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Filippo Bracci

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Complex Analysis and Geometry

KSCV10, Gyeongju, Korea, August 2014

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Hervé Gaussier · Kengo Hirachi
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Editors

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Preface

The KSCV Symposium, the Korean conference on Several Complex Variables, started in 1997 at Pohang University of Science and Technology (POSTECH) in an effort to promote the study of complex analysis and geometry in all dimensions. Since then, the conference met semi-regularly for about 10 years and then settled as a biannual conference. The sixth conference was held in 2002 as a satellite conference to the Beijing ICM. The symposia have been successful in the sense that many leading scholars in the field have participated from all over the world, and more importantly, many new researchers in this field, especially from Korea, have been brought up along with this effort.

The KSCV10 (the 10th) Symposium was held during 7–11 July 2014, as a satellite conference to the ICM again; this time the ICM was held in Seoul, Korea. It was clearly noticed by many that not only has the research level of the Korean SCV community but also that of the conference improved so much that the contents of the lectures will be useful to mathematicians in the field of complex analysis and geometry of the world. Therefore, the organizers as well as the participants of the conference agreed to have them organized into a book form; therefore, this proceedings volume was composed.

I would like to express deep thanks to all those who contributed their articles to this volume. We, the committee, of course, wish their study to flourish greatly.

April 2015

Filippo Bracci
Jisoo Byun
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Kengo Hirachi
Kang-Tae Kim
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Fatou Flowers and Parabolic Curves

Marco Abate

Abstract In this survey we collect the main results known up to now (July 2015) regarding possible generalizations to several complex variables of the classical Leau-Fatou flower theorem about holomorphic parabolic dynamics.

Keywords Local holomorphic dynamical systems · Parabolic points · Fatou flowers · Parabolic curves · Germs tangent to the identity

1 The Original Leau-Fatou Flower Theorem

In this survey we shall present the known generalizations of the classical Leau-Fatou theorem describing the local holomorphic dynamics about a parabolic point. But let us start with a number of standard definitions.

Definition 1.1 A local n -dimensional discrete holomorphic dynamical system (in short, a local dynamical system) is a holomorphic germ f of self-map of a complex n -dimensional manifold M at a point $p \in M$ such that $f(p) = p$; we shall denote by $\text{End}(M, p)$ the set of such germs.

If f, g belongs to $\text{End}(M, p)$ their composition $g \circ f$ is defined as germ in $\text{End}(M, p)$; in particular, we can consider the sequence $\{f^k\} \subset \text{End}(M, p)$ of iterates of $f \in \text{End}(M, p)$, inductively defined by $f^0 = \text{id}_M$ and $f^k = f \circ f^{k-1}$ for $k \geq 1$. The aim of local discrete dynamics is exactly the study of the behavior of the sequence of iterates.

Remark 1.1 In practice, we shall work with representatives, that is with holomorphic maps $f: U \rightarrow M$, where $U \subseteq M$ is an open neighborhood of $p \in U$, such that $f(p) = p$. The fact we are working with germs will be reflected in the freedom we have in taking U as small as needed. We shall also mostly (but not always) take $M = \mathbb{C}^n$ and $p = O$; indeed a choice of local coordinates φ for M centered at p

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yields an isomorphism $\varphi_*: \text{End}(M, p) \rightarrow \text{End}(\mathbb{C}^n, O)$ preserving the composition by setting $\varphi_*(f) = \varphi \circ f \circ \varphi^{-1}$.

Definition 1.2 Let $f: U \rightarrow M$ be a representative of a germ in $\text{End}(M, p)$. The *stable set* $K_f \subseteq U$ of f is the set of points $z \in U$ such that $f^k(z)$ is defined for all $k \in \mathbb{N}$; clearly, $p \in K_f$. If $z \in K_f$, the set $\{f^k(z)\}$ is the *orbit* of z ; if $z \in U \setminus K_f$ we shall say that z *escapes*. The stable set depends on the chosen representative, but its germ at p does not; so we shall freely talk about the stable set of an element of $\text{End}(M, p)$. An *f*-invariant set is a subset $P \subseteq U$ such that $f(P) \subseteq P$; clearly, the stable set is *f*-invariant.

Definition 1.3 A local dynamical system $f \in \text{End}(M, p)$ is *parabolic* (and sometimes we shall say that p is a *parabolic fixed point* of f) if df_p is diagonalizable and all its eigenvalues are roots of unity; is *tangent to the identity* if $df_p = \text{id}$. We shall denote by $\text{End}_1(M, p)$ the set of local dynamical systems tangent to the identity in p .

Remark 1.2 If $f \in \text{End}(M, p)$ is parabolic then a suitable iterate f^q is tangent to the identity; for this reason we shall mostly concentrate on germs tangent to the identity. Furthermore, if $f \in \text{End}(M, p)$ is tangent to the identity then f^{-1} is a well-defined germ in $\text{End}(M, p)$ still tangent to the identity.

Definition 1.4 The *order* $\text{ord}_p(f)$ of a holomorphic function $f: M \rightarrow \mathbb{C}$ at $p \in M$ is the order of vanishing at p , that is the degree of the first non-vanishing term in the Taylor expansion of f at p (computed in any set of local coordinates centered at p). The *order* $\text{ord}_p(F)$ of a holomorphic map $F: M \rightarrow \mathbb{C}^n$ at $p \in M$ is the minimum order of its components.

A germ $f \in \text{End}(\mathbb{C}^n, O)$ can be represented by a n -tuple of convergent power series in n variables; collecting terms of the same degree we obtain the homogeneous expansion.

Definition 1.5 A *homogeneous map* of degree $d \geq 1$ is a map $P: \mathbb{C}^n \rightarrow \mathbb{C}^n$ where P is a n -tuple of homogeneous polynomials of degree d in n variables. The *homogeneous expansion* of a germ tangent to the identity $f \in \text{End}_1(\mathbb{C}^n, O)$, $f \neq \text{id}_{\mathbb{C}^n}$, is the (unique) series expansion

$$f(z) = z + P_{\nu+1}(z) + P_{\nu+2}(z) + \cdots \quad (1.1)$$

where P_k is a homogeneous map of degree k , and $P_{\nu+1} \neq O$. The number $\nu \geq 1$ is the *order* (or, sometimes, *multiplicity*) $\nu(f)$ of f at O , and $P_{\nu+1}$ is the *leading term* of f . It is easy to check that the order is invariant under change of coordinates, and thus it can be defined for any germ tangent to the identity $f \in \text{End}_1(M, p)$; we shall denote by $\text{End}_\nu(M, p)$ the set of germs tangent to the identity with order at least ν .

In the rest of this section we shall discuss the 1-dimensional case, where the homogeneous expansion reduces to the usual Taylor expansion

$$f(z) = z + a_{\nu+1}z^{\nu+1} + O(z^{\nu+2}) \tag{1.2}$$

with $a_{\nu+1} \neq 0$.

Definition 1.6 Let $f \in \text{End}_1(\mathbb{C}, 0)$ be tangent to the identity given by (1.2). A unit vector $\nu \in S^1$ is an *attracting* (respectively, *repelling*) *direction* for f at 0 if $a_{\nu+1}\nu^\nu$ is real and negative (respectively, positive). Clearly, there are ν equally spaced attracting directions, separated by ν equally spaced repelling directions.

Example 1.1 To understand this definition, let us consider the particular case $f(z) = z + az^{\nu+1}$. If $\nu \in S^1$ is such that $a\nu^\nu > 0$ then for every $z \in \mathbb{R}^+\nu$ we have $f(z) \in \mathbb{R}^+\nu$ and $|f(z)| > |z|$; in other words, the half-line $\mathbb{R}^+\nu$ is f -invariant and repelled from the origin. Conversely, if $\nu \in S^1$ is such that $a\nu^\nu < 0$ then $\mathbb{R}^+\nu$ is again f -invariant but now $|f(z)| < |z|$ if $z \in \mathbb{R}^+\nu$ is small enough; so there is a segment of $\mathbb{R}^+\nu$ attracted by the origin.

Remark 1.3 If $f \in \text{End}_1(\mathbb{C}, 0)$ is given by (1.2) then

$$f^{-1}(z) = z - a_{\nu+1}z^{\nu+1} + O(z^{\nu+2}).$$

In particular, if $\nu \in S^1$ is attracting (respectively, repelling) for f then it is repelling (respectively, attracting) for f^{-1} , and conversely.

To describe the dynamics of a tangent to the identity germ two more definitions are needed.

Definition 1.7 Let $\nu \in S^1$ be an attracting direction for a $f \in \text{End}_1(\mathbb{C}, 0)$ tangent to the identity. The *basin* centered at ν is the set of points $z \in K_f \setminus \{0\}$ such that $f^k(z) \rightarrow 0$ and $f^k(z)/|f^k(z)| \rightarrow \nu$ (notice that, up to shrinking the domain of f , we can assume that $f(z) \neq 0$ for all $z \in K_f \setminus \{0\}$). If z belongs to the basin centered at ν , we shall say that the orbit of z *tends to 0 tangent to ν* .

A slightly more specialized (but more useful) object is the following:

Definition 1.8 Let $f \in \text{End}_1(\mathbb{C}, 0)$ be tangent to the identity. An *attracting petal* with attracting *central direction* $\nu \in S^1$ for f is an open simply connected f -invariant set $P \subseteq K_f \setminus \{0\}$ with $0 \in \partial P$ such that a point $z \in K_f \setminus \{0\}$ belongs to the basin centered at ν if and only if its orbit intersects P . In other words, the orbit of a point tends to 0 tangent to ν if and only if it is eventually contained in P . A *repelling petal* (with repelling central direction) is an attracting petal for the inverse of f .

We can now state the original *Leau-Fatou flower theorem*, describing the dynamics of a one-dimensional tangent to the identity germ in a full neighborhood of the origin (see, e.g., [M] for a modern proof):

Theorem 1.1 ([Le, F1, F2, F3]) *Let $f \in \text{End}_1(\mathbb{C}, 0)$ be tangent to the identity of order $\nu \geq 1$. Let $\nu_1^+, \dots, \nu_\nu^+ \in S^1$ be the ν attracting directions of f at the origin, and $\nu_1^-, \dots, \nu_\nu^- \in S^1$ the ν repelling directions. Then:*

- (i) for each attracting (repelling) direction v_j^+ (v_j^-) we can find an attracting (repelling) petal P_j^+ (P_j^-) such that the union of these 2ν petals together with the origin forms a neighborhood of the origin. Furthermore, the 2ν petals are arranged cyclically so that two petals intersect if and only if the angle between their central directions is π/ν .
- (ii) $K_f \setminus \{0\}$ is the (disjoint) union of the basins centered at the ν attracting directions.
- (iii) If B is a basin centered at one of the attracting directions, then there is a function $\chi : B \rightarrow \mathbb{C}$ such that $\chi \circ f(z) = \chi(z) + 1$ for all $z \in B$. Furthermore, if P is the corresponding petal constructed in part (i), then $\chi|_P$ is a biholomorphism with an open subset of the complex plane containing a right half-plane — and so $f|_P$ is holomorphically conjugated to the translation $z \mapsto z + 1$.

Definition 1.9 The function $\chi : B \rightarrow \mathbb{C}$ constructed in Theorem 1.1. (iii) is a Fatou coordinate on the basin B .

Remark 1.4 Up to a linear change of variable, we can assume that $a_{\nu+1} = -1$ in (1.2), so that the attracting directions are the ν -th roots of unity. Given $\delta > 0$, the set

$$D_{\nu,\delta} = \{z \in \mathbb{C} \mid |z^\nu - \delta| < \delta\} \quad (1.3)$$

has exactly ν connected components (each one symmetric with respect to a different ν -th root of unity), and it turns out that when $\delta > 0$ is small enough these components can be taken as attracting petals for f —even though to cover a neighborhood of the origin one needs slightly larger petals. The components of $D_{\nu,\delta}$ are distributed as petals in a flower; this is the reason why Theorem 1.1 is called “flower theorem”.

So the union of attracting and repelling petals gives a pointed neighborhood of the origin, and the dynamics of f on each petal is conjugated to a translation via a Fatou coordinate. The relationships between different Fatou coordinates is the key to Écalle-Voronin holomorphic classification of parabolic germs (see, e.g., [A4] and references therein for a concise introduction to Écalle-Voronin invariants), which is however outside of the scope of this survey. We end this section with the statement of the Leau-Fatou flower theorem for general parabolic germs:

Theorem 1.2 ([Le, F1, F2, F3]) *Let $f \in \text{End}_1(\mathbb{C}, 0)$ be of the form $f(z) = \lambda z + O(z^2)$, where $\lambda \in S^1$ is a primitive root of the unity of order q . Assume that $f^q \neq \text{id}$. Then there exists $\mu \geq 1$ such that f^q has order $q\mu$, and f acts on the attracting (respectively, repelling) petals of f^q as a permutation composed by μ disjoint cycles. Finally, $K_f = K_{f^q}$.*

In the subsequent sections we shall discuss known generalizations of Theorem 1.1 to several variables.

2 Écalle-Hakim Theory

From now on we shall work in dimension $n \geq 2$. So let $f \in \text{End}_1(\mathbb{C}^n, O)$ be tangent to the identity; we would like to find a multidimensional version of the petals of Theorem 1.1.

If f had a non-trivial one-dimensional f -invariant curve passing through the origin, that is an injective holomorphic map $\psi: \Delta \rightarrow \mathbb{C}^n$, where $\Delta \subset \mathbb{C}$ is a neighborhood of the origin, such that $\psi(0) = O$, $\psi'(0) \neq O$ and $f(\psi(\Delta)) \subseteq \psi(\Delta)$ with $f|_{\psi(\Delta)} \not\equiv \text{id}$, we could apply Leau-Fatou flower theorem to $f|_{\psi(\Delta)}$ obtaining a one-dimensional Fatou flower for f inside the invariant curve. In particular, if $z^o \in \psi(\Delta)$ belongs to an attractive petal, we would have $f^k(z^o) \rightarrow O$ and $[f^k(z^o)] \rightarrow [\psi'(0)]$, where $[\cdot]: \mathbb{C}^n \setminus \{O\} \rightarrow \mathbb{P}^{n-1}(\mathbb{C})$ is the canonical projection. The first observation we can make is that then $[\psi'(0)]$ cannot be any direction in $\mathbb{P}^{n-1}(\mathbb{C})$. Indeed:

Proposition 2.1 ([H2]) *Let $f(z) = z + P_{v+1}(z) + \dots \in \text{End}_1(\mathbb{C}^n, O)$ be tangent to the identity of order $v \geq 1$. Assume there is $z^o \in K_f$ such that $f^k(z^o) \rightarrow O$ and $[f^k(z^o)] \rightarrow [v] \in \mathbb{P}^{n-1}(\mathbb{C})$. Then $P_{v+1}(v) = \lambda v$ for some $\lambda \in \mathbb{C}$.*

Definition 2.1 Let $P: \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a homogeneous map. A direction $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$ is *characteristic* for P if $P(v) = \lambda v$ for some $\lambda \in \mathbb{C}$. Furthermore, we shall say that $[v]$ is *degenerate* if $P(v) = O$, and *non-degenerate* otherwise.

Remark 2.1 From now on, given $f \in \text{End}_1(\mathbb{C}^n, O)$ tangent to the identity of order $v \geq 1$, every notion/object/concept introduced for its leading term P_{v+1} will be introduced also for f ; for instance, a (degenerate/non-degenerate) characteristic direction for P_{v+1} will also be a (degenerate/non-degenerate) characteristic direction for f .

Remark 2.2 If $f \in \text{End}_1(\mathbb{C}^n, O)$ is given by (1.1), then $f^{-1} \in \text{End}_1(\mathbb{C}^n, O)$ is given by

$$f^{-1}(z) = z - P_{v+1}(z) + \dots .$$

In particular, f and f^{-1} have the same (degenerate/non-degenerate) characteristic directions.

Remark 2.3 If $\psi: \Delta \rightarrow \mathbb{C}^n$ is a one-dimensional curve with $\psi(0) = O$ and $\psi'(0) \neq O$ such that $f|_{\psi(\Delta)} \equiv \text{id}$, it is easy to see that $[\psi'(0)]$ must be a degenerate characteristic direction for f .

So if we have an f -invariant one-dimensional curve ψ through the origin then $[\psi'(0)]$ must be a characteristic direction. However, in general the converse is false: there are non-degenerate characteristic directions which are not tangent to any f -invariant curve passing through the origin.

Example 2.1 ([H2]) Let $f \in \text{End}(\mathbb{C}^2, O)$ be given by

$$f(z, w) = \left(\frac{z}{1+z}, w + z^2 \right),$$

so that f is tangent to the identity of order 1, and $P_2(z, w) = (-z^2, z^2)$. In particular, f has a degenerate characteristic direction $[0 : 1]$ and a non-degenerate characteristic direction $[v] = [1 : -1]$. The degenerate characteristic direction is tangent to the curve $\{z = 0\}$, which is pointwise fixed by f , in accord with Remark 2.3. We claim that no f -invariant curve can be tangent to $[v]$.

Assume, by contradiction, that we have an f -invariant curve $\psi: \Delta \rightarrow \mathbb{C}^2$ with $\psi(0) = O$ and $[\psi'(0)] = [v]$. Without loss of generality, we can assume that $\psi(\zeta) = (\zeta, u(\zeta))$ with $u \in \text{End}(\mathbb{C}, 0)$. Then the condition of f -invariance becomes $f_2(\zeta, u(\zeta)) = u(f_1(\zeta, u(\zeta)))$, that is

$$u(\zeta) + \zeta^2 = u\left(\frac{\zeta}{1+\zeta}\right). \quad (2.1)$$

Put $g(\zeta) = \zeta/(1+\zeta)$, so that $g^k(\zeta) = \zeta/(1+k\zeta)$; in particular, $g^k(\zeta) \rightarrow 0$ for all $\zeta \in \mathbb{C} \setminus \{-\frac{1}{n} \mid n \in \mathbb{N}^*\}$. This means that by using (2.1) we can extend u to $\mathbb{C} \setminus \{-\frac{1}{n} \mid n \in \mathbb{N}^*\}$ by setting

$$u(\zeta) = u(g^k(\zeta)) - \sum_{j=0}^{k-1} [g^j(\zeta)]^2$$

where $k \in \mathbb{N}$ is chosen so that $g^k(\zeta) \in \Delta$. Analogously, (2.1) implies that for $|\zeta|$ small enough one has

$$u(g^{-1}(\zeta)) + (g^{-1}(\zeta))^2 = u(\zeta);$$

so we can use this relation to extend u to all of \mathbb{C} , and then to $\mathbb{P}^1(\mathbb{C})$, because $g^{-1}(\infty) = -1$. So u is a holomorphic function defined on $\mathbb{P}^1(\mathbb{C})$, that is a constant; but no constant can satisfy (2.1), contradiction.

Remark 2.4 Ribón [R] has given examples of germs having no holomorphic invariant curves at all. For instance, this is the case for germs of the form $f(z, w) = (z + w^2, w + z^2 + \lambda z^5)$ for all $\lambda \in \mathbb{C}$ outside a polar Borel set.

The first important theorem we would like to quote is due to Écalle [E] and Hakim [H2], and it says that we do always have a Fatou flower tangent to a non-degenerate characteristic direction, even when there are no invariant complex curves containing the origin in their relative interior. To state it, we need to define what is the correct multidimensional notion of petal.

Definition 2.2 A parabolic curve for $f \in \text{End}_1(\mathbb{C}^n, O)$ tangent to the identity is an injective holomorphic map $\varphi: D \rightarrow \mathbb{C}^n \setminus \{O\}$ satisfying the following properties:

- (a) D is a simply connected domain in \mathbb{C} with $0 \in \partial D$;
- (b) φ is continuous at the origin, and $\varphi(0) = O$;
- (c) $\varphi(D)$ is f -invariant, and $(f|_{\varphi(D)})^k \rightarrow O$ uniformly on compact subsets as $k \rightarrow +\infty$.

Furthermore, if $[\varphi(\zeta)] \rightarrow [v]$ in $\mathbb{P}^{n-1}(\mathbb{C})$ as $\zeta \rightarrow 0$ in D , we shall say that the parabolic curve φ is *tangent* to the direction $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$. Finally, a *Fatou flower* with ν petals tangent to a direction $[v]$ is a holomorphic map $\Phi : D_{\nu, \delta} \rightarrow \mathbb{C}$, where $D_{\nu, \delta}$ is given by (1.3), such that Φ restricted to any connected component of $D_{\nu, \delta}$ is a parabolic curve tangent to $[v]$, a *petal* of the Fatou flower. If ν is the order of f then we shall talk of a Fatou flower for f without mentioning the number of petals.

Then Écalle, using his resurgence theory (see, e.g., [S] for an introduction to Écalle's resurgence theory in one dimension), and Hakim, using more classical methods, have proved the following result (see also [W]):

Theorem 2.2 ([E, H2, H3]) *Let $f \in \text{End}_1(\mathbb{C}^n, O)$ be tangent to the identity, and $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$ a non-degenerate characteristic direction for f . Then there exists (at least) one Fatou flower tangent to $[v]$. Furthermore, for every petal $\varphi : \Delta \rightarrow \mathbb{C}^n$ of the Fatou flower there exists a injective holomorphic map $\chi : \varphi(\Delta) \rightarrow \mathbb{C}$ such that $\chi(f(z)) = \chi(z) + 1$ for all $z \in \varphi(\Delta)$.*

Definition 2.3 The function χ constructed in the previous theorem is a *Hakim-Fatou coordinate*.

Remark 2.5 A characteristic direction is a *complex* direction, not a real one; so it should not be confused with the attracting/repelling directions of Theorem 1.1. All petals of a Fatou flower are tangent to the same characteristic direction, but each petal is tangent to a different real direction inside the same complex (characteristic) direction. In particular, Fatou flowers of f and f^{-1} are tangent to the same characteristic directions (see Remark 2.2) but the corresponding petals are tangent to different real directions, as in Theorem 1.1.

In particular there exist parabolic curves tangent to $[1 : -1]$ for the system of Example 2.1 even though there are no invariant curves passing through the origin tangent to that direction.

Parabolic curves are one-dimensional objects in an n -dimensional space; it is natural to wonder about the existence of higher dimensional invariant subsets. A sufficient condition for their existence has been given by Hakim; to state it we need to introduce another definition.

Definition 2.4 Let $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$ be a non-degenerate characteristic direction for a homogeneous map $P : \mathbb{C}^n \rightarrow \mathbb{C}^n$ of degree $\nu + 1 \geq 2$; in particular, $[v]$ is a fixed point for the meromorphic self-map $[P]$ of $\mathbb{P}^{n-1}(\mathbb{C})$ induced by P . The *directors* of P in $[v]$ are the eigenvalues $\alpha_1, \dots, \alpha_{n-1} \in \mathbb{C}$ of the linear operator

$$\frac{1}{\nu} (d[P]_{[v]} - \text{id}) : T_{[v]}\mathbb{P}^{n-1}(\mathbb{C}) \rightarrow T_{[v]}\mathbb{P}^{n-1}(\mathbb{C}).$$

As usual, if $f \in \text{End}_1(\mathbb{C}^n, O)$ is of the form (1.2), then the *directors* of f in a non-degenerate characteristic direction $[v]$ are the directors of P_{v+1} in $[v]$.

Remark 2.6 Definition 2.4 is equivalent to the original definition used by Hakim (see, e.g., [ArR]). Furthermore, in dimension 2 if $[v] = [1 : 0]$ is a non-degenerate characteristic direction of $P = (P_1, P_2)$ we have $P_1(1, 0) \neq 0$, $P_2(1, 0) = 0$ and the director is given by

$$\frac{1}{v} \frac{d}{d\zeta} \frac{P_2(1, \zeta) - \zeta P_1(1, \zeta)}{P_1(1, \zeta)} \Big|_{\zeta=0} = \frac{1}{v} \left[\frac{\frac{\partial P_2}{\partial z_2}(1, 0)}{P_1(1, 0)} - 1 \right].$$

Remark 2.7 Recalling Remark 2.2 one sees that a germ $f \in \text{End}_1(\mathbb{C}^n, O)$ tangent to the identity and its inverse f^{-1} have the same directors at their non-degenerate characteristic directions.

Remark 2.8 The proof of Theorem 2.2 becomes simpler when no director is of the form $\frac{k}{v}$ with $k \in \mathbb{N}^*$; furthermore, in this case the parabolic curves enjoy additional properties (in the terminology of [AT1] they are *robust*; see also [Ro3]).

Definition 2.5 A *parabolic manifold* for a germ $f \in \text{End}_1(\mathbb{C}^n, O)$ tangent to the identity is an f -invariant complex submanifold $M \subset \mathbb{C}^n \setminus \{O\}$ with $O \in \partial M$ such that $f^k(z) \rightarrow O$ for all $z \in M$. A *parabolic domain* is a parabolic manifold of dimension n . We shall say that M is *attached* to the characteristic direction $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$ if furthermore $[f^k(z)] \rightarrow [v]$ for all $z \in M$.

Then Hakim has proved (see also [ArR] for the details of the proof) the following theorem:

Theorem 2.3 ([H3]) *Let $f \in \text{End}_1(\mathbb{C}^n, O)$ be tangent to the identity of order $v \geq 1$. Let $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$ be a non-degenerate characteristic direction, with directors $\alpha_1, \dots, \alpha_{n-1} \in \mathbb{C}$. Furthermore, assume that $\text{Re}\alpha_1, \dots, \text{Re}\alpha_d > 0$ and $\text{Re}\alpha_{d+1}, \dots, \text{Re}\alpha_{n-1} \leq 0$ for a suitable $d \geq 0$. Then:*

- (i) *There exist (at least) v parabolic $(d+1)$ -manifolds M_1, \dots, M_v of \mathbb{C}^n attached to $[v]$;*
- (ii) *$f|_{M_j}$ is holomorphically conjugated to the translation $\tau(w_0, w_1, \dots, w_d) = (w_0 + 1, w_1, \dots, w_d)$ defined on a suitable right half-space in \mathbb{C}^{d+1} .*

Remark 2.9 In particular, if all the directors of $[v]$ have positive real part, there is at least one parabolic domain. However, the condition given by Theorem 2.3 is not necessary for the existence of parabolic domains; see [Ri1, Us, AT3] for examples, and [Ro8] for conditions ensuring the existence of a parabolic domain when some directors have positive real part and all the others are equal to zero. Moreover, Lapan [L1] has proved that if $n = 2$ and f has a unique characteristic direction $[v]$ which is non degenerate then there exists a parabolic domain attached to $[v]$ even though the director is necessarily 0.

Two natural questions now are: how many characteristic directions are there? Does there always exist a non-degenerate characteristic direction? To answer the first question, we need to introduce the notion of multiplicity of a characteristic direction. To do so, notice that $[v] = [v_1 : \dots : v_n] \in \mathbb{P}^{n-1}(\mathbb{C})$ is a characteristic direction for the homogeneous map $P = (P_1, \dots, P_n)$ if and only if $v_h P_k(v) - v_k P_h(v) = 0$ for all $h, k = 1, \dots, n$. In particular, the set of characteristic directions of P is an algebraic subvariety of $\mathbb{P}^{n-1}(\mathbb{C})$.

Definition 2.6 If the maximal dimension of the irreducible components of the subvariety of characteristic directions of a homogeneous map $P: \mathbb{C}^n \rightarrow \mathbb{C}^n$ is k , we shall say that P is *k-dicritical*; if $k = n$ we shall say that P is *dicritical*; if $k = 0$ we shall say that P is *non-dicritical*.

Remark 2.10 A homogeneous map $P: \mathbb{C}^n \rightarrow \mathbb{C}^n$ of degree d is dicritical if and only if $P(z) = p(z)z$ for some homogenous polynomial $p: \mathbb{C}^n \rightarrow \mathbb{C}$ of degree $d - 1$. In particular, the degenerate characteristic directions are the zeroes of the polynomial p .

In the non-dicritical case we can count the number of characteristic directions, using a suitable multiplicity.

Definition 2.7 Let $[v] = [v_1 : \dots : v_n] \in \mathbb{P}^{n-1}(\mathbb{C})$ be a characteristic direction of a homogeneous map $P = (P_1, \dots, P_n): \mathbb{C}^n \rightarrow \mathbb{C}^n$. Choose $1 \leq j_0 \leq n$ so that $v_{j_0} \neq 0$. The *multiplicity* $\mu_P([v])$ of $[v]$ is the local intersection multiplicity at $[v]$ in $\mathbb{P}^{n-1}(\mathbb{C})$ of the polynomials $z_{j_0} P_j - z_j P_{j_0}$ with $j \neq j_0$ if $[v]$ is an isolated characteristic direction; it is $+\infty$ if $[v]$ is not isolated.

Remark 2.11 The local intersection multiplicity $I(p_1, \dots, p_k; z^o)$ of a set $\{p_1, \dots, p_k\}$ of holomorphic functions at a point $z^o \in \mathbb{C}^n$ can be defined (see, e.g., [GH]) as

$$I(p_1, \dots, p_k; z^o) = \dim \mathcal{O}_{n, z^o} / (p_1, \dots, p_k),$$

where \mathcal{O}_{n, z^o} is the local ring of germs of holomorphic functions at z^o , and the dimension is as vector space. It is easy to check that the definition of multiplicity of a characteristic direction does not depend on the index j_0 chosen. Furthermore, since the local intersection multiplicity is invariant under change of coordinates, we can use local charts to compute the local intersection multiplicity on complex manifolds.

Remark 2.12 When $n = 2$, the multiplicity of $[v] = [1 : v_2]$ as characteristic direction of $P = (P_1, P_2)$ is the order of vanishing at $t = v_2$ of $P_2(1, t) - tP_1(1, t)$; analogously, the multiplicity of $[0 : 1]$ is the order of vanishing at $t = 0$ of $P_1(t, 1) - tP_2(t, 1)$.

Then we have the following result (see, e.g., [AT1]):

Proposition 2.4 *Let $P: \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a non-dicritical homogeneous map of degree $v + 1 \geq 2$. Then P has exactly*

$$\frac{1}{\nu}((\nu + 1)^n - 1) = \sum_{j=0}^{n-1} \binom{n}{j+1} \nu^j$$

characteristic directions, counted according to their multiplicity.

In particular, when $n = 2$ then a homogeneous map of degree $\nu + 1$ either is dicritical (and all directions are characteristic) or has exactly $\nu + 2$ characteristic directions. But all of them can be degenerate; an example is the following (but it is easy to build infinitely many others).

Example 2.2 Let $P(z, w) = (z^2w + zw^2, zw^2)$. Then the characteristic directions of P are $[1 : 0]$ and $[0 : 1]$, both degenerate. Using Remark 2.12, we see that $\mu_P([1 : 0]) = 3$ and $\mu_P([0 : 1]) = 1$.

So we cannot apply Theorem 2.2 to any germ of the form $f(z) = z + P(z) + \dots$ when P is given by Example 2.2. However, as soon as the higher order terms are chosen so that the origin is an isolated fixed point then f does have parabolic curves:

Theorem 2.5 ([A2]) *Let $f \in \text{End}_1(\mathbb{C}^2, O)$ be tangent to the identity such that O is an isolated fixed point. Then f admits at least one Fatou flower tangent to some characteristic direction.*

In the next section we shall explain why this theorem holds, we shall give more general statements, and we shall give an example (Example 3.1) showing the necessity of the hypothesis that the origin is an isolated fixed point.

3 Blow-Ups, Indices and Fatou Flowers

In the previous section we saw that for studying the dynamics of a germ tangent to the identity it is useful to consider the tangent directions at the fixed point. A useful way for dealing with tangent directions consists, roughly speaking, in replacing the fixed point by the projective space of the tangent directions, in such a way that the new space is still a complex manifold, where the tangent directions at the original fixed point are now points. We refer to, e.g., [GH] or [A1] for a precise description of this construction; here we shall limit ourselves to explain how to work with it.

Definition 3.1 Let M be a complex n -dimensional manifold, and $p \in M$. The *blow-up* of M of center p is a complex n -dimensional manifold \tilde{M} equipped with a surjective holomorphic map $\pi : \tilde{M} \rightarrow M$ such that

- (i) $E = \pi^{-1}(p)$ is a compact submanifold of \tilde{M} , the *exceptional divisor* of the blow-up, biholomorphic to $\mathbb{P}(T_p M)$;
- (ii) $\pi|_{\tilde{M} \setminus E} : \tilde{M} \setminus E \rightarrow M \setminus \{p\}$ is a biholomorphism.

Let us describe the construction for $(M, p) = (\mathbb{C}^n, O)$; using local charts one can repeat the construction for any manifold. As a set, $\tilde{\mathbb{C}}^n$ is the disjoint union of $\mathbb{C}^n \setminus \{O\}$ and $E = \mathbb{P}^{n-1}(\mathbb{C})$; we shall define a manifold structure using charts. For

$j = 1, \dots, n$ let $U'_j = \{[v_1 : \dots : v_n] \in \mathbb{P}^{n-1}(\mathbb{C}) \mid v_j \neq 0\}$, $U''_j = \{w \in \mathbb{C}^n \mid w_j \neq 0\}$ and $\tilde{U}_j = U'_j \cup U''_j \subset \widetilde{\mathbb{C}^n}$. Define $\chi_j: \tilde{U}_j \rightarrow \mathbb{C}^n$ by setting

$$\chi_j(q) = \begin{cases} \left(\frac{v_1}{v_j}, \dots, \frac{v_{j-1}}{v_j}, 0, \frac{v_{j+1}}{v_j}, \dots, \frac{v_n}{v_j} \right) & \text{if } q = [v_1 : \dots : v_n] \in U'_j, \\ \left(\frac{w_1}{w_j}, \dots, \frac{w_{j-1}}{w_j}, w_j, \frac{w_{j+1}}{w_j}, \dots, \frac{w_n}{w_j} \right) & \text{if } q = (w_1, \dots, w_n) \in U''_j. \end{cases}$$

We have

$$\chi_j^{-1}(w) = \begin{cases} [w_1 : \dots : w_{j-1} : 1 : w_{j+1} : \dots : w_n] & \text{if } w_j = 0, \\ (w_j w, \dots, w_j w_{j-1}, w_j, w_j w_{j+1}, \dots, w_j w_n) & \text{if } w_j \neq 0, \end{cases}$$

and it is easy to check that $\{(\tilde{U}_1, \chi_1), \dots, (\tilde{U}_n, \chi_n)\}$ is an atlas for $\widetilde{\mathbb{C}^n}$, with $\chi_j([0 : \dots : 1 : \dots : 0]) = O$ and $\chi_j(\tilde{U}_j \cap \mathbb{P}^{n-1}(\mathbb{C})) = \{w_j = 0\} \subset \mathbb{C}^n$. We can then define the projection $\pi: \widetilde{\mathbb{C}^n} \rightarrow \mathbb{C}^n$ in coordinates by setting

$$\pi \circ \chi_j^{-1}(w) = (w_1 w_j, \dots, w_{j-1} w_j, w_j, w_{j+1} w_j, \dots, w_n w_j);$$

it is easy to check that π is well-defined, that $\pi^{-1}(O) = \mathbb{P}^{n-1}(\mathbb{C})$ and that π induces a biholomorphism between $\widetilde{\mathbb{C}^n} \setminus \mathbb{P}^{n-1}(\mathbb{C})$ and $\mathbb{C}^n \setminus \{O\}$. Notice furthermore that $\widetilde{\mathbb{C}^n}$ has a canonical structure of line bundle over $\mathbb{P}^{n-1}(\mathbb{C})$ given by the projection $\tilde{\pi}: \widetilde{\mathbb{C}^n} \rightarrow \mathbb{P}^{n-1}(\mathbb{C})$ defined by

$$\tilde{\pi}(q) = \begin{cases} [v] & \text{if } q = [v] \in \mathbb{P}^{n-1}(\mathbb{C}), \\ [w_1 : \dots : w_n] & \text{if } q = w \in \mathbb{C}^n \setminus \{O\}; \end{cases}$$

the fiber over $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$ is given by the line $\mathbb{C}v \subset \mathbb{C}^n$.

Two more definitions we shall need later on:

Definition 3.2 Let $\pi: \tilde{M} \rightarrow M$ be the blow-up of a complex manifold M at $p \in M$. Given a subset $S \subset M$, the *full* (or *total*) *transform* of S is $\pi^{-1}(S)$, whereas the *strict transform* of S is the closure in \tilde{M} of $\pi^{-1}(S \setminus \{O\})$.

Clearly, the full and the strict transform coincide if $p \notin S$; if $p \in S$ then the full transform is the union of the strict transform and the exceptional divisor. Furthermore, if S is a submanifold at p then its strict transform is $(S \setminus \{p\}) \cup \mathbb{P}(T_p S)$.

Definition 3.3 Let $f \in \text{End}(M, p)$ be a germ such that df_p is invertible. Choose a representative (U, f) of the germ such that f is injective in U . Then the *blow-up* of f is the map $\tilde{f}: \pi^{-1}(U) \rightarrow \tilde{M}$ defined by

$$\tilde{f}(q) = \begin{cases} [df_p(v)] & \text{if } q = [v] \in E = \mathbb{P}(T_p M), \\ f(w) & \text{if } q = w \in U \setminus \{p\}. \end{cases}$$

In this way we get a germ about the exceptional divisor of a holomorphic self-map of the blow-up, given by the differential of f along the exceptional divisor and by f itself elsewhere, satisfying $\pi \circ \tilde{f} = f \circ \pi$. In particular, $K_{\tilde{f}} = \pi^{-1}(K_f) = (K_f \setminus \{O\}) \cup E$, and to study the dynamics of \tilde{f} in a neighborhood of the exceptional divisor is equivalent to studying the dynamics of f in a neighborhood of p .

If $f \in \text{End}_1(\mathbb{C}^n, O)$ is tangent to the identity, its blow-up \tilde{f} in the chart (U_1, χ_1) is given by

$$\chi_1 \circ \tilde{f} \circ \chi_1^{-1}(w) = \begin{cases} w & \text{if } w_1 = 0, \\ \left(f_1(w_1, w_1 w_2, \dots, w_1 w_n), \frac{f_2(w_1, w_1 w_2, \dots, w_1 w_n)}{f_1(w_1, w_1 w_2, \dots, w_1 w_n)}, \dots, \frac{f_n(w_1, w_1 w_2, \dots, w_1 w_n)}{f_1(w_1, w_1 w_2, \dots, w_1 w_n)} \right) & \text{if } w_1 \neq 0; \end{cases}$$

similar formulas hold in the other charts. In particular, writing $w = (w_1, w')$, and $\chi_1 \circ \tilde{f} \circ \chi_1^{-1} = (\tilde{f}_1, \dots, \tilde{f}_n)$, if f is tangent to the identity of order $\nu \geq 1$ and leading term $P_{\nu+1} = (P_{\nu+1,1}, \dots, P_{\nu+1,n})$, we get

$$\begin{cases} \tilde{f}_1(w) = w_1 + w_1^{\nu+1} P_{\nu+1,1}(1, w') + O(w_1^{\nu+2}), \\ \tilde{f}_j(w) = w_j + w_1^\nu (P_{\nu+1,j}(1, w') - w_j P_{\nu+1,1}(1, w')) + O(w_1^{\nu+1}) & \text{if } j \neq 1. \end{cases} \quad (3.1)$$

It follows immediately that:

- if $\nu \geq 2$ then \tilde{f} is tangent to the identity in all points of the exceptional divisor;
- if $\nu = 1$ then \tilde{f} is tangent to the identity in all characteristic directions of f ; in other points of the exceptional divisor the eigenvalues of the differential of \tilde{f} are all equal to 1 but the differential is not diagonalizable.

This means that we can always repeat the previous construction blowing-up \tilde{f} at a characteristic direction of f ; this will be important in the sequel.

As a first application of the blow-up construction, let us use it for describing the dynamics of dicritical maps. If $f \in \text{End}_1(\mathbb{C}^n, O)$ is dicritical, Theorem 2.2 yields a parabolic curve tangent to all directions outside a hypersurface of $\mathbb{P}^{n-1}(\mathbb{C})$ (notice that all directors are zero), and the same holds for f^{-1} . One can then summarize the situation as follows:

Proposition 3.1 ([Br1, Br2]) *Let $f \in \text{End}_1(\mathbb{C}^n, O)$ be a dicritical germ tangent to the identity of order $\nu \geq 1$. Write $P_{\nu+1}(z) = p(z)z$, and let $D = \{[v] \in \mathbb{P}^{n-1}(\mathbb{C}) \mid p(v) = 0\}$. Then there are two open sets $U^+, U^- \subset \mathbb{C}^n \setminus \{O\}$ such that:*

- (i) $\overline{U^+ \cup U^-}$ is a neighborhood of $\mathbb{P}^{n-1}(\mathbb{C}) \setminus D$ in the blow-up $\widetilde{\mathbb{C}^n}$ of O ;
- (ii) the orbit of any $z \in U^+$ converges to the origin tangent to a direction $[v] \in \mathbb{P}^{n-1}(\mathbb{C}) \setminus D$;
- (iii) the inverse orbit (that is, the orbit under f^{-1}) of any $z \in U^-$ converges to the origin tangent to a direction $[v] \in \mathbb{P}^{n-1}(\mathbb{C}) \setminus D$.

Coming back to the general situation, when $f \in \text{End}_1(\mathbb{C}^n, O)$ is tangent to the identity its blow-up \tilde{f} fixes pointwise the exceptional divisor; more precisely, the fixed point set of \tilde{f} is the full transform of the fixed point set of f , and in particular \tilde{f} has at least a hypersurface of fixed points. This is a situation important enough to deserve a special notation.

Definition 3.4 Let E be a connected (possibly singular) hypersurface in a complex manifold M . We shall denote by $\text{End}(M, E)$ the set of germs about E of holomorphic self-maps of M fixing pointwise E .

If E is a hypersurface in a complex manifold M , we shall denote by \mathcal{O}_M the sheaf of holomorphic functions on M , and by \mathcal{S}_E the subsheaf of functions vanishing on E . Given $f \in \text{End}(M, E)$, $f \neq \text{id}_M$, take $p \in E$. For every $h \in \mathcal{O}_{M,p}$, the germ $h \circ f$ is well-defined, and $h \circ f - h \in \mathcal{S}_{E,p}$. Following [ABT1] (see also [ABT2, ABT3]), we can then introduce a couple of important notions.

Definition 3.5 Let E be a connected hypersurface in a complex manifold M . Given $f \in \text{End}(M, E)$, $p \in E$ and $h \in \mathcal{O}_{M,p}$, let $v_f(h) = \max\{\mu \in \mathbb{N} \mid h \circ f - h \in \mathcal{S}_{E,p}^\mu\}$. Then the *order of contact* v_f of f with the identity along E is

$$v_f = \min\{v_f(h) \mid h \in \mathcal{O}_{M,p}\};$$

it can be shown ([ABT1]) that v_f does not depend on $p \in E$. Furthermore, we say that f is *tangential* if $\min\{v_f(h) \mid h \in \mathcal{S}_{E,p}\} > v_f$ for some (and hence any; see again [ABT1]) $p \in E$.

Let (z_1, \dots, z_n) be local coordinates in M centered at $p \in E$, and $\ell \in \mathcal{S}_{E,p}$ a reduced equation of E at p (that is, a generator of $\mathcal{S}_{E,p}$). If (f_1, \dots, f_n) is the expression of f in local coordinates, it turns out [ABT1] that we can write

$$f_j(z) = z_j + \ell(z)^{v_f} g_j^o(z) \tag{3.2}$$

for $j = 1, \dots, n$, where there is a j_0 such that ℓ does not divide $g_{j_0}^o$; furthermore, f is tangential if and only if $v_f(\ell) > v_f$.

Remark 3.1 If E is smooth at p , we can choose local coordinates so that locally E is given by $\{z_1 = 0\}$, that is $\ell = z_1$. Then we can write

$$f_j(z) = z_j + z_1^{v_f} g_j^o(z)$$

with z_1 not dividing some g_j^o ; and f is tangential if z_1 divides g_1^o , that is if $f_1(z) = z_1 + z_1^{v_f+1} h_1^o(z)$. More generally, if E has a normal crossing at p with $1 \leq r \leq n$ smooth branches, then we can choose local coordinates so that $\ell = z_1 \cdots z_r$, so that $f_j(z) = z_j + (z_1 \cdots z_r)^{v_f} g_j^o(z)$ with some $g_{j_0}^o$ not divisible by $z_1 \cdots z_r$; in this case f is tangential if and only if z_j divides g_j^o for $j = 1, \dots, r$. In particular, in the terminology of [A2] f is tangential if and only if it is nondegenerate and $b_f = 1$.

Definition 3.6 We say that $p \in E$ is a *singular point* for $f \in \text{End}(M, E)$ (with respect to E) if $g_1^o(p) = \cdots = g_n^o(p)$ in (3.2); it turns out [ABT1] that this definition is independent of the local coordinates. Furthermore, the *pure order* (or *pure multiplicity*) $v_o(f, E)$ of f along E at p is

$$v_o(f, E) = \min\{\text{ord}_O(g_1^o), \dots, \text{ord}_O(g_n^o)\}.$$

It is easy to see that the pure order does not depend on the local coordinates; in particular, p is singular for f with respect to E if and only if $v_o(f, E) \geq 1$. If E is the fixed point set of f at p then we shall talk of the *pure order* $v_o(f)$ of f at p .

Remark 3.2 When f is the blow-up of a germ $f_o \in \text{End}_1(\mathbb{C}^n, O)$ tangent to the identity of order $\nu \geq 1$, then (3.1) implies that:

- f is tangential if and only if f_o is not dicritical; in particular, in this case being tangential is a generic condition;
- $v_f = \nu$ if f_o is not dicritical, and $v_f = \nu + 1$ if f_o is dicritical;
- if f_o is non dicritical, then $[\nu] \in \mathbb{P}^{n-1}(\mathbb{C})$ is singular for f if and only if it is a characteristic direction of f_o .

Using the notion of singular points we can generalize Proposition 2.1 as follows:

Proposition 3.2 ([ABT1]) *Let $E \subset M$ be a hypersurface in a complex manifold M , and $f \in \text{End}(M, E)$, $f \neq \text{id}_M$, tangential to E . Let $p \in E$ be a smooth point of E which is non-singular for f . Then no infinite orbit of f can stay arbitrarily close to p , that is, there exists a neighborhood U of p such that for all $q \in U$ either the orbit of q lands on E or $f^{n_0}(q) \notin U$ for some $n_0 \in \mathbb{N}$. In particular, no infinite orbit is converging to p .*

More generally, we have:

Proposition 3.3 ([AT1]) *Let $f \in \text{End}(\mathbb{C}^n, O)$ be of the form*

$$f_j(z) = \begin{cases} z_j + z_j \left(\prod_{h=1}^r z_h^{v_h} \right) g_j(z) & \text{for } 1 \leq j \leq r, \\ z_j + \left(\prod_{h=1}^r z_h^{v_h} \right) g_j(z) & \text{for } r+1 \leq j \leq n, \end{cases} \quad (3.3)$$

for suitable $1 \leq r < n$, with $v_1, \dots, v_r \geq 1$ and $g_1, \dots, g_n \in \mathcal{O}_{\mathbb{C}^n, O}$. Assume that $g_{j_0}(O) \neq 0$ for some $r+1 \leq j_0 \leq n$. Then no infinite orbit can stay arbitrarily close to O .

A very easy example of this phenomenon, promised at the end of the previous section, is the following:

Example 3.1 Let $f(z, w) = (z, w + z^2)$. Then f is tangent to the identity at the origin; the fixed point set is $\{z = 0\}$, and thus O is not an isolated fixed point. We have $f^k(z, w) = (z, w + kz^2)$; therefore all orbits outside the fixed point set escape to infinity, and in particular no orbit converges to the origin. Notice that this germ

has only one characteristic direction, which is degenerate (and tangential to the fixed point set). Moreover, f is tangential with order of contact 2 to its fixed point set, but the origin is not singular.

After these generalities, in the rest of this section we specialize to the case $n = 2$ and to tangential maps (because of Remark 3.2; see anyway [ABT1] for information on the dynamics of non-tangential maps). Take $f \in \text{End}(M, E)$, where M is a complex surface and $E \subset M$ is a 1-dimensional curve smooth at $p \in E$, and assume that f is tangential to E with order of contact $v_f \geq 1$. Then we can choose local coordinates centered at p so that we can write

$$\begin{cases} f_1(z) = z_1 + z_1^{v_f+1} h_1^o(z), \\ f_2(z) = z_2 + z_1^{v_f} g_2^o(z), \end{cases} \quad (3.4)$$

where z_1 does not divide g_2^o ; notice that $h_1^o(0, \cdot) = \frac{\partial g_1^o}{\partial z_1}(0, \cdot)$, where $g_1^o = z_1 h_1^o$. In particular, O is singular if and only if $g_2^o(O) = 0$. We then introduce the following definitions:

Definition 3.7 Let $f \in \text{End}(M, E)$ be written in the form (3.4). Then:

- the *multiplicity* μ_p of f along E at p is $\mu_p = \text{ord}_0(g_2^o(0, \cdot))$, so that p is a singular point if and only if $\mu_p \geq 1$;
- the *transversal multiplicity* τ_p of f along E at p is $\tau_p = \text{ord}_0(h_1^o(0, \cdot))$;
- p is an *apparent singularity* if $1 \leq \mu_p \leq \tau_p$;
- p is a *Fuchsian singularity* if $\mu_p = \tau_p + 1$;
- p is an *irregular singularity* if $\mu_p > \tau_p + 1$;
- p is a *non-degenerate singularity* if $\mu_p \geq 1$ but $\tau_p = 0$;
- p is a *degenerate singularity* if $\mu_p, \tau_p \geq 1$;
- the *index* $\iota_p(f, E)$ of f at p along E is

$$\iota_p(f, E) = v_f \text{Res}_0 \frac{h_1^o(0, \cdot)}{g_2^o(0, \cdot)};$$

- the *induced residue* $\text{Res}_p^0(f)$ of f along E at p is

$$\text{Res}_p^0(f) = -\iota_p(f, E) - \mu_p.$$

It is possible to prove (see [A2, ABT1, AT3]; notice that our index is v_f times the residual index introduced in [A2]) that these definitions are independent of the local coordinates.

Remark 3.3 Recalling (3.1), we see that if f is obtained as the blow-up of a non-dicritical map f_o , and E is the exceptional divisor of the blow-up, then:

- the multiplicity of $[v]$ as characteristic direction of f_o is equal to the multiplicity of f along E at $[v]$;

- $[v]$ is a degenerate/non-degenerate characteristic direction of f_o if and only if it is a degenerate/non-degenerate singularity of f .

Furthermore, if we write $P_{v+1,1}(1, w) = \sum_{k=0}^{v+1} a_k w^k$ and $P_{v+1,2}(1, w) = \sum_{k=0}^{v+1} b_k w^k$ then $[1 : 0]$ is a characteristic direction if and only if $b_0 = 0$, non-degenerate if and only if moreover $a_0 \neq 0$, and (setting $b_{v+2} = 0$)

$$\frac{h_1^o(0, \zeta)}{g_2^o(0, \zeta)} = \frac{1}{\zeta} \frac{a_0 + \sum_{k=1}^{v+1} a_k \zeta^k}{(b_1 - a_0) + \sum_{k=1}^{v+1} (b_{k+1} - a_k) \zeta^k}.$$

So if $b_1 \neq a_0$ we have

$$\mu_O = 1, \quad \iota_{[1:0]}(f, E) = \frac{va_0}{b_1 - a_0}, \quad \text{Res}_{[1:0]}^0(f) = \frac{(v-1)a_0 + b_1}{a_0 - b_1};$$

moreover, if $a_0 \neq 0$ then $[1 : 0]$ is a non-degenerate characteristic direction with director $\alpha = (b_1 - a_0)/va_0$. More generally, we have $\tau_{[1:0]} = \text{ord}_0(P_{v+1,1}(1, w))$, $\mu_{[1:0]} = \text{ord}_0(P_{v+1,2}(1, w) - wP_{v+1,1}(1, w))$ and

$$\iota_{[1:0]} = \frac{va_{\mu-1}}{b_{\mu} - a_{\mu-1}}, \quad \text{Res}_{[1:0]}^0 = \frac{(v-\mu)a_{\mu-1} + \mu b_{\mu}}{a_{\mu-1} - b_{\mu}},$$

where $\mu = \mu_{[1:0]}$. In particular we obtain:

- if $[v]$ is non-degenerate characteristic direction of f_o with director $\alpha \neq 0$ then

$$\iota_{[v]}(f, E) = \frac{1}{\alpha};$$

- $[v]$ is a non-degenerate characteristic direction with non-zero director for f_o if and only if it is a Fuchsian singularity of multiplicity 1 for f .

Residues and indices are important for two reasons. First of all, we have the following *index theorem*:

Theorem 3.4 ([A2, ABT1]) *Let $E \subset M$ be a smooth compact Riemann surface in a complex surface M . Let $f \in \text{End}(M, E)$, $f \neq \text{id}_M$, be tangential with order of contact v ; denote by $\text{Sing}(f) \subset E$ the finite set of singular points of f in E . Then*

$$\sum_{p \in \text{Sing}(f)} \iota_p(f) = vc_1(N_E), \quad \sum_{p \in \text{Sing}(f)} \text{Res}_p^0(f) = -\chi(E),$$

where $c_1(N_E)$ is the first Chern class of the normal bundle N_E of E in M , and $\chi(E)$ is the Euler characteristic of E . In particular, when f is the blow-up of a nondicritical germ tangent to the identity and $E = \mathbb{P}^1(\mathbb{C})$ is the exceptional divisor we have

$$\sum_{p \in \text{Sing}(f)} \iota_p(f) = -v, \quad \sum_{p \in \text{Sing}(f)} \text{Res}_p^0(f) = -2.$$

Remark 3.4 Bracci and Tovena [BT] have defined a notion of index at non-smooth points of E allowing the generalization of Theorem 3.4 to non necessarily smooth compact Riemann surfaces, where in the statement $c_1(N_E)$ is replaced by the self-intersection $E \cdot E$.

The second reason is that the index can be used to detect the presence of parabolic curves. To state this precisely, we need a definition.

Definition 3.8 Let $f \in \text{End}_1(\mathbb{C}^2, O)$ be tangent to the identity. We say that O is a *corner* if the germ of the fixed point set at the origin is locally reducible, that is has more than one irreducible component.

Then

Theorem 3.5 ([A2]) *Let $E \subset M$ be a smooth Riemann surface in a complex surface M , and take $f \in \text{End}(M, E)$ tangential. Let $p \in E$ be a singular point, not a corner, such that $\iota_p(f) \notin \mathbb{Q}^+ \cup \{0\}$. Then there exists a Fatou flower with v_f petals for f at p .*

Corollary 3.6 ([A2]) *Let $f \in \text{End}_1(\mathbb{C}^2, O)$ be tangent to the identity, and assume that O is a nondicritical singular point. Let $[v] \in \mathbb{P}^1(\mathbb{C})$ be a characteristic direction, and \tilde{f} the blow-up of f . If $[v]$ is not a corner for \tilde{f} and $\iota_{[v]}(\tilde{f}) \notin \mathbb{Q}^+ \cup \{0\}$ then there exists a Fatou flower for f tangent to $[v]$.*

Theorem 2.5 is then a consequence of Theorem 3.4 and Corollary 3.6. Indeed, take $f \in \text{End}_1(\mathbb{C}^2, O)$ tangent to the identity with an isolated fixed point at the origin. If O is dicritical, we can directly apply Theorem 2.2. Assume then O non-dicritical, and let $\tilde{f} \in \text{End}(\tilde{\mathbb{C}}^2, E)$ be the blow-up of f . Since O is non-dicritical, \tilde{f} is tangential; Theorem 3.4 then implies that at least one characteristic direction $[v]$ has negative index. Since O is an isolated fixed point, the fixed point set of \tilde{f} coincides with the exceptional divisor; therefore $[v]$ is not a corner, and Corollary 3.6 yields the Fatou flower we were looking for.

Remark 3.5 Bracci and Degli Innocenti (see [B, D]), using the definition of index introduced in [BT], have shown that Theorem 3.5 still holds when E is not smooth at p . Bracci and Suwa [BS] have also obtained a version of Theorem 3.5 when M has a (sufficiently tame) singularity at p .

Example 3.2 Let $f(z, w) = (z + z^2w + zw^2 + w^4, w + zw^2 + z^4)$. Then f is tangent to the identity at the origin of order 2, and the origin is an isolated fixed point. Furthermore, f is non-dicritical and it has (see Example 2.2) two characteristic directions, $[v_1] = [1 : 0]$ and $[v_2] = [0 : 1]$, both degenerate. Working as in Remark 3.4 it is easy to see that $[v_1]$ is an irregular singularity of multiplicity 3 with index -2 and induced residue -1 , and that $[v_2]$ is an apparent singularity of multiplicity 1, vanishing index, and induced residue -1 . In particular, f admits a Fatou flower with 2 petals tangent to $[v_1]$.

Example 3.3 Let $f(z, w) = (z + w^2, w + z^3)$. Then f is tangent to the identity at the origin of order 1, and the origin is an isolated fixed point. Furthermore, f is non-dicritical with only one characteristic direction $[v] = [1 : 0]$, which is degenerate of multiplicity 3, Fuchsian, with index -1 and induced residue -2 . Therefore f admits a Fatou flower with one petal tangent to $[v]$; compare with Example 3.4.

There are still instances where Theorem 3.5 cannot be applied:

Example 3.4 Let $f(z, w) = (z + zw + w^3, w + 2w^2 + bz^3)$ with $b \neq 0$. This map is tangent to the identity, with order 1, and the origin is an isolated fixed point. Moreover, it has two characteristic directions: $[1 : 0]$, degenerate Fuchsian with multiplicity 2, index 1 and induced residue -3 ; and $[0 : 1]$, non-degenerate Fuchsian with multiplicity 1, index -2 and induced residue 1. Theorem 2.2 (as well as Corollary 3.6) yields a Fatou flower tangent to $[0 : 1]$; on the other hand, none of the results proven up to now say anything about direction $[1 : 0]$.

However, a deep result by Molino gives the existence of a Fatou flower in the latter case too:

Theorem 3.7 ([Mo]) *Let $E \subset M$ be a smooth Riemann surface in a complex surface M , and take $f \in \text{End}(M, E)$ tangential with order of contact ν . Let $p \in E$ be a singular point, not a corner, such that $\nu_o(f) = 1$ and $\iota_p(f) \neq 0$. Then there exists a Fatou flower for f at p . More precisely:*

- (i) *if p is an irregular singularity, or a Fuchsian singularity with $\iota_p(f) \neq \nu\mu_p$, then there exists a Fatou flower for f with $\nu + \tau_p(\nu + 1)$ petals;*
- (ii) *if p is a Fuchsian singularity with $\iota_p(f) = \nu\mu_p$ then there exists a Fatou flower for f with ν petals.*

Remark 3.6 Even more precisely, when p is Fuchsian with $\mu_p \geq 2$ and $\iota_p(f) = \nu\mu_p$ then Molino constructs parabolic curves defined on the connected components of a set of the form

$$D_{\nu+1-\frac{1}{\mu_p}, \delta} = \left\{ \zeta \in \mathbb{C} \mid |\zeta^{\nu+1-1/\mu_p} (\log \zeta)^{1-1/\mu_p} - \delta| < \delta \right\},$$

which has at least ν connected components with the origin in the boundary.

Corollary 3.8 ([Mo]) *Let $f \in \text{End}_1(\mathbb{C}^2, O)$ be tangent to the identity of order $\nu \geq 1$, and assume that O is a nondicritical singular point. Let $[v] \in \mathbb{P}^1(\mathbb{C})$ be a characteristic direction, and \tilde{f} the blow-up of f . If $[v]$ is not a corner for \tilde{f} , $\nu_o(\tilde{f}) = 1$ and $\iota_{[v]}(\tilde{f}) \neq 0$ then there exists a Fatou flower for f with at least ν petals tangent to $[v]$.*

The assumption on the pure order in these statements seems to be purely technical; so it is natural to advance the following

Conjecture 3.9 *Let $E \subset M$ be a smooth Riemann surface in a complex surface M , and take $f \in \text{End}(M, E)$ tangential of order of contact ν . Let $p \in E$ be a singular point, not a corner, such that $\iota_p(f) \neq 0$. Then there exists a Fatou flower for f at p .*

See Sect. 5, and in particular (5.3), for examples of systems having Fatou flowers at singular points with vanishing index.

Instrumental in the proofs of Theorems 3.5 and 3.7 is a reduction of singularities statement. We shall need a few definitions:

Definition 3.9 Let $f \in \text{End}_1(\mathbb{C}^n, O)$ be tangent to the identity. A *modification* of f is a $\tilde{f} \in \text{End}(M, E)$ obtained as the lifting of f to a finite sequence of blow-ups, where the first one is centered in O and the remaining ones are centered in singular points of the intermediate lifted maps contained in the exceptional divisor. A modification is *non-dicritical* if none of the centers of the blow-ups is dicritical. Associated to a modification $\tilde{f} \in \text{End}(M, E)$ of f we have a holomorphic map $\pi : M \rightarrow \mathbb{C}^n$ such that $\pi^{-1}(O) = E$, $\pi|_{M \setminus E}$ is a biholomorphism between $M \setminus E$ and $\mathbb{C}^n \setminus \{O\}$, and $f \circ \pi = \pi \circ \tilde{f}$. The exceptional divisor E is the union of a finite number of copies of $\mathbb{P}^{n-1}(\mathbb{C})$, crossing transversally.

Definition 3.10 Let $f \in \text{End}_1(M, p)$ be tangent to the identity, where M is a complex surface. In local coordinates centered at p , we can write $f(z) = z + \ell(z)g^o(z)$, where $\ell = \text{gcd}(f_1 - z_1, f_2 - z_2)$ is defined up to units. We shall say that p is an *irreducible singularity* if:

- (a) $\text{ord}_p(\ell) \geq 1$ and $v_o(f) = 1$; and
- (b) if λ_1, λ_2 are the eigenvalues of the linear part of g^o then either
 - (\star_1) $\lambda_1, \lambda_2 \neq 0$ and $\lambda_1/\lambda_2, \lambda_2/\lambda_1 \notin \mathbb{N}$, or
 - (\star_2) $\lambda_1 \neq 0, \lambda_2 = 0$.

It turns out that there always exists a modification with only dicritical or irreducible singularities:

Theorem 3.10 ([A2]) *Let $f \in \text{End}_1(\mathbb{C}^2, O)$ be tangent to the identity, and assume that O is a singular point. Then there exists a non-dicritical modification $\tilde{f} \in \text{End}(M, E)$ of f such that the singular points of \tilde{f} on E are either irreducible or dicritical.*

Definition 3.11 Let $f \in \text{End}_1(\mathbb{C}^2, O)$ be tangent to the identity. The modification of f satisfying the conclusion of Theorem 3.10 obtained with the minimum number of blow-ups is the *minimal resolution* of f .

It is easy to see that the techniques of the proof of Theorem 2.2 yield the existence of a Fatou flower at dicritical singularities, and at irreducible singularities of type (\star_1) which are not a corner; then the proof of Theorem 3.5 amounts to showing that if the index is not a non-negative rational number then the minimal resolution contains at least a singularity which is either dicritical or of type (\star_1) and not a corner. The proof of Theorem 3.7(i) consists in showing that, under those hypotheses, the minimal resolution must contain a non-degenerate singularity, which is not a corner and where one can apply Theorem 2.2; the proof of Theorem 3.7(ii) requires instead a technically hard extension of Theorem 2.2.

See also [Ro4, Ro7] and [LS] for other approaches to resolution of singularities for germs tangent to the identity in arbitrary dimension, and [AT2, AR] for the somewhat related problem of the identification of formal normal forms for germs tangent to the identity.

4 Parabolic Domains

Theorem 2.3 yields conditions ensuring the existence of parabolic domains attached to a non-degenerate characteristic direction. In dimension 2, Vivas has found conditions ensuring the existence of a parabolic domain attached to Fuchsian and irregular degenerate characteristic directions, and Rong has found conditions ensuring the existence of a parabolic domain attached to apparent degenerate characteristic directions. Very recently, Lapan [L2] has extended Rong's approach to cover more types of degenerate characteristic directions.

More precisely, Vivas has proved the following result:

Theorem 4.1 ([V1]) *Let $f \in \text{End}_1(\mathbb{C}^2, O)$ be tangent to the identity of order $\nu \geq 1$, with O nondicritical. Let $[v] \in \mathbb{P}^1(\mathbb{C})$ be a degenerate characteristic direction, and \tilde{f} the blow-up of f . Denote by $\mu \geq 1$ the multiplicity, by $\tau \geq 0$ the transversal multiplicity, by $\iota \in \mathbb{C}$ the index, and by $\nu_o \geq 1$ the pure order of \tilde{f} at $[v]$. Assume that either*

(a) *$[v]$ is Fuchsian (thus necessarily $\tau \geq 1$ because $[v]$ is degenerate) and*

$$\text{Re } \iota + \tau > 0, \quad \left| \iota + \frac{\tau}{2} - \frac{\nu\mu}{2} \right| > \frac{\tau}{2} + \frac{\nu\mu}{2};$$

or

(b) *$[v]$ is Fuchsian, $\nu_o = 1$ and*

$$\left| \iota - \frac{\mu\nu}{2} \right| < \frac{\mu\nu}{2};$$

or

(c) *$[v]$ is Fuchsian, $\nu_o > 1$ and*

$$\text{Re } \iota + \tau > 0, \quad \left| \iota - \frac{(\nu+1)\tau}{2} \right| > \frac{(\nu+1)\tau}{2};$$

or

(d) *$[v]$ is irregular.*

Then there is a parabolic domain attached to $[v]$.

See also Remark 6.4 for a comment about the conditions on ι and τ .

To state Rong's theorem, consider a germ $f \in \text{End}_1(\mathbb{C}^2, O)$ tangent to the identity of order $\nu \geq 1$, and assume that $[1 : 0]$ is a characteristic direction of f . Then we can write

$$\begin{cases} f_1(z, w) = z + az^{r+1} + O(z^{r+2}) + w\alpha(z, w), \\ f_2(z, w) = w + bz^\nu w + dz^{s+1} + O(z^{s+2}) + O(wz^{\nu+1}) + w^2\beta(z, w), \end{cases} \quad (4.1)$$

with $\nu \leq r \leq +\infty$, $\nu + 1 \leq s \leq +\infty$, $\text{ord}_O(\alpha) \geq \nu$, $\text{ord}_O(\beta) \geq \nu - 1$, and $a \neq 0$ if $r < +\infty$ (respectively, $d \neq 0$ if $s < +\infty$). The characteristic direction $[1 : 0]$ is non-degenerate if and only if $r = \nu$; in this case the director is given by $\frac{1}{\nu}(\frac{b}{a} - 1)$. On the other hand, saying that $[1 : 0]$ is degenerate with $b \neq 0$ is equivalent to saying that $r > \nu$ and that $[1 : 0]$ has multiplicity 1 and transversal multiplicity at least 1; in particular, in this case it is an apparent singularity. Then Rong's theorem can be stated as follows:

Theorem 4.2 ([Ro9]) *Let $f \in \text{End}_1(\mathbb{C}^2, O)$ be tangent to the identity of order $\nu \geq 1$ and written in the form (4.1), so that $[1 : 0]$ is a characteristic direction. Assume that $r > \nu$ and $b \neq 0$, so that $[1 : 0]$ is an apparent degenerate characteristic direction. Suppose furthermore that $s > r$, and that $b^2/a \notin \mathbb{R}^+$ if $r = 2\nu$. Then there is a parabolic domain attached to $[1 : 0]$.*

To state Lapan's result we need to introduce a few definitions.

Definition 4.1 Let $f \in \text{End}_1(\mathbb{C}^2, O)$ be tangent to the identity of order $\nu \geq 1$ with homogeneous expansion (1.1). We say that $[\nu] \in \mathbb{P}^1(\mathbb{C})$ is a *characteristic direction of degree $s \geq \nu + 1$* if it is a characteristic direction of $P_{\nu+1}, \dots, P_s$. We shall say that it is *non-degenerate in degree $r + 1$* , with $\nu < r < s$, if it is degenerate for $P_{\nu+1}, \dots, P_r$ and non-degenerate for P_{r+1} .

For instance, if f is in the form (4.1) with $s < +\infty$, then $[1 : 0]$ is a characteristic direction of degree s . If furthermore $r + 1 \leq s$ then it is non-degenerate in degree $r + 1$.

Definition 4.2 Let $f \in \text{End}_1(\mathbb{C}^2, O)$ be tangent to the identity of order $\nu \geq 1$ with homogeneous expansion (1.1). Assume that $[1 : 0] \in \mathbb{P}^1(\mathbb{C})$ is a characteristic direction of degree $s \geq \nu + 1$. Given $\nu + 1 \leq j \leq s$, the *j -order* of $[1 : 0]$ is the order of vanishing at 0 of $P_{j,2}(1, \cdot)$, where $P_j = (P_{j,1}, P_{j,2})$. We say that $[1 : 0]$ is *of order one in degree $t + 1$* , with $\nu \leq t < s$, if the j -order of $[1 : 0]$ is larger than one for $\nu + 1 \leq j \leq t$ and of $(t + 1)$ -order exactly equal to 1.

For instance, if $b \neq 0$ in (4.1) then $[1 : 0]$ is of order one in degree $\nu + 1$. More generally, if $b = 0$ and $[1 : 0]$ is of order one in degree $t + 1$ then we can replace the term $O(wz^{\nu+1})$ by $O(wz^{t+1})$.

Assume that $[1 : 0]$ is a characteristic direction of degree $s < +\infty$, non-degenerate in degree $r + 1 \leq s$ and of order one in degree $t + 1 \leq s$. Then we can write

$$\begin{cases} f_1(z, w) = z + az^{r+1} + O(z^{r+2}) + w\alpha(z, w), \\ f_2(z, w) = w + bz^{t+1}w + dz^{s+1} + O(z^{s+2}) + O(wz^{t+2}) + w^2\beta(z, w), \end{cases} \quad (4.2)$$