing project, "Hybrid modelling of uncertainty analysis in environmental risk assessments" under BRNS, Department of Atomic Energy (DAE), Govt. of India. His research interests include operations research and optimization, soft computing, uncertainty theory and financial modelling.

Part I
Uncertainty Modelling

Rough Sets and Other Mathematics: Ten Research Programs

Piero Pagliani

Abstract Since its inception, interesting connections between Rough Set Theory and different mathematical and logical topics have been investigated. This paper is a survey of some less known although highly interesting connections, which extend from Rough Set Theory to other mathematical and logical fields. The survey is primarily thought of as a guide for new directions to be explored.

Keywords Rough sets · Algebraic logic · Topology

1 Information from Data and Information as Metaphor

As is well known, the starting point of Rough Set Theory is an *indiscernible space* $\langle U, E\rangle$, where U is a set and $E \subseteq U \times U$ is an equivalence relation such that $\langle x, y\rangle \in E$ states that items x and y take exactly the same attribute-values according to an evaluation recorded in an *Information System*.

Given any relational structure $\langle U, R\rangle$, with $R \subseteq U \times U$, and $X \subseteq U$, the set $R(X) = \{y : \exists x \in X(\langle x, y\rangle \in R)\}$ will by named the *R-neighborhood of* X. If $X = \{a\}$ we shall write $R(a)$. Thus, by means of $E-$neighborhoods, from any indiscernibility space the following operators are defined on $\wp(U)$:

$$(lE)(X) = \{x : E(x) \subseteq X\} = \{x : \forall y(\langle x, y\rangle \in E \Rightarrow y \in X)\} \tag{1}$$

$$(uE)(X) = \{x : E(x) \cap X \neq \emptyset\} = \{x : \exists y(\langle x, y\rangle \in E \wedge y \in X)\} \tag{2}$$

$(lE)(X)$ is called the *lower approximation* of X (*via* E), while $(uE)(X)$ is called the *upper approximation* of X (*via* E), and $\langle U, (uE), (lE)\rangle$ is called an *approximation space*. Any equivalence class modulo E is a neighborhood $E(a)$ for some $a \in U$, and represents a "basic property," that is, a unique array of attribute-values, hence a subset of U *definable* by means of the given evaluation. Moreover, forany X, $(lE)(X)$ and

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$(uE)(X)$ output either an E−equivalence class or a union of equivalence classes, which represents nonelementary definable set.

The well-known properties of (lE) and (uE) depend on the fact that E is an equivalence relation. Particularly, it is possible to show that $\{(lE)(X) : X \subseteq U\} = \{(uE)(X) : X \subseteq U\}$. This set will be denoted by $Df(U)$. It collects the *definable subsets* of U. If $X \in Df(U)$, then $X = (lE)(X) = (uE)(X)$: a definable set does not need to be approximated.

All the concepts not introduced in the paper can be found in [16][1]

Observation 1 *The second part of definitions* (1) *and* (2) *displays the dual logical constructions* $(\forall, \Rightarrow)$ *and* $(\exists, \wedge)$. *They are the backbone to a number of mathematical concepts. Notably,* $(\forall, \Rightarrow)$ *is the logical core of* interior *and* necessity *operators, while* $(\exists, \wedge)$ *is that of* closure *and* possibility *operators.*

We remind that an operator ϕ on a lattice $\mathbf{L}$ is an interior (resp. closure) operator if it is (i) decreasing: $\phi(x) \leq x$ (resp. increasing: $x \leq \phi(x)$), (ii) monotone: $x \leq y$ implies $\phi(x) \leq \phi(y)$ and (iii) idempotent: $\phi(\phi(x)) = \phi(x)$. An interior (resp. closure) operator ϕ is *topological* if it is (iv) multiplicative: $\phi(x \wedge y) = \phi(x) \wedge \phi(y)$ (resp. additive: $\phi(x \vee y) = \phi(x) \vee \phi(y)$) and (v) co-normal: $\phi(1) = 1$ (resp. normal: $\phi(0) = 0$).

Indeed, (lE) and (uE) are interior and, respectively, closure topological operators. This was one of the first results of Pawlak's approximation spaces and it can be stated under different points of views:

Facts 1
1. (lE) is an interior, $\mathbb{I}$, and (uE) a closure, $\mathbb{C}$, operator of a topological space with a basis of clopen - closed and open - subsets.
2. $\mathbf{AS}(U) = \langle Df(U), \cap, \cup, -, \emptyset, U\rangle$ is a subalgebra of the Boolean on $\wp(U)$.
3. $\langle \wp(U), \cap, \cup, -, \emptyset, U, (uE), (lE)\rangle$ is a topological Boolean algebra.
4. $\langle \wp(U), \cap, \cup, -, \emptyset, U, (uE), (lE)\rangle$ is a model for S5 modal logic, where (lE) stands for the necessity operator $\Box$, and (uE) for the possibility operator $\Diamond$.

By extension, $\mathbf{AS}(U)$ will also be called an approximation space. Since the two approximation operators are defined by means of E-neighborhoods, straightforward generalizations were proposed in the Rough Set literature since its inception, by considering other types of binary relations, R. However, problems arise if definitions (1) and (2) are merely traced. For instance, $(lR)(X)$ could fail to be decreasing, which is "strange" for a *lower* approximation. In view of these problems, usually generalized approximation operators do not mechanically trace the original definitions (see for instance [6]). Anyway, in view of Observation 1, when definitions (1) and (2) are used, properties of generalized upper and lower approximations can be easily derived from the literature on modal logics with Kripke models (see [16], Chap. 4.13). The following issue arises:

[1]Except for [7], this book is the only work of the author's that will be cited. The story of the results can be found in the mentioned chapters.

ISSUE A. INFORMATIONAL INTERPRETATION OF KRIPKEAN MODAL LOGICS, THROUGH ROUGH SETS: *For any binary relation* R, *give* (lR) and (uR) *a meaningful informational interpretation. That is, a concretely justified interpretation of* R, *such as that of relative accessibility relation (as defined in* [15]*). Conversely, give a modal interpretation to generalized approximation operators.*

2 Algebras of Rough Sets

Given an approximation space $\mathbf{AS}(U)$, a *rough set* is an equivalence class modulo (lE) and (uE) on the powerset $\wp(U)$. Thus, the rough set of X can be identified by the ordered pair $\langle (uE)(X), (lE)(X)\rangle$, called *decreasing representation*, or $\langle (lE)(X), -(uE)(X)\rangle$, called *disjoint representation*. The symbol $rs(X)$ will denote both of these representations. However, not all the ordered pairs of decreasing (disjoint) elements of $\mathbf{AS}(U)$ represent a rough set. In fact, if $S \subseteq U$ is a singleton equivalence class, then for any $X \subseteq U$ the following equivalent conditions hold (cf. [16], Chap. 7):

$$\text{(a) } S \subseteq (uE)(X) \textit{ iff } S \subseteq (lE)(X); \;\; \text{(b) } S \subseteq (lE)(X) \textit{ or } S \subseteq -(uE)(X) \qquad (3)$$

The informational explanation of this fact is that singletons represent completely defined objects. Thus, U divides into two parts: an *exact* part, given by the union B of all singleton equivalence classes, and an *uncertain* part, given by its complement $P = U \cap -B$. Indeed, a clause equivalent to conditions (3) is $(uE)(X) \cap B = (lE)(X) \cap B$. It states a *local* property: on B there is no roughness because lower and upper approximations coincide, which is the characteristic of definable sets. Consequently, the set of all and only the rough sets of an approximation space $\mathbf{AS}(U)$ is definable as follows.

In decreasing representation:

$$Dc_{\equiv_{JB}}(\mathbf{AS}(U)) = \{\langle A_1, A_2\rangle \in \mathbf{AS}(U)^2 : A_2 \Rightarrow A_1 = U, A_1 \Rightarrow A_2 \equiv_{JB} U\} \quad (4)$$

where for all $X, Y \in \mathbf{AS}(U)$, $X \Rightarrow Y$ is $-X \cup Y$ and $X \equiv_{JB} Y$ if and only if $B \Rightarrow X = B \Rightarrow Y$. So, the first clause just means $A_2 \subseteq A_1$, while conditions (3) follow from the second clause.

For the disjoint representation we have:

$$Dj_{\equiv_{JB}}(\mathbf{AS}(U)) = \{\langle A_1, A_2\rangle \in \mathbf{AS}(U)^2 : A_1 \cap A_2 = \emptyset, A_1 \cup A_2 \equiv_{JB} U\}. \quad (5)$$

With $D_{\equiv_{JB}}(\mathbf{AS}(U))$ we denote either of these collections. Now, notice that $\equiv_{JB}$ is a (Boolean) congruence on $\mathbf{AS}(U)$. In general, given any Heyting algebra $\mathbf{H}$ and a Boolean congruence $\equiv$ on it (i.e. $\mathbf{H}/_{\equiv}$ is a Boolean algebra), the operations in the following table are definable on the set $Dj_{\equiv}(\mathbf{H})$. If $\mathbf{H}$ is a Boolean algebra, corresponding operations are definable on the set $Dc_{\equiv}(\mathbf{H})$. It is understood that

$a = \langle a_1, a_2 \rangle, b = \langle b_1, b_2 \rangle$ and the operations between elements of the pairs are those of $\mathbf{H}$:

Symbol	$Dc_{\equiv}(\mathbf{H})$	$Dj_{\equiv}(\mathbf{H})$	Name
$0, 1$	$\langle 0, 0 \rangle, \langle 1, 1 \rangle$	$\langle 0, 1 \rangle, \langle 1, 0 \rangle$	Bottom, resp. Top
$\sim a$	$\langle \neg a_2, \neg a_1 \rangle$	$\langle a_2, a_1 \rangle$	Strong negation
$a \longrightarrow b$	$\langle a_2 \Rightarrow b_1, a_2 \Rightarrow b_2 \rangle$	$\langle a_1 \Rightarrow b_1, a_1 \wedge b_2 \rangle$	Weak implication
$a \wedge b$	$\langle a_1 \wedge b_1, a_2 \wedge b_2 \rangle$	$\langle a_1 \wedge b_1, a_2 \vee b_2 \rangle$	Inf
$a \vee b$	$\langle a_1 \vee b_1, a_2 \vee b_2 \rangle$	$\langle a_1 \vee b_1, a_2 \wedge b_2 \rangle$	Sup

Derived operations:

Symbol	Definition	Name
$\lrcorner a$	$a \longrightarrow 0$	Weak negation
$a \supset b$	$\sim \lrcorner \sim a \vee b \vee (\lrcorner a \wedge \lrcorner \sim b)$	Pre-relative pseudocomplementation
$\neg a$	$a \supset 0 = \sim \lrcorner \sim a$	Pre-pseudocomplementation
$a \subset b$	$\sim (\sim a \supset \sim b)$	Pre-relative co-pseudocomplementation
$a \overset{c}{\Longrightarrow} b$	$\sim \lrcorner (a \supset b)$	Pre-relative pseudosupplementation
$a \overset{c}{\Longleftarrow} b$	$\lrcorner \sim (a \subset b)$	Pre-relative co-pseudosupplementation
!	$1 \overset{c}{\Longrightarrow} a$	Pre-pseudosupplementation
¡	$0 \overset{c}{\Longleftarrow} a$	Pre-copseudosupplementation

Facts 2 (cf. [16], Chaps. 7, 8 and 9.6) (1) $\sim\sim a = a$; (2) $\sim (a \wedge b) = \sim a \vee \sim b$; (3) $\sim (a \vee b) = \sim a \wedge \sim b$; (4) $\lrcorner a = \langle \neg a_2, \neg a_2 \rangle$in $Dc_{\equiv}(\mathbf{H})$ and $\langle \neg a_1, a_1 \rangle$ in $Dj_{\equiv}(\mathbf{H})$, (5) $\neg a = \langle \neg a_1, \neg a_1 \rangle$ in $Dc_{\equiv}(\mathbf{H})$ and $\langle a_2, \neg a_2 \rangle$ in $Dj_{\equiv}(\mathbf{H})$. If $\mathbf{H}$ is a Boolean algebra: (6) $\supset$ is a *relative pseudocomplementation* in the lattices $\langle Dc_{\equiv}(\mathbf{H}), \leq \rangle$ and $\langle Dj_{\equiv}(\mathbf{H}), \leq \rangle$, where $a \leq b$ if and only if $a \wedge b = a$. Hence, for all a, b, c of these lattices, $c \wedge a \leq b$ iff $c \leq a \supset b$. As a consequence, $\neg$ is a *pseudocomplementation.* (7) $\subset$ is a *relative co-pseudocomplementation,* that is, $c \vee a \geq b$ iff $c \geq a \subset b$. Since $\lrcorner a = a \subset 1$, $\lrcorner$ is a *co-pseudocomplementation;* (8) $¡a = \sim \neg a = \lrcorner \sim a = \neg\neg a = \lrcorner \neg a$; (9) $!a = \sim \lrcorner a = \neg \sim a = \lrcorner \lrcorner a = \neg \lrcorner a$.

In what follows we set $D_1 = \phi_1 = ¡$, $D_2 = \phi_2 = !$, $e_0 = 0$, $e_2 = 1$ and $e_1 = \langle U, B \rangle$ if $D_{\equiv_{JB}}$ is $Dc_{\equiv_{JB}}$, while $e_1 = \langle B, \emptyset \rangle$ if $D_{\equiv_{JB}}$ is $Dj_{\equiv_{JB}}$.
Since any approximation space $\mathbf{AS}(U)$ is a Boolean algebra one can prove:

Facts 3 (cf. [16], Chaps. 6–10) Let $B \in \mathbf{AS}(U)$, then:

1. $\langle D_{\equiv_{JB}}(\mathbf{AS}(U)), \wedge, \vee, \longrightarrow, \sim, \lrcorner, 0, 1 \rangle$ is a *semi-simple Nelson algebra.*
2. $\langle D_{\equiv_{JB}}(\mathbf{AS}(U)), \wedge, \vee, \sim, \phi_1, \phi_2, 0, 1 \rangle$, is a *three-valued Łukasiewicz algebra.*
3. $\langle D_{\equiv_{JB}}(\mathbf{AS}(U)), \wedge, \vee, \neg, \supset, 0, 1 \rangle$, is a *Heyting algebra.*
4. $\langle D_{\equiv_{JB}}(\mathbf{AS}(U)), \wedge, \vee, \lrcorner, \subset, 0, 1 \rangle$, is a *co-Heyting algebra.*
5. $\langle D_{\equiv_{JU}}(\mathbf{AS}(U)), \wedge, \vee, \sim, 0, 1 \rangle$ is a *Boolean algebra* isomorphic to $\mathbf{AS}(U)$.
6. $\langle D_{\equiv_{J\emptyset}}(\mathbf{AS}(U)), \wedge, \vee, \neg, \supset, D_1, D_2, e_0, e_1, e_2 \rangle$, is a *Post algebra of order three.*

We have enough material for a number of interesting observations.

Observation 2 *Given a topological space on a set U and $X \subseteq U$, the boundary $\mathbb{B}(X)$ of X is given by $\mathbb{C}(X) \cap -\mathbb{I}(X)$. In [10] William Lawvere pointed out that in co-Heyting algebras the topological notion of a boundary is definable as $\partial(x) = x \wedge \lrcorner x$, and observed that for all x, y:*

(1) $\partial(x \wedge y) = (\partial(x) \wedge y) \vee (x \wedge \partial(y))$*;*   *(2)* $\partial(x \wedge y) \vee \partial(x \vee y) = \partial(x) \vee \partial(y)$.

The first equation is essentially the usual Leibniz rule for differentiation of a product. Lawvere emphasizes that though its validity for boundaries of closed sets is supported by our space intuition, nevertheless it is virtually unknown in general topology literature. Moreover, given an element x of a co-Heyting algebra, Lawvere calls $\lrcorner \lrcorner x$ the regular core *of x. In the context of Continuum Physics, he claimed that a part x may be considered a* sub-body *(or shortly a* body*) if and only if $\lrcorner \lrcorner x = x$ and noticed that any element x is the join of its core and its boundary: $x = \lrcorner \lrcorner x \vee \partial(x)$.*

In view of Lawvere's observations and Fact 3.(4), the notion of a co-Heyting boundary was exploited by the author in the context of Rough Set analysis. Given an Approximation Space $\mathbf{AS}(U)$, $X \subseteq U$ and $a = \langle (uE)(X), (lE)(X)\rangle$, $a \wedge \lrcorner a$ (or, equivalently, $a \wedge \sim a$), is $\langle \mathbb{B}(X), \emptyset\rangle$. In order to obtain the rough set of $\mathbb{B}(X)$ it is sufficient to compute $\neg\neg(a \wedge \lrcorner a)$. Moreover, $\lrcorner \lrcorner a = \langle (lE)(X), (lE)(X)\rangle$, which is the rough set of $(lE)(X)$. But $(lE)(X)$ is the *internal* or *necessary* part of X, (in a literal sense when $\mathbf{AS}(U)$ is interpreted as an S5 modal space). This part is stable because (lE) is idempotent. This means that $\delta(\lrcorner \lrcorner a) = 0$; that is, the boundary of $(lE)(X)$ is empty.

In a private communication, Lawvere said that to his knowledge this was the first nontrivial, albeit simple, example of his algebraic characterization of topological boundaries. A new issue arises, thus:

ISSUE B. MORE GENERAL ALGEBRAIC CHARACTERIZATION OF TOPOLOGICAL BOUNDARIES THROUGH GENERALIZED ROUGH SETS: *Exhibit more general examples of rough set systems in which Lawvere's algebraic descriptions of a "body", a "core" and a "boundary" can be expressed.*

Rough set systems induced by pre or partial orders $\mathbf{P} = \langle U, \leq\rangle$ are natural candidates, because the set $F(\mathbf{P})$ of order filters of $\mathbf{P}$ is a Heyting algebra $\mathbf{H}(\mathbf{P})$ (for $X \subseteq U$ the *order filter* $\uparrow X$, or $\uparrow x$ if $X = \{x\}$, is nothing but the $\leq -$neighborhood of X). But, given a Heyting algebra $\mathbf{H}$, and a Boolean congruence $\equiv$ on it, $\mathbf{N}_{\equiv}(\mathbf{H}) = \langle Dj_{\equiv}(\mathbf{H}), \longrightarrow, \sim, \lrcorner, \wedge, \vee, 0, 1\rangle$ is a Nelson algebra, which is a model for *Constructive Logic with Strong Negation*, $\mathcal{CLSN}$ (from this result one obtains Fact 3.(1)). But when is $\mathbf{N}_{\equiv}(\mathbf{H})$ a Heyting algebra? When a bi-Heyting algebra? Is it possible a characterization of these cases and, moreover, a rough set, hence informational, interpretation as it is done in [7] for finite algebras and particular infinite cases (see Facts 6)?

Observation 3 *An operator J on a Heyting algebra* $\mathbf{H}$*, is a* Lawvere-Tierney operator *if it is idempotent, increasing and multiplicative. The operator J^B of definitions*

(4) *and* (5) *is such an operator. A Lawvere-Tierney operator* J *on the dual Heyting algebra* $\mathbf{H}(\mathbf{P})$ *of a preorder* $\mathbf{P} = \langle U, \leq \rangle$ *defines an association between elements* p *of* U *and subfilters of* $\uparrow p$*, in the following manner:*

$$J_{[p]} = \{\uparrow p \cap X : p \in J(X), X \in \mathbf{H}(\mathbf{P})\}. \tag{6}$$

The family $\Gamma = \{J_{[p]} : p \in U\}$ *is called a* Grothendieck topology *and the system* $\langle \mathbf{P}, \Gamma \rangle$ *an* ordered site *(see [16], Chap. 7.3). The logical importance of ordered sites is the following.* $\mathbf{H}(\mathbf{P})$ *is the set of possible evaluations* $[\![A]\!]$ *from intuitionistic formulas* A *to the Kripke model* $\langle \mathbf{P}, \models \rangle$*. Given an element* $x \in U$*, a formula* A *is said to be* locally valid *on* x*,* $x \models \langle l \rangle A$*, if* $[\![A]\!] \in J_{[x]}$ *for some Grothendieck topology* Γ *on* $\mathbf{P}$*.*

This formalizes our intuition that *locally* on B sets are not rough but exact (see [16], Chap. 7).

Issue C: Rough Set Systems and Logic: *Find a faithful logic for rough set systems. The problem is that, for instance, three-valued Łukasiewicz logic, which is often proposed as the logic of rough set systems, is not able to grasp the distinction between the exact behaviour on* B *and the inexact behaviour on* P*. In fact, this logic encompasses the cases in which* $B = U$*,* $B = \emptyset$ *and* $B \neq U$*,* $B \neq \emptyset$*. Maybe, Labeled Deduction Systems could be useful (see* [4] *and subsequent works).*

Observation 4 *From Facts* 2.(8)–(9)*, one derives* $\sim \neg\neg = \lrcorner\lrcorner \sim$ *and* $\sim \lrcorner\lrcorner = \neg\neg \sim$*, which suggest that the double negations* $\neg\neg$ *and* $\lrcorner\lrcorner$ *behave in modal* $(\sim \Box = \Diamond \sim, \sim \Diamond = \Box \sim)$ *and topological* $(-\mathbb{I} = \mathbb{C}-, -\mathbb{C} = \mathbb{I}-)$ *manners.*

The rough set explanation of this fact is given by the following equations:

$$\neg\neg rs(X) = rs((uE)(X)); \ \lrcorner\lrcorner rs(X) = rs((lE)(X)) \tag{7}$$

From Facts 2.(8)-(9), $\neg\neg = \lrcorner\neg$ and $\lrcorner\lrcorner = \neg\lrcorner$. Thus they are particular cases of two more general operators definable in bi-Heyting algebras. Indeed, let us define the following sequences in a $\sigma-$complete bi-Heyting algebra **BH**: (i) $\Box_0 = \Diamond_0 = Id$; (ii) $\Box_{n+1} = \neg\lrcorner\Box_n$, $\Diamond_{n+1} = \lrcorner\neg\Diamond_n$; (iii) $\blacksquare(a) = \bigwedge_{i=1}^{n} \Box_i(a)$; (iv) $\blacklozenge(a) = \bigvee_{i=1}^{n} \Diamond_i(a)$, $\forall a \in \mathbf{BH}$. In [19] it is shown that for any a, $\blacksquare(a)$ is the largest complemented element of **BH** below a, while $\blacklozenge(a)$ is the smallest complemented element above a. From Facts 3.(3)-(4), a rough set system is a bi-Heyting algebra where $\blacksquare = \Box_1$ and $\blacklozenge = \Diamond_1$ (from (7), because (lE) and (uE) are idempotent or directly from idempotency of $\neg\neg$ and $\lrcorner\lrcorner$).

This property is related to the following laws: (DM1) Let **H** be a Heyting algebra. **H** satisfies the De Morgan law for $\neg$, if $\forall x, y$, $\neg(x \wedge y) = \neg x \vee \neg y$. (DM2) Let **CH** be a co-Heyting algebra. **CH** satisfies the De Morgan law for $\lrcorner$, if $\forall x, y$, $\lrcorner(x \vee y) = \lrcorner x \wedge \lrcorner y$. It can be shown that in bi-Heyting algebras the law for $\neg$ implies $\blacksquare(a) = \neg\lrcorner a$ and that the law for $\lrcorner$ implies $\blacklozenge(a) = \lrcorner\neg a$ (the reverse of the implications does not hold). Actually, both laws hold in rough set systems. The proof is in the following list, as well as some consequences:

Facts 4 1. (DM1) is equivalent to the fact that $Reg(\mathbf{H}) = \{x \in \mathbf{H} : x = \neg\neg x\}$ is a sublattice of $\mathbf{H}$ (see [8]). Dually for (DM2) and $coReg(\mathbf{CH}) = \{x \in \mathbf{CH} : x = \lrcorner\lrcorner x\}$.

2. Since $\neg\neg$ is a Lawvere-Tierney operator, it is multiplicative but, in general, not additive. This means that generally $Reg(\mathbf{H})$ is not a sublattice of $\mathbf{H}$. Dually, $coReg(\mathbf{CH})$ is not a sublattice of a generic co-Heyting algebra $\mathbf{CH}$.
3. But in rough set systems, from (7) and Facts 1.(1), $\neg\neg$ is a topological closure, hence additive, while $\lrcorner\lrcorner$ is a topological interior, hence multiplicative. It follows that both $Reg(D_{\equiv_{JB}}(\mathbf{AS}(U)))$ and $coReg(D_{\equiv_{JB}}(\mathbf{AS}(U)))$ are sublattices of $D_{\equiv_{JB}}(\mathbf{AS}(U))$.
4. The set of the complemented elements of a lattice $\mathbf{L}$ is called the *center of* $\mathbf{L}$, $Ctr(\mathbf{L})$. One can prove that $Reg(D_{\equiv_{JB}}(\mathbf{AS}(U))) = coReg(D_{\equiv_{JB}}(\mathbf{AS}(U))) = Ctr(D_{\equiv_{JB}}(\mathbf{AS}(U)))$. Of course, if $a \in Ctr(D_{\equiv_{JB}}(\mathbf{AS}(U)))$, then $\delta(a) = 0$.

ISSUE D: INFORMATIONAL INTERPRETATION OF THE SITUATION WHERE THE SEQUENCES $\Box_n$ AND $\Diamond_n$ DO NOT STABILIZE AT STEP 2: *Longer steps for stabilization reflect the fact that new boundaries must be included after each application of closure and smaller internal parts must be grasped after any interior application. Informational interpretations of this situation should be provided. In Sect.* 4 *a first answer is suggested.*

Facts 5 1. In a Heyting algebra $\mathbf{H}$, an element x is called *dense* if $\neg\neg x = 1$. If $\mathbf{H}$ has a least dense element d, then $Reg(\mathbf{H})$ is isomorphic to $\mathbf{H}/_{\equiv_{Jd}}$. It can be proved ([16] Chap. 7) that in $Dj_{\equiv_{JB}}(\mathbf{AS}(U))$ the least dense element is $\langle B, \emptyset\rangle$, while in $Dc_{\equiv_{JB}}(\mathbf{AS}(U))$ is $\langle U, B\rangle$.

2. Thus, one can prove that $\langle D_{\equiv_{JB}}(\mathbf{AS}(U)), \wedge, \vee, !, e_0, e_1, e_2\rangle$, is a *$P_2$-lattice of order three,* if $B \neq U$. Here, $e_0 \le e_1 \le e_2$ is the chain of values. Moreover, if A is a classical tautology (with $\sim$, $\neg$ or $\lrcorner$ as negation), then $e_1 \le [\![A]\!] \le e_2$, while if A is a classical contradiction, then $e_0 \le [\![A]\!] \le {\sim} e_1$ (see [16] Chap. 9.6). Thus, e_1 is a local classical top, and $\sim e_1$ is a local classical bottom.
3. Eventually, $\langle D_{\equiv_{JB}}(\mathbf{AS}(U)), \wedge, \vee, \overset{c}{\Longrightarrow}, \overset{c}{\Longleftarrow}, \supset, \subset, !, ¡, 0, 1\rangle$, is a *P-algebra.* In this case $a \overset{c}{\Longrightarrow} b$ is the largest element e of the center such that $e \wedge a \le b$, while $a \overset{c}{\Longleftarrow} b$ is the least element e of the center such that $e \vee a \ge b$.

The last result leads us to a new issue:

ISSUE E: ROUGH SET SYSTEMS AND TOPOS THEORY: *In* [3] *it is shown that a lattice* $\mathbf{L}$ *has a stalk (etalé) space representation when for each* $s \in \mathbf{L}$, *the mapping* $\varphi_s : Ctr(\mathbf{L}) \longrightarrow \mathbf{L}$; $\varphi_s(e) = e \wedge s$ *is residuated, or, equivalently, if* $\eta_s : Ctr(\mathbf{L}) \longrightarrow \mathbf{L}$; $\eta_s(e) = e \vee s$ *is residuated. But from Fact 5.(3)* $\overset{c}{\Longrightarrow}$ *and* $\overset{c}{\Longleftarrow}$ *are the required residuations, respectively. Moreover, the definition of a bi-Heyting algebra in* [19] *is given in terms of a topos* $\mathcal{E}$, *a Boolean topos* $\mathcal{B}$, *and a surjective geometric morphism* $\Gamma : \mathcal{E} \longrightarrow \mathcal{B}$, *such that the canonical map* $\delta : \Omega_{\mathcal{E}} \longrightarrow \Omega_{\mathcal{B}}$ *has a lax adjoint. Finally, also the construction of rough set systems through Grothendieck (Lawvere-Tierney) topologies is a signal that these systems are some sort of topos. Therefore,* *a unitary and coherent description of (generalized) rough sets from the point of view of etalé spaces and topos theory would be welcome (some hints: the representation of a rough set system in terms of the dual of a set of disjoint chains of max length 2, or as a product of a Boolean algebra and a Post algebra of order three (see* [16] *Chaps. 10.4 and 8.3); the fact that the prime ideals of a P−algebra lie in disjoint maximal chains (*[3]*)).*

We have seen that rough set systems induced by pre or partial orders $\mathbf{P}$ are well-behaving because the dual $\mathbf{H}(\mathbf{P})$ of a preorder $\mathbf{P}$ is a Heyting algebra. Not only they are useful in data-mining (cf. [5]), but in this case the construction of rough set systems assumes an unexpected amazing meaning (see [7]) (on the topic, see also [14]). In fact, if $\mathbf{P}$ is a *partial order* upper bounded by a set M of maximal elements (always if it is finite), then for all $m \in M$, $\uparrow m$ is a singleton definable set. In view of the previous discussion, the corresponding rough set system is given by $Dj_{\equiv_{JM}}(\mathbf{H}(\mathbf{P}))$. One has, thus:

Facts 6
1. M is the least dense element of $\mathbf{H}(\mathbf{P})$ (i.e. $\neg\neg M = U$).
2. $\equiv_{JM}$ is the Glivenko congruence: i.e. $X \equiv_{JM} Y$ iff $\neg X = \neg Y$.
3. $Dj_{\equiv_{JM}}(\mathbf{H}(\mathbf{P}))$ belongs to the subvariety of Nelson algebras of pairs $\langle a, b\rangle$ determined by the equation $\neg a = \neg\neg b$.
4. $Dj_{\equiv_{JM}}(\mathbf{H}(\mathbf{P}))$ is a model for the logic E_0, which is $\mathcal{CLSN}$ plus the following modal axioms: $(\sim A \longrightarrow \perp) \longrightarrow \mathbf{T}(A)$ and $(A \longrightarrow \perp) \longrightarrow \sim\mathbf{T}(A)$. Logic E_0 was introduced in [11] where it is proved $\vdash_{\mathcal{CL}} A$ if and only if $\vdash_{\mathcal{E}_0} \mathbf{T}(A)$, also in the predicative case, thus extending the well-known Gödel-Glivenko theorem stating $\vdash_{\mathcal{CL}} A$ if and only if $\vdash_{\mathcal{INT}} \neg\neg A$, for A any classical *propositional* formula, $\mathcal{CL}$ a classical logic system and $\mathcal{INT}$ an intuitionistic system.

We arrive at a new issue:
ISSUE F: ROUGH SET SYSTEMS AND SUBSTRUCTURAL LOGICS: *$\mathcal{CLSN}$ is a substructural logic* ([24], *but see also* [23]). *Is there any rough set-based informational interpretation of this fact? (for some suggestions see* [12, 13], *and* [25]*).*

3 Purely Relational Approximation Operators

Consider a structure $\mathbf{P} = \langle U, M, R\rangle$, with U, M sets and $R \subseteq U \times M$. We interpret U and M as sets of objects and, respectively, properties, so that $\langle g, m\rangle \in R$ means that object g fulfills property m. $\mathbf{P}$ will be called a *property system*. Let us define the following functions, where $R^{\smile}$ is the reverse of R (see [22], cf. [16], Chap. 2.):

- $\langle e\rangle : \wp(M) \longmapsto \wp(U)$; $\langle e\rangle(Y) = \{a \in U : \exists b(b \in Y \wedge b \in R(a))\}$;
- $[e] : \wp(M) \longmapsto \wp(U)$; $[e](Y) = \{a \in U : \forall b(b \in R(a) \Rightarrow b \in Y)\}$;
- $\langle i\rangle : \wp(U) \longmapsto \wp(M)$; $\langle i\rangle(X) = \{b \in M : \exists a(a \in X \wedge a \in R^{\smile}(b))\}$;
- $[i] : \wp(U) \longmapsto \wp(M)$; $[i](X) = \{b \in M : \forall a(a \in R^{\smile}(b) \Rightarrow a \in X)\}$.

A function is decorated by 'e' when its application gives an *extension*, i.e., a set of objects, and it is decorated by 'i' when it outputs an *intension*. From Observation 1, it is clear why two of them are $\Diamond$-shaped (*possibility*), and two are $\Box$-shaped (*necessity*). These functions fulfill a strategic property: $\langle\langle i\rangle, [e]\rangle$ and $\langle\langle e\rangle, [i]\rangle$ are *Galois adjunctions*: $\langle i\rangle(X) \subseteq Y$ iff $X \subseteq [e](Y)$, $\langle e\rangle(Y) \subseteq X$ iff $Y \subseteq [i](X)$, for all $X \subseteq G, Y \subseteq M$. Exploiting this fact one immediately obtains that $\langle i\rangle[e]$ and $\langle e\rangle[i]$ are pre-topological interior operators, while $[i]\langle e\rangle$ and $[e]\langle i\rangle$ are pre-topological closure operators, on M and U, respectively. For this reason we set, for all $X \subseteq U, Y \subseteq M$:

$$\text{(a) } int(X) = \langle e\rangle([i](X)); \text{ (b) } cl(X) = [e](\langle i\rangle(X)).$$
$$\text{(c) } \mathcal{A}(Y) = [i](\langle e\rangle(Y)); \text{ (d) } \mathcal{C}(Y) = \langle i\rangle([e](Y)).$$

$\mathcal{A}$ and $\mathcal{C}$ are the "formal" counterparts of cl and, respectively, int. One has:

$$int(X) \subseteq X \subseteq cl(X), \quad any\ X \subseteq U. \tag{8}$$

If $R(U) = M$ and $R^{\smile}(M) = U$, then int is co-normal and cl is normal (in this case we shall say that the property system is *normal*). It can be proved that (lR) and (uE) are special cases of int, respectively, cl.

Issue G. Rough Set Systems from adjoint operators: *What are the logico-algebraic properties of the set of ordered pairs of the form $\langle cl(X), int(X)\rangle$ or $\langle int(X), -cl(X)\rangle$? (Some hints from* [1] *or* [2]).

Observation 5 *It is worth noticing that the above machinery can be rephrased in the framework of* Chu spaces. *Since they provide models for Linear Logic (see [17, 18]), one could add this ingredient to Issue F for a more comprehensive description of the "substructural picture".*

4 Approximation by Means of Neighborhoods

Consider a structure $\mathbf{N} = \langle U, \wp(U), R\rangle$, with $R \subseteq U \times \wp(U)$. It can be considered a concrete instance of a neighborhood system. If $u' \in N \in R(u)$, we say that u' is a *neighbor* and N a *neighborhood* of u. We call $\mathcal{N}(U) = \{R(u) : u \in U\}$ a *neighborhood system*. Let us define the following operators on $\wp(U)$:

$$\text{(a) } G(X) = \{u : X \in R(u)\}; \text{ (b) } (X) = -G(-X) = \{u : -X \notin R(u)\}.$$

Consider the following conditions on $\mathcal{N}(U)$, for any $x \in U$, $A, N, N' \subseteq U$:
1. $U \in R(x)$; **0**. $\emptyset \notin R(x)$; **Id**. if $x \in G(A)$ then $G(A) \in R(x)$;
N1. $x \in N$, for all $N \in R(x)$; **N2**. if $N \in R(x)$ and $N \subseteq N'$, then $N' \in R(x)$;
N3. if $N, N' \in R(x)$, then $N \cap N' \in R(x)$.

They induce the following properties of the operators G and F:

Condition	Equivalent properties of G	Equivalent properties of F
1	$G(U) = U$	$F(\emptyset) = \emptyset$
0	$G(\emptyset) = \emptyset$	$F(U) = U$
Id	$G(X) \subseteq G(G(X))$	$F(F(X)) \subseteq F(X)$
N1	$G(X) \subseteq X$	$X \subseteq F(X)$
	$X \subseteq Y \Rightarrow G(X) \subseteq G(Y)$	$X \subseteq Y \Rightarrow F(X) \subseteq F(Y)$
N2	$G(X \cap Y) \subseteq G(X) \cap G(Y)$	$F(X \cup Y) \supseteq F(X) \cup F(Y)$
N3	$G(X \cap Y) \supseteq G(X) \cap G(Y)$	$F(X \cup Y) \subseteq F(X) \cup F(Y)$

But **N** is a property system, too. So it is possible to define int and cl. One can prove that $int = G$ and $cl = F$ if conditions **Id**, **N1** and **N2** are satisfied. Moreover, if **N** is normal, then **1** and **0** are satisfied, too. Neighborhood systems satisfying these conditions will be classified as $\mathcal{N}_{2Id}$. A topology is a $\mathcal{N}_{2Id}$ neighborhood system which fulfills **N3** in addition.

Now, we compare formal and concrete pre-topological spaces by exploiting the *formal semi-cover* relation $\blacktriangleleft$ introduced in [20]. Let $b \in M$ and $Y, Y' \subseteq M$:

$$(basis)\ b \blacktriangleleft Y \text{ iff } b \in \mathcal{A}(Y), \qquad (step)\ Y \blacktriangleleft Y' \text{ iff } \forall y \in Y, y \blacktriangleleft Y'.$$

Moreover, we assume M to be a monoid with a binary operation "$\cdot$" and unity 1. The operation "$\cdot$" is a formal counterpart of intersection. Then we put for $X, Y \subseteq M$:

$$\text{(a) } X \cdot Y = \{x \cdot y : x \in X \ \&\ y \in Y\}; \quad \text{(b) } X \bullet Y = \mathcal{A}(X \cdot Y). \tag{9}$$

Let us put $Sat_{\mathcal{A}}(M) = \{X \subseteq M : \mathcal{A}(X) = X\}$ and let $\perp$ be any subset of M. Then we say that a pre-topological formal system $\langle M, \cdot, 1, \perp, \blacktriangleleft\rangle$ is topological if $\langle Sat_{\mathcal{A}}(M), \bullet, \vee, M, \mathcal{A}(\perp)\rangle$ is a complete lattice with complete distributivity and ordering $\subseteq$. It can be proved that a pre-topological formal system is topological if the following (*left*) and (*right*) properties hold:

$$(left)\ \ \frac{b \blacktriangleleft Y}{b \cdot b' \blacktriangleleft Y}; \quad (right)\ \ \frac{b \blacktriangleleft Y \qquad b \blacktriangleleft Y'}{b \blacktriangleleft Y \cdot Y'}.$$

Since **N** is a property system, we can define the operation $\mathcal{A}$ on $\wp(U)$ obtaining the pre-topological system, $\tau = \langle \wp(U), \cap, U, \emptyset, \blacktriangleleft\rangle$ which is in between a formal and a concrete system. Thus, the question is (see [16], Chap. 14.2): given a relational structure $\langle U, \wp(U), R\rangle$, is there any connection between the properties **1**, **0**, **Id**, **N1**, **N2**, **N3** and **N4**, of $\mathcal{N}(U)$ and the properties (*left*) and (*right*) of τ? There are two answers, for the present: (A): If $\mathcal{N}(U)$ fulfills **N3**, then (*right*) holds in τ (the converse does not hold). (B) $\mathcal{N}(U)$ fulfills **N2** if and only if (*left*) holds.

At this point a couple of issues arises:

ISSUE H. G, F, $\blacktriangleleft$, int AND cl IN PARTNERSHIP: *Since* $G = int$ *and* $F = cl$ when conditions **Id**, **N1** and **N2** hold in $\mathcal{N}(U)$, the above results provide us with a glimpse of the relationships between those operators and the properties of $\blacktriangleleft$. However, it is worthwhile a thorough investigation of the connections which link the couples of "concrete" operators (int, cl) and (G, F) and the abstract relation $\blacktriangleleft$, along with their informational interpretations.

ISSUE I. *$Sat_{\mathcal{A}}(M)$ can be made into a logical model in which $\blacktriangleleft$ plays the role of the sequent relation $\vdash$. It happens that in such a system, $(right)$ corresponds to the* contraction rule *and* (*left*) to the *weakening rule* (cf. [20, 21]). *Thus an amazing task would be to put this fact and those of Issue F and Observation 5 into a sound comprehensive picture.*

If one collects different observations about the same sets of objects and properties, a composite system $\langle U, M, \{R_i\}_{1 \le i \le n}\rangle$ is obtained. If $M = U$, we call it a *Dynamic System* (see [16], Chap. 12). In this case, one can ask what are the lower and the upper approximations of a subset of U according to a certain number of observations. Let us then set the following operators $\varkappa^m$ and ε^m, for $1 \le m \le n$:

1. (*Contraction*): We say that $x \in \varkappa^m(A)$, if $R_i(x) \subseteq A$ for at least m indices.
2. (*Expansion*): We say that $x \in \varepsilon^m(A)$, if $R_i(x) \cap A \neq \emptyset$ for at least $n + 1 - m$ indices.

In [16], Chap. 12.6, it is explained how to compute these operators. However an issue arises about them:

ISSUE J: ALGEBRAIC AND TOPOLOGICAL PROPERTIES OF DYNAMIC OPERATORS: *What are the topological and algebraic properties of $\varkappa^m$ and ε^m? Do graded operators int^m, cl^m, G^m and F^m make any sense?*

A few results are available, and just for simple cases (namely if all $R \in \{R_i\}_{1 \le i \le n}$ are preorders, then $\varkappa^1$ is a pretopological interior operator and ε^n is a pretopological closure operator—see [16], Chap. 12). This topic is connected with multiple-source approximation spaces (see [9]).

5 Conclusions

The above connections are not exhaustive. Rough Set Theory is productive of new unexpected intersections and partnerships with surprising fields. We just mention that it suggested a semantic interpretation of the *Logic of conjectures and assertions* (see G. Bellin's page *profs.sci.univr.it/∼bellin/papers.html*) and a tool for *spatial reasoning* (see the works by T. Bittner and J. S. Stell at *www.comp.leeds.ac.uk/jsg* and the works on spatial reasoning by I. Düntsch and E. Orłowska). Eventually, a lot of work is still required to understand the logico-algebraic properties of approximations of relations (for some results in simple cases, see [16], Chap. 15.18).

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Dealing with Uncertainty: From Rough Sets to Interactive Rough-Granular Computing

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*If you thought that science was certain-
well, that is just an error on your part.*

—Richard P. Feynman,
The Nobel Prize in Physics (1965)

Abstract We discuss an approach for dealing with uncertainty in complex intelligent systems. The approach is based on interactive computations over complex objects called here complex granules (c-granules, for short). C-granules are defined relative to a given agent. Any c-granule of a given agent specifies a perceived structure of local environment of physical objects, called hunks. There are three kinds of such hunks: (i) hunks in the agent external environment creating the hard_suit of c-granule, (ii) internal hunks of agent, creating the soft_suit of c-granule, some of which can be represented by agent as infogranules, and (iii) hunks creating the link_suit of c-granule and playing the role of links between hunks from the hard_suit and soft_suit. This structure is used for recording by means of infogranules the results of interactions of hunks from the local environment. We begin from the discussion on dealing with uncertainty in the rough set approach and next we move toward interactive computations on c-granules. In particular, from our considerations

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it follows that the fundamental issues of intelligent systems based on interactive computations concern the efficiency management in controlling of computations performed by such systems. Our approach is a step toward realization of the Wisdom Technology (WisTech) program. The approach was developed over years of work, based on the work on different real-life projects.

Keywords Information granule · Physical object · Interaction · Complex granule · Granular computing · Rough set · Complex vague concept approximation · Adaptive judgment · Efficiency management

1 Introduction

There are quite many well-known different approaches for dealing with uncertainty (e.g., [13, 16, 17, 23, 24, 43, 44]). We emphasize some basic issues related to uncertainty in: (i) object perception, (ii) concept perception as well as (iii) reasoning about concepts. In real-life applications, the objects and concepts we are dealing with are complex. Moreover, they are often vague what causes additional problems in coping with them.

We start from the rough set approach proposed by Professor Pawlak [23, 24, 27] as a tool for dealing with imperfect knowledge, in particular with vague concepts. Rough set theory has attracted the attention of many researchers and practitioners all over the world. We discuss uncertainty issues in object and concept perception in the rough set framework.

Granular Computing (GC) is now an active area of research [29]. Objects we are dealing with in GC are *information granules* (or *infogranules*, for short). Such granules are obtained as the result of information granulation [47]:

> Information granulation can be viewed as a human way of achieving data compression and it plays a key role in implementation of the strategy of divide-and-conquer in human problem-solving.

The concept of granulation is rooted in the concept of a linguistic variable introduced by Professor Lotfi Zadeh in 1973. Information granules are constructed starting from some elementary ones. More compound granules are composed of finer granules that are drawn together by distinguishability, similarity, and functionality [45].

Understanding of interactions of objects on which are performed computations is fundamental for modeling of complex systems [3]. For example, in [21] this is expressed in the following way:

> [...] interaction is a critical issue in the understanding of complex systems of any sorts: as such, it has emerged in several well-established scientific areas other than computer science, like biology, physics, social and organizational sciences.

When we move to dealing with perception of interacting complex objects in observed situations one should consider that due to resource bounds only some parts of complex objects may be perceived at a given moment of time. These parts are perceived as values of compound attributes computed on the basis of the delivered (e.g., by control of the agent) parameters of sensors and recorded in relevant information (decision) systems as the results of sensory measurements. Hence, uncertainty in identification of the environment state often causes that results of interactions with and within the environment cannot be predicted with certainty. As a consequence, e.g., results of performed actions may be different than the predicted ones.

In this paper, we outline an extension of Interactive Rough-Granular Computing (IRGC) approach (see, e.g., [29, 38, 41, 42]) by introducing *complex granules* (*c-granules*, for short) making it possible to model interactive computations performed by an agent. In such computations, interactions among physical objects and interactions of these physical objects with information granules possessed by the agent are represented.

In IRGC, the rough set approach in combination with other soft computing approaches are used for inducing approximations of complex vague concepts.

Different problems related to dealing with uncertainty in IRGC are outlined in the paper.

Let us mention here that our discussion on IRGC based on c-granules is strongly related to the following sentences:

> As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality. (Albert Einstein, [2])

> Constructing the physical part of the theory and unifying it with the mathematical part should be considered as one of the main goals of statistical learning theory. (Vladimir Vapnik, [43] p. 721)

This paper covers some issues presented in the invited talk at ICFUA 2013.

In Sect. 2, we discuss some basic problems related to dealing with uncertainty in the rough set approach. Section 3 outlines the approach to IRGC based on c-granules and reports some issues concerning uncertainty in IRGC. In particular, due to uncertainty, e.g., in identification of the global environment state, development of the efficiency management techniques for controlling by agent computations performed over c-granules for achieving goals is crucial for intelligent systems based on IRGC.

2 Rough Sets and Uncertainty

2.1 Uncertainty in Object Perception

The rough set philosophy [23, 24, 27] is founded on the assumption that with every object of the universe of discourse, we associate some information (data, knowledge) called the object signature. Objects characterized by the same information are indiscernible (similar) in view of the available information about them. The *indiscernibility relation* generated in this way is the mathematical basis of rough set theory. This understanding of indiscernibility is related to the idea of Gottfried Wilhelm Leibniz that objects are indiscernible if and only if all available functionals take