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Weakly Wandering Sequences in Ergodic Theory

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This monograph is dedicated to
Professor Shizuo Kakutani 1911–2004

Foreword

Weakly wandering (*ww*) sets made their first appearance over 50 years ago in [23, 39] (see also [32, 40]). The late Professor Shizuo Kakutani of Yale University was instrumental in advising and directing us in the development of the above works. Initially, the appearance of *ww* sets and sequences was a surprising event, yet at the same time quite useful in the study of problems connected with the existence of finite invariant measures. Soon it was realized that *ww* sequences were always present for all ergodic transformations that did not preserve a finite measure. Professor Kakutani felt that this was an important fact and strongly encouraged us to study the role of these sequences in the classification of infinite ergodic transformations. During the years that followed, we would meet periodically with him in the southern New England region from New Haven, to Providence, Boston, and Amherst, spend long periods studying various problems in ergodic theory, and often discuss questions connected with properties of *ww* sequences. He was aware that the *ww* and related sequences associated with infinite ergodic transformations were powerful isomorphism invariants, and he urged us to investigate their properties. More than anyone else he had a keen sense of understanding the nature of infinite measure spaces and properties of the transformations defined on them.

During the following 40 years we continued our joint work and published several articles on the properties of infinite ergodic transformations. It was during one of our frequent meetings that Professor Kakutani suggested the writing of a monograph which gathered most of the published and unpublished results that we had obtained. We had just started on that project when it was interrupted by his untimely departure.

Professor Kakutani was a constant force guiding and encouraging us to continue working and looking into the effect of *ww* and related sequences on the behavior of infinite ergodic transformations. This monograph is a result of that. Eliminating the contribution of any of the co-authors from this monograph would make it noticeably weaker. On the other hand, without Professor Kakutani's contribution and constant encouragement this monograph would not exist.

Preface

The material in this monograph is self-contained. A basic knowledge of measure theory as taught to beginning graduate students is the only prerequisite needed to read and understand the material presented. Prior knowledge of ergodic theory is useful but not necessary. Some fundamental properties of ergodic transformations preserving a σ-finite infinite measure as discussed in Chap. 3 follow easily from Birkhoff's Individual Ergodic Theorem. However, even these properties are developed and proven directly.

In Chap. 1 we discuss in some detail various conditions for the existence of a finite invariant measure. In 1932, E. Hopf [37] presented an interesting geometric condition that was necessary and sufficient for the existence of a finite invariant measure for a measurable and nonsingular transformation. Later in 1956, Y. Dowker [8] discussed the same problem and presented a different condition involving the measure of iterates of the images of measurable sets. Initially, the two conditions, the one presented by Hopf (**H**) and the other by Dowker (**D1**), did not seem to be obviously related except for the fact that they were both necessary and sufficient conditions for the solution of the same problem. The attempt to prove their equivalence by direct arguments on the other hand revealed the interesting and unexpected fact that all infinite ergodic transformations possessed weakly wandering (*ww*) sets: these are sets of positive measure with an infinite number of mutually disjoint images under a sequence of integers called a *ww* sequence. We also mention some minor facts that emerged during our attempt to show by direct arguments the equivalence of the various conditions of the Finite Invariant Measure Theorem 1.2.1. One of these is Proposition (subadditive) 1.2.2, which is a slight generalization related to a well-known result on the equivalence of finite measures, and another is Proposition (additive) 1.2.4, which exhibits the additive nature of the Cesaro sums of the measure of iterates of measurable sets. We also point out that condition (**W***), which is a (seemingly) stronger condition than condition (**W**), happens to be equivalent to it. The following interesting-sounding remark is a consequence of Theorems 1.1.3 and 1.2.1: a simple strengthening of the statement

> A measurable transformation is recurrent if and only if it does not possess wandering sets

is

> A measurable transformation is strongly recurrent if and only if it does not possess weakly wandering sets,

and both statements are true.

In Chap. 2 we discuss properties of transformations that do not possess a finite invariant measure. While writing this monograph we were often tempted to relabel this chapter: "The Non-Existence of a Finite Invariant Measure." The appearance of *ww* sequences for such transformations turned out to be a powerful tool in the classification of infinite ergodic transformations. In time the existence of these sequences implied the existence of even more interesting sequences connected with ergodic transformations without finite invariant measure. One such was the appearance of an equally unexpected sequence that we called an exhaustive weakly wandering (*eww*) sequence: this is a *ww* sequence where the images of a corresponding *ww* set cover the whole space. In Definition 2.1.1 we introduce a more complicated sequence for measurable transformations, which we call a strongly weakly wandering (*sww*) sequence. At first encounter it seems that *sww* sequences have unnecessarily complicated properties and are difficult to understand. However, for ergodic transformations without a finite invariant measure we are able to show without too much effort the existence of *sww* sequences, and these in turn imply the existence of a special kind of *eww* sequence. Moreover, we do not know a better way of proving the existence of *eww* sequences for transformations without a finite invariant measure. Initially, after we were confronted with the existence of *ww* and *eww* sequences, for a very long time we were not aware of any general property of a *ww* sequence that insured its possessing an *eww* subsequence. To our surprise Proposition 2.2.4 accomplishes that.

In Chap. 3 we discuss infinite ergodic transformations and mention the following important property that these transformations possess: for any two sets A and B of finite measure $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} m(T^i A \cap B) = 0$. This property follows immediately from Birkhoff's Individual Ergodic Theorem. However, we present a direct and elementary proof of it. Next we introduce *recurrent* sequences for infinite ergodic transformations. These are sequences of integers that have a finite intersection with every *ww* sequence for an infinite ergodic transformation. As we shall see in Chap. 4 there exist infinite ergodic transformations that possess recurrent sequences and others that do not. We discuss both classes of infinite ergodic transformations. The infinite ergodic transformations that possess recurrent sequences are interesting in connection with the various conditions discussed in Theorem 1.2.1 (the Finite Invariant Measure Theorem). The infinite ergodic transformations that do not possess recurrent sequences happen to be even more interesting. For such transformations we prove Theorems 3.3.11 and 3.3.12 where we show that these transformations possess *ww* and *eww* growth sequences. A consequence of this is the fact that for infinite ergodic transformations without recurrent sequences every infinite sequence of integers contains an *eww* subsequence. Incidentally, after working with the presence and absence of recurrent sequences and discussing their connection with *ww* and *eww* sequences, we urge the reader to refrain from naming any feature of an infinite ergodic transformation as being some sort of mixing. As tempting as it may seem, we feel that labeling any property of an infinite ergodic transformation as some type of "mixing" is wrong.

In Chap. 4 we present three important and basic examples of infinite ergodic transformations. The First Basic Example was constructed soon after it was realized that a consequence of the main theorem of [32] was the fact that every infinite ergodic transformation possesses *ww* sets. Initially, this fact was a bit difficult to digest. In particular, among all the existing examples of infinite ergodic transformations we could not exhibit a single *ww* set. This prompted us to construct the First Basic Example in [33]. Our object was to see a concrete *ww* set and the *ww* sequence associated with it. Employing the machinery of induced transformations we succeeded in constructing the desired transformation together with the *ww* sequence and set. Subsequently, we were compensated with a few extra and unexpected rewards. The *ww* set that we had constructed happened to be, in fact, an *eww* set as well. Up to that point the existence of an *eww* set for any infinite ergodic transformation was unthinkable. Our surprise was even greater when we noticed that this *eww* set was actually a set of finite measure. A number of years later the Second Basic Example was constructed in [31] with the purpose of showing that the existence of an *eww* set of finite measure was not shared by all infinite ergodic transformations. This was accomplished by showing that the commutators of the Second Basic Example contained a non-measure-preserving transformation. This fact was used in showing how to construct in a systematic way an ergodic transformation that does not preserve any σ-finite invariant measure. The Third Basic Example that we present next was actually constructed well before the two preceding examples. It was originally constructed in [23] to show that the further weakening of condition (**D3**) of Theorem 1.2.1 of Chap. 1 as a necessary and sufficient condition for the existence of a finite invariant measure was not possible. Later it was also realized that, unlike the previous two examples, this basic example was an infinite ergodic transformation that did not possess recurrent sequences. Finally, a variant of an example discussed by E. Hopf in his book [38] is sketched as another example of an infinite ergodic transformation that does not possess recurrent sequences. This example can also be regarded as a simple realization of symmetric random walk on the integers as an infinite ergodic transformation.

In Chap. 5 we consider various collections of infinite subsets of the integers $\mathbb{Z}$ associated to an infinite ergodic transformation, and discuss a number of properties of these collections. In particular we give descriptions of collections of *ww*, recurrent, and dissipative sequences for the transformation T. We end the chapter with a topological description of these and other collections in terms of the Stone–Čech compactification $\beta\mathbb{Z}$ of $\mathbb{Z}$. Theorem 5.1.2 of this chapter is a particularly interesting theorem concerning the behavior of transformations that possess an *eww* set of finite measure.

In Chap. 6 we examine various isomorphism invariants for infinite ergodic transformations. We begin with *eww* sequences and note that an isomorphism, besides leaving an *eww* sequence invariant, must also map *eww* sets to each other. We then show by example that two such sets for a common sequence may sit quite differently within a transformation. We then introduce the α-type of an ergodic transformation as an isomorphism invariant and show its relation to the recurrent sequences of the transformation. In the second part of the chapter we examine a class of transformations for which a complete characterization of the recurrent sequences can be described. We end the chapter with a result on how the growth rate of the *ww* sequences for a transformation is also an isomorphism invariant.

In Chap. 7 we show that *eww* sequences are related to complementing pairs of subsets of integers which tile the set of integers. We begin with a review of known results for tilings of the integers and point out that the tools used when one member of a pair is finite are not applicable when both members are infinite. We then show how such tilings of the integers arise in ergodic theory and use the fact that one member is the hitting times of a generic point to an *eww* set to obtain a characterization of *eww* sequences. A number of examples are given which indicate the difficulties of the subject. Finally, we conclude the chapter by showing how p-adic analysis is related to *eww* sequences for some infinite ergodic transformations.

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Contents

Chapter 1
Existence of Finite Invariant Measure

In this chapter we discuss properties of a transformation T that are equivalent to the transformation being recurrent. We show that strengthened versions of these properties, together with a few more properties of T, are necessary and sufficient conditions for the existence of a finite invariant measure μ for T.

We consider transformations T that are invertible (1-1, onto) maps defined on a σ-finite Lebesgue measure space $(X, \mathscr{B}, m)$. Even when not mentioned explicitly, all the transformations we consider are assumed to be measurable ($A \in \mathscr{B}$ if and only if $TA \in \mathscr{B}$) and nonsingular ($m(A) = 0$ if and only if $m(TA) = 0$). Throughout this monograph all the sets we mention are assumed to be measurable, and often we make statements ignoring sets of measure 0.

We say that m is an invariant measure for a transformation T if $m(TA) = m(A)$ for all $A \in \mathscr{B}$. Two measures m and μ are *equivalent* ($m \sim \mu$) if m and μ have the same sets of measure zero. When an invariant measure $\mu \sim m$ exists for T we say that T preserves the measure μ, or T is a measure-preserving transformation.

In this section we study necessary and sufficient conditions for the existence of a finite T-invariant measure $\mu \sim m$.

We remark that since $(X, \mathscr{B}, m)$ is a σ-finite measure space, it is always possible to find a finite measure $m' \sim m$. Namely, since $X = \bigcup_{i=1}^{\infty} A_i\,(disj)$, where A_i are sets of finite positive measure for $i = 1, 2, \ldots$, we define

$$m'(B) = \sum_{i=1}^{\infty} \frac{m(B \cap A_i)}{2^i m(A_i)} \quad \text{for} \quad B \in \mathscr{B}.$$

Therefore, whenever a transformation T is not assumed to be measure-preserving, without loss of generality we may assume that T is defined on a finite measure space $(X, \mathscr{B}, m)$ with $m(X) = 1$.

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1.1 Recurrent Transformations

Let us make the following definitions:

Definition 1.1.1. Let T be a measurable and nonsingular transformation defined on the measure space $(X, \mathscr{B}, m)$.

- T is a *recurrent* transformation if
  $m(A) > 0 \implies$ for a.a. $x \in A$ there is an integer $n > 0$ such that $T^n x \in A$.
- A is a *wandering* set for T if
  $m(A) > 0$, and $T^i A \cap T^j A = \emptyset$ for $i, j \in \mathbb{Z}$ and $i \neq j$.
- Two sets A and B are *finitely equivalent*, $A \approx B$, if for some integer $p > 0$
  $A = \bigcup_{i=1}^{p} A_i\,(disj)$, $B = \bigcup_{i=1}^{p} B_i\,(disj)$, and for a set of p integers $\{n_i : 1 \leq i \leq p\}$ $T^{n_i} A_i = B_i$ for $1 \leq i \leq p$.
- Two sets A and B are *countably equivalent*, $A \sim B$, if
  $A = \bigcup_{i=1}^{\infty} A_i\,(disj)$, $B = \bigcup_{i=1}^{\infty} B_i\,(disj)$, and for a sequence of integers $\{n_i : i \geq 1\}$ $T^{n_i} A_i = B_i$ for $1 \leq i < \infty$.
- A set A is *strongly recurrent* if $\{n : m(T^n A \cap A) > 0\}$ is relatively dense in $\mathbb{Z}$, or equivalently:
  there is an integer $k > 0$ such that $\max_{0 \leq i \leq k} m(T^{n+i} A \cap A) > 0$ for all $n \in \mathbb{Z}$.
- An infinite sequence of integers $\{n_i : i \geq 0\}$ is a *weakly wandering* (*ww*) *sequence* for T if there is a set W of positive measure such that
  $T^{n_i} W \cap T^{n_j} W = \emptyset$ for $i, j \geq 0$ and $i \neq j$.
  Often we will say W is a *ww* set (for T) (with the sequence $\{n_i\}$),
  or $\{n_i\}$ is a *ww* sequence (for T) (with the set W).

The following lemma about wandering sets is used in the proof of the Recurrence Theorem that follows.

Lemma 1.1.2 (Wandering Sets). *The following two conditions for a nonsingular transformation T on $(X, \mathscr{B}, m)$ are equivalent.*

(1) *T does not admit any wandering sets.*
(2) *For a measurable function $f(x)$, if $f(Tx) \leq f(x)$ a.e., then $f(Tx) = f(x)$ a.e.*

Proof. **(1) ⇒ (2):** Assume condition **(2)** does not hold. Then there is a measurable function $f(x)$ so that $f(Tx) \leq f(x)$ a.e. and $m\{x : f(Tx) < f(x) < \infty\} > 0$. Therefore there exists a constant c such that if $W = \{x : f(Tx) \leq c < f(x)\}$ then $m(W) > 0$.
For $x \in W$ we have: $f(T^n x) \leq f(T^{n-1} x) \leq \cdots \leq f(Tx) \leq c$.
For $x \in T^{-n} W$, since $T^n x \in W$, we have: $c < f(T^n x)$.
Then $T^n W \cap W = \emptyset$ for all $n > 0$. Thus $T^i W \cap T^j W = T^j(T^{i-j} W \cap W) = \emptyset$ for $i > j$. In other words, W is a wandering set for T. This is a contradiction to **(1)**.

(2) ⇒ (1): Assume condition **(1)** does not hold.
Let W be a wandering set for T, and let $W^* = \bigcup_{n=0}^{\infty} T^{-n} W$.