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Zhang Functions and Various Models

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To our ancestors and parents, as always

Preface

Time-varying mathematical problems are frequently encountered in scientific and engineering applications, such as circuit parameters in electronic circuits, aerodynamic coefficients in high-speed aircraft, and mechanical parameters in machinery. How to solve the time-varying problem effectively is becoming more and more necessary and important (as it is usually an essential part of many solutions). As we know, the common way to solve the time-varying problem is to treat such a problem as a static problem within a small time period (i.e., assume the short-time invariance of the problem). Then, the related numerical algorithms and/or neural-dynamics methods are developed to solve the problem at each single time instant, where the change trend of the time-varying coefficient(s) is not exploited. As for these conventional approaches, the computation is based on the present data, and the computed result is directly used for future. Thus, there exist lagging-error phenomena, when they are directly exploited to solve time-varying problems. In other words, the aforementioned approaches, which are designed theoretically/intrinsically for solving the static (or say, time-invariant, constant) problems, are less effective and efficient on time-varying problems solving.

Since March 12, 2001, Zhang et al. have formally proposed, investigated, and developed a special class of recurrent neural networks (i.e., Zhang neural network), which have been analyzed theoretically and substantiated comparatively for solving time-varying problems precisely and efficiently. By following the previous research on Zhang neural network, Zhang dynamics (ZD) has been generalized and further developed since 2008, whose state dimension can be multiple or one. It is viewed as a systematic approach to solving time-varying problems with the scalar situation included. It differs from conventional gradient dynamics (GD) in terms of the problem to be solved, error function, design formula, dynamic equation, and the utilization of time-derivative information. Besides, Zhang function (ZF), which is also referred to as Zhangian, is the design basis of ZD. It differs from the usual error/energy functions in the study of conventional approaches. Specifically, compared with the norm-based scalar-valued positive or at least lower-bounded energy function usually used in the GD design, ZF (1) is indefinite (i.e., can be positive, zero, or negative, in addition to being bounded, unbounded, or even lower unbounded), (2) can be matrix- or vector-valued (when solving a time-varying matrix- or vector-valued problem), and (3) can be real- or complex-valued (corresponding to a real- or complex-valued time-varying problem solving) to monitor and control the process of time-varying problems solving fully.

In this book, focusing on solving different types of time-varying problems, we design, propose, develop, analyze, model, and simulate various ZD models by defining various ZFs in real and complex domains. Specifically, in the real domain, we define three different classes of ZFs, i.e., scalar-valued ZFs, vector-valued ZFs and matrix-valued ZFs, for developing the resultant ZD models to solve the corresponding time-varying (scalar/vector/matrix-valued) problems. In the complex domain, we define different complex-valued ZFs for developing the resultant ZD models to solve three different types of complex-valued time-varying problems (one is with scalar formulation, and the rest are with matrix formulations). As for these ZD models, the related theoretical analyses are given, and the corresponding modeling (together with block diagrams) is illustrated. Computer simulations with various illustrative examples are performed to substantiate the efficacy of the proposed ZD models for time-varying problems solving. The simulation results also show the feasibility of the presented ZD approach (i.e., different ZFs leading to different ZD models) for real-time solution of time-varying problems. Based on these successful researches, we further apply such a ZD approach to repetitive motion planning (RMP) of redundant robot manipulators (including fixed-base and mobile ones). The corresponding results show the application prospect of the presented ZD approach to robots RMP.

The idea for this book on neural dynamics was conceived during classroom teaching as well as during research discussion in the laboratory and at international scientific meetings. Most of the materials in this book are derived from the authors' papers published in journals and proceedings of international conferences. In fact, since the early 1980s, the field of neural networks/dynamics has undergone phases of exponential growth, generating many new theoretical concepts and tools (including the authors' ones). At the same time, these theoretical results have been applied successfully to the solution of many practical problems. Our first priority is thus to cover each central topic in enough detail to make the material clear and coherent; in other words, each part (and even each chapter) is written in a relatively self-contained manner.

This book contains 15 chapters which are classified into the following five parts.

Part I: Scalar-Valued ZF in Real Domain (Chaps. 1–3);
Part II: Vector-Valued ZF in Real Domain (Chaps. 4–6);
Part III: Matrix-Valued ZF in Real Domain (Chaps. 7–10);
Part IV: ZF in Complex Domain (Chaps. 11–13);
Part V: ZF Application to Robot Control (Chaps. 14 and 15).

Chapter 1—In this chapter, we propose and develop four different indefinite ZFs as the error-monitoring functions, which lead to four different ZD models for time-varying reciprocal finding. In addition, theoretical analyses and Simulink modeling of such different ZD models are presented. Computer simulation results with three illustrative examples further substantiate the efficacy of the ZD models for time-varying reciprocal finding.

Chapter 2—In this chapter, by introducing six different ZFs, we propose, develop, and investigate six different ZD models to solve for time-varying inverse square root. In addition, this chapter presents theoretical analyses and Simulink modeling of such ZD models. Computer simulation results with two illustrative examples further substantiate the efficacy of the ZD models for time-varying inverse square root finding.

Chapter 3—In this chapter, six different ZD models are proposed, developed, and investigated by introducing six different ZFs for time-varying square root finding. In addition, the Simulink modeling of such ZD models is presented. Computer simulation results with two illustrative examples further substantiate the efficacy of the ZD models for time-varying square root finding.

Chapter 4—In this chapter, by following the idea of ZF, two ZD models are proposed, developed, and investigated for solving system of time-varying linear equations. In addition, it is theoretically proved that such two ZD models globally and exponentially converge to the theoretical time-varying solution of system of time-varying linear equations. Computer simulation results with three illustrative examples further substantiate the efficacy (as well as theoretical analyses) of the ZD models for solving system of time-varying linear equations.

Chapter 5—In this chapter, focusing on solving over-determined system of time-varying linear equations, we first propose, develop, and investigate two ZD models based on two different ZFs. Then, by introducing anther two different ZFs, another two ZD models are proposed, developed, and investigated for solving under-determined system of time-varying linear equations. Computer simulation results with four illustrative examples further substantiate the efficacy of such ZD models for solving over-determined and under-determined systems of time-varying linear equations.

Chapter 6—In this chapter, by introducing three different ZFs, we propose, develop, and investigate three different ZD models for solving time-varying linear matrix-vector inequality. Theoretical analyses and results are presented as well to show the excellent convergence performance of such ZD models. Computer simulation results with two illustrative examples further substantiate the efficacy of the ZD models for time-varying linear matrix-vector inequality solving.

Chapter 7—In this chapter, focusing on time-varying matrix inversion, we propose and develop six different ZFs that lead to six different ZD models. Meanwhile, a specific relationship between the ZD model and the Getz and Marsden (G-M) dynamic system is discovered. Eventually, theoretical analyses and Simulink modeling of such different ZD models are presented. Computer simulation results with two illustrative examples further substantiate the efficacy of the ZD models for time-varying matrix inversion.

Chapter 8—In this chapter, by introducing five different ZFs, we propose, develop, and investigate five different ZD models for time-varying matrix left pseudoinversion. In addition, the link between the ZD model and G-M dynamic system is discovered for time-varying matrix left pseudoinverse solving.

Theoretical analyses and computer simulation results with three illustrative examples further substantiate the efficacy of the ZD models on solving for the time-varying matrix left pseudoinverse.

Chapter 9—In this chapter, by introducing four different ZFs, four different ZD models are proposed, developed, and investigated for time-varying right pseudoinversion. In addition, the link between the ZD model and G-M dynamic system is discovered to solve for time-varying matrix right pseudoinverse. Theoretical results and computer simulations with three illustrative examples further substantiate the efficacy of the ZD models for time-varying matrix right pseudoinversion.

Chapter 10—In this chapter, eight different indefinite ZFs, which lead to eight different ZD models, are proposed and developed as the error-monitoring functions for time-varying matrix square root finding. In addition, theoretical analyses and Simulink modeling of such ZD models are presented. Computer simulation results with two illustrative examples further substantiate the efficacy of the ZD models for time-varying matrix square root finding.

Chapter 11—In this chapter, by introducing four different ZFs in complex domain, four different ZD models are proposed, developed, and investigated to solve for time-varying complex reciprocal. Computer simulation results with three illustrative examples further substantiate the efficacy of the complex ZD models for time-varying complex reciprocal finding.

Chapter 12—In this chapter, focusing on time-varying complex matrix inversion, we propose, develop, and investigate three different complex ZD models by introducing three different complex ZFs. Computer simulation results with four illustrative examples further substantiate the efficacy of the complex ZD models for time-varying complex matrix inversion.

Chapter 13—In this chapter, by introducing five different complex ZFs, five different complex ZD models are proposed, developed, and investigated to solve for time-varying complex matrix generalized inverse (in most cases, the complex pseudoinverse). Meanwhile, theoretical analyses and results are presented to show the convergence properties of such complex ZD models. In addition, we discover the link between the complex ZD model and G-M dynamic system in complex domain. Computer simulation results with four illustrative examples further substantiate the efficacy of the complex ZD models for time-varying complex matrix generalized inverse solving.

Chapter 14—In this chapter, by introducing two different ZFs and by exploiting the ZD design formula, an acceleration-level RMP performance index is proposed, developed and investigated for fixed-base redundant robot manipulators. The resultant RMP scheme, which incorporates joint-angle, joint-velocity, and joint-acceleration limits, is further presented and investigated to remedy the joint-angle drift phenomenon of fixed-base redundant robot manipulators. Such a scheme is then reformulated as a quadratic program (QP), which is solved by a primal–dual neural network. With three path-tracking examples, computer simulation results based on PUMA560 robot manipulator substantiate well the effectiveness and accuracy of the acceleration-level RMP scheme, as well as show the application prospect of the presented ZD approach (i.e., different ZFs leading to different ZD models).

Chapter 15—In this chapter, by introducing three different ZFs and by exploiting the ZD design formula, we propose, develop, and investigate a velocity-level RMP performance index for mobile redundant robot manipulators. Then, based on such a performance and with physical limits considered, the resultant RMP scheme is presented and investigated to remedy the joint-angle drift phenomenon of mobile redundant robot manipulators. Such a scheme is reformulated as a QP, which is solved by a numerical algorithm. With two path-tracking examples, computer simulation results based on a wheeled mobile robot manipulator substantiate well the effectiveness and accuracy of the velocity-level RMP scheme, and show the application prospect of the presented ZD approach once again.

In summary, this book presents a novel approach (i.e., different ZFs resulting in different ZD models) for solving various time-varying problems in real and complex domains, and further applies such an approach to RMP control of different types of robot manipulators (showing its application prospect). This book is written for graduate students as well as academic and industrial researchers studying in the developing fields of neural dynamics, computer mathematics, time-varying computation, simulation and modeling, analog hardware, and robotics. It provides a comprehensive view of the combined research of these fields, in addition to its accomplishments, potentials, and perspectives. We do hope that this book will generate curiosity and also happiness to its readers for learning more in the fields and the research, and that it will provide new challenges to seek new theoretical tools and practical applications.

At the end of this Preface, it is worth pointing out that, in this book, a new and inspiring direction on the definition of the error function (or say, the energy function involved in convention researches) is provided for the neural-dynamics construction. This opens the door on defining the error function from a single definition equation of the specific problem to be solved to various appropriate formulations (resulting in various neural-dynamics models that can be chosen for practitioners in accordance with specific requests). It may promise to become a major inspiration for studies and researches in neural dynamics, time-varying problems solving, prediction, and dynamic decision making. Without doubt, this book can be extended. Any comments or suggestions are welcome. The authors can be contacted via e-mail: zhynong@mail.sysu.edu.cn, and gdongsh2008@126.com. The web page of Yunong Zhang is http://sist.sysu.edu.cn/~zhynong/.

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Contents

Acronyms

ASIC	Application-specific integrated circuit
DNN	Dual neural network
DOF	Degrees of freedom
FPGA	Field programmable gate array
GD	Gradient dynamics
MSSMRE	Maximal steady-state modeling residual error
PDNN	Primal–dual neural network
QP	Quadratic program
RMP	Repetitive motion planning
RT	Relative tolerance
VLSI	Very large-scale integration
ZD	Zhang dynamics
ZF	Zhang function

Part I
Scalar-Valued ZF in Real Domain

Chapter 1
Time-Varying Reciprocal

Abstract Along with neural dynamics (based on analog solvers) widely arising in scientific computation and optimization fields in recent decades which attracts extensive interest and investigation of researchers, a special type of neural dynamics, called Zhang dynamics (ZD), has been formally proposed by Zhang et al. for real-time solution of time-varying problems. By following Zhang et al.'s neural-dynamics design method, the ZD model, which is based on an indefinite Zhang function (ZF), can guarantee the exponential convergence performance for time-varying problems solving. In this chapter, for time-varying reciprocal finding, we propose, generalize, develop, and investigate different indefinite ZFs as the error-monitoring functions, which can lead to different ZD models. In addition, for the goal of developing the floating-point processors or coprocessors for the future generation of computers, the MATLAB Simulink modeling and simulative verifications of such different ZD models are presented. The modeling results further substantiate the efficacy of the proposed ZD models for time-varying reciprocal finding.

1.1 Introduction and Preliminaries

The reciprocal computation, which is described in the form of $f(x) = ax - 1 = 0$, is considered to be an important operation in a floating-point divider/processor. Thus, many researches on the reciprocal computation are conducted and presented [1–6]. However, these researches are just for the static reciprocal computation, thereby making the corresponding methods less accurate enough to solve the time-varying reciprocal problem in the following form:

$$f(x(t), t) = a(t)x(t) - 1 = 0 \in \mathbb{R},\ \ t \in [0, +\infty), \tag{1.1}$$

where $a(t) \neq 0 \in \mathbb{R}$ denotes a smoothly time-varying scalar with $\dot{a}(t) \in \mathbb{R}$ denoting the time derivative of $a(t)$, both of which are assumed to be known numerically or could be measured accurately. In this chapter, we aim at finding the $x(t) \in \mathbb{R}$ to make (1.1) hold true at any time instant $t \in [0, +\infty)$. Furthermore, $x^*(t)$ is used to denote the theoretical time-varying reciprocal of $a(t)$ [i.e., mathematically, $x^*(t) = 1/a(t)$].

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Remark 1.1 The above $x^*(t)$ is given symbolically for better understanding and solution comparison, whose the computation of $1/a(t)$ at every single time instant t is less practical in real-life applications. Specifically, when we compute $1/a(t)$ at a time instant t, as the computation consumes time Δt inevitably, the value of $a(t)$ is changing during the computation procedure. Thus, the computed result is less accurate and less effective, since the value of the theoretical time-varying reciprocal has actually changed to $x^*(t+\Delta t)$ after the computation. This is the so-called lagging-error phenomenon. Note that, as for other time-varying problems solving (via the conventional computation approaches), such types of lagging-error phenomena still exist. This propels us to develop and investigate an effective computation approach for real-time solution of various time-varying problems (e.g., the ones presented and investigated in this chapter as well as Chaps. 2–13).

Generally speaking, in the solving process of (1.1), a real-time solver first receives the specific data of $a(t)$ at one single time instant; then the solver does computations based on the present and/or the stored previous data; and finally, it outputs the result to the user. Note that, in this process, the solver cannot use the future data because they are unknown and have not come yet at the present time instant. Furthermore, at every single time instant, we, based on the present and/or previous data, compute the result for future. This is also because computation consumes time inevitably. As for the conventional approaches, the computation is based on the present data, and the computed result is directly used for future. Thus, there exists the lagging-error problem (see also Remark 1.1), when they are directly exploited to solve the time-varying reciprocal problem. In other words, these approaches are less effective on the time-varying reciprocal problem solving. This is the reason why we need to develop and investigate an effective approach for time-varying reciprocal finding (and further, for real-time solution of various time-varying problems).

Being different from the conventional neural-dynamics approach (i.e., gradient dynamics, GD), a special type of neural dynamics, called Zhang dynamics (ZD), has been formally proposed by Zhang et al. for various time-varying problems solving [7–13]. According to Zhang et al.'s neural-dynamics design method, the ZD is designed based on an indefinite Zhang function (ZF) as the error-monitoring function (where the word "indefinite" here means that such an error-monitoring function can be positive, zero, negative or even lower-unbounded). This differs from the situation involved in the design of conventional approaches; for example, a norm-based positive-definite energy function is generally used in the GD design [8, 11, 12]. Thus, by making use of the time-derivative information of the time-varying coefficient(s) involved in the time-varying problem, the resultant ZD models can methodologically avoid the lagging errors generated by the conventional approaches. Note that such ZD models can guarantee much better convergence performance to the theoretical time-varying solution of the time-varying problem in an error-free manner. Besides, for better understanding and to lay a basis for further investigation, the concepts of ZD and ZF are presented as follows.

Concept 1.1 Zhang dynamics (ZD) has been generalized from Zhang neural network formally since 2008 [12], of which the state dimension can be multiple or one. It is viewed as a systematic approach to real-time solution of time-varying problems with scalar situation included as well. It differs from the conventional GD in terms of the problem to be solved, error function, design formula, dynamic equation, and the utilization of time-derivative information.

Concept 1.2 Zhang function (ZF), which is also referred to as Zhangian, is the design basis of ZD. It differs from the usual error/energy functions in the study of conventional approaches. Specifically, compared with the norm-based scalar-valued positive or at least lower-bounded energy function usually used in the GD design, ZF (1) is indefinite (i.e., can be positive, zero, or negative, in addition to being bounded, unbounded, or even lower unbounded), (2) can be matrix- or vector-valued (when solving a time-varying matrix- or vector-valued problem), and (3) can be real- or complex-valued (corresponding to a real- or complex-valued time-varying problem solving) to monitor and control the process of time-varying problems solving fully.

In this chapter, focusing on time-varying reciprocal finding, we propose, generalize, develop, and investigate different ZD models by defining different ZFs as the error-monitoring functions. In addition to the theoretical analyses and verifications of the convergence characteristics of the proposed ZD models, the MATLAB Simulink modeling [14–16] and illustrative examples are presented and investigated with the goal of developing the floating-point processors or coprocessors for the future generation of computers. From the modeling results, the efficacy of the proposed ZD models based on different ZFs for time-varying reciprocal finding is substantiated. To the best of the author's knowledge, almost all reported computation approaches [1–6] are theoretically/intrinsically designed for static/time-invariant reciprocal finding. There is almost no other literature handling such a specific problem solving, i.e., real-time solution of time-varying reciprocal, at present stage.

1.2 ZFs and ZD Models

In this section, we introduce four different ZFs and propose the resultant ZD models for solving the time-varying reciprocal problem (1.1).

Because the ZF is the design basis for deriving a ZD model and for presentation convenience, we denote the ZF by $e(t)$ with $\dot{e}(t)$ being the time derivative of $e(t)$. Note that, in this chapter and also in Chaps. 2 and 3, $e(t)$ and $\dot{e}(t)$ are used as the notations of the scalar-valued ZF and its time derivative, respectively. Besides, to lay a basis for further discussion, the design procedure for a ZD model is presented as follows.

- First, we define an indefinite ZF as the error-monitoring function to monitor the process of time-varying reciprocal finding.
- Second, to force $e(t)$ globally and exponentially converge to zero, we choose its time derivative $\dot{e}(t)$ via the following ZD design formula: [7–13]:

$$\dot{e}(t) = \frac{\mathrm{d}e(t)}{\mathrm{d}t} = -\gamma e(t), \tag{1.2}$$

where design parameter $\gamma > 0 \in \mathbb{R}$ corresponds to the reciprocal of a capacitance parameter, which should be set as large as the hardware would permit [10, 12, 13, 17], or selected appropriately for the simulative purpose.

- Finally, by expanding the ZD design formula (1.2), the dynamic equation of a ZD model is thus established for time-varying reciprocal finding.

For the excellent property of global and exponential convergence of the ZD design formula (1.2), we have the following theorem.

Theorem 1.1 *As for the ZD design formula (1.2) which is also a dynamic system, starting from an initial error $e(0) \in \mathbb{R}$, the error function $e(t) \in \mathbb{R}$ globally and exponentially converges to zero with rate γ.*

Proof For (1.2), by calculus, we obtain its analytical solution as $e(t) = e(0) \exp(-\gamma t)$. Based on the definition of global and exponential convergence, we can draw the conclusion that, starting from any $e(0)$, $e(t)$ globally and exponentially converges to zero with rate γ, as time t tends to infinity. The proof is thus complete. □

Besides, it is worth pointing out here that the aforementioned design procedure for the scalar situation can also be generalized for deriving the ZD models to solve other time-varying problems with matrix or vector formulations (e.g., the ones presented and investigated in Chaps. 4–10).

Specifically, for real-time solution of time-varying reciprocal problem (1.1), in this chapter, we define the following four different ZFs:

$$e(t) = x(t) - \frac{1}{a(t)}, \tag{1.3}$$

$$e(t) = a(t) - \frac{1}{x(t)}, \tag{1.4}$$

$$e(t) = a(t)x(t) - 1, \tag{1.5}$$

$$e(t) = \frac{1}{a(t)x(t)} - 1. \tag{1.6}$$

According to the ZD design formula (1.2), different ZFs lead to different ZD models, which is detailed as below.

- Let us consider the ZD design formula (1.2) and ZF (1.3). Then, we have

$$\dot{x}(t) + \frac{1}{a^2(t)}\dot{a}(t) = -\gamma \left(x(t) - \frac{1}{a(t)} \right),$$

which is rewritten as

$$a^2(t)\dot{x}(t) = -\dot{a}(t) - \gamma \left(a^2(t)x(t) - a(t)\right). \tag{1.7}$$

Thus, we obtain ZD model (1.7) for time-varying reciprocal finding.

- Considering the ZD design formula (1.2) and ZF (1.4), we have

$$\dot{a}(t) + \frac{1}{x^2(t)}\dot{x}(t) = -\gamma \left(a(t) - \frac{1}{x(t)}\right)$$

which is reformulated as

$$\dot{x}(t) = -\dot{a}(t)x^2(t) - \gamma \left(a(t)x^2(t) - x(t)\right). \tag{1.8}$$

Therefore, ZD model (1.8) for time-varying reciprocal finding is obtained.

- By combining the ZD design formula (1.2) and ZF (1.5), we have

$$\dot{a}(t)x(t) + a(t)\dot{x}(t) = -\gamma\,(a(t)x(t) - 1),$$

and then

$$a(t)\dot{x}(t) = -\dot{a}(t)x(t) - \gamma\,(a(t)x(t) - 1). \tag{1.9}$$

ZD model (1.9) for time-varying reciprocal finding is thus obtained.

- With the ZD design formula (1.2) and ZF (1.6) combined, we have

$$-\frac{1}{a^2(t)x^2(t)}\,(\dot{a}(t)x(t) + a(t)\dot{x}(t)) = -\gamma \left(\frac{1}{a(t)x(t)} - 1\right),$$

which is rewritten as

$$a(t)\dot{x}(t) = -\dot{a}(t)x(t) + \gamma \left(a(t)x(t) - a^2(t)x^2(t)\right). \tag{1.10}$$

Therefore, we come to ZD model (1.10) for time-varying reciprocal finding.

As a result, we have obtained four different types of ZD models [i.e. (1.7)–(1.10)] for time-varying reciprocal finding, which correspond to four different types of ZFs [i.e., (1.3)–(1.6)]. For readers' convenience and also for comparison, such four different ZD model based on four different ZFs for time-varying reciprocal finding are listed in Table 1.1.

Table 1.1 Different ZFs resulting in different ZD models for time-varying reciprocal finding

ZF	ZD model
(1.3)	$a^2(t)\dot{x}(t) = -\dot{a}(t) - \gamma \left(a^2(t)x(t) - a(t)\right)$
(1.4)	$\dot{x}(t) = -\dot{a}(t)x^2(t) - \gamma \left(a(t)x^2(t) - x(t)\right)$
(1.5)	$a(t)\dot{x}(t) = -\dot{a}(t)x(t) - \gamma \left(a(t)x(t) - 1\right)$
(1.6)	$a(t)\dot{x}(t) = -\dot{a}(t)x(t) + \gamma \left(a(t)x(t) - a^2(t)x^2(t)\right)$

1.3 Theoretical Results and Analyses

In this section, four propositions (viewed as the special cases of 1.1) are presented, which show the convergence properties of the proposed ZD models (1.7)–(1.10) for time-varying reciprocal finding.

Proposition 1.1 *Consider a smoothly time-varying scalar $a(t) \neq 0 \in \mathbb{R}$ involved in time-varying reciprocal problem (1.1). Starting from randomly-generated initial state $x(0) \neq 0 \in \mathbb{R}$ which has the same sign as $a(0)$, the neural state $x(t)$ of ZD model (1.7) derived from ZF (1.3) exponentially converges to the theoretical time-varying reciprocal $x^*(t)$ of $a(t)$ [i.e., $a^{-1}(t)$].*

Proof We use the well-known Lyapunov method to prove the exponential convergence of ZD model (1.7).

First, starting with ZF (1.3), we define a Lyapunov candidate

$$V(x(t), t) = \frac{1}{2}\left(x(t) - \frac{1}{a(t)}\right)^2 \geqslant 0,$$

where $V(x(t), t) = 0$ for any $x(t) = a^{-1}(t)$, and $V(x(t), t) > 0$ for any $x(t) \neq a^{-1}(t)$. Then, we derive its time derivative as

$$\begin{aligned}\dot{V}(x(t), t) = \frac{\mathrm{d}V(x(t), t)}{\mathrm{d}t} &= \left(x(t) - \frac{1}{a(t)}\right)\left(\dot{x}(t) + \frac{1}{a^2(t)}\dot{a}(t)\right)\\ &= -\gamma\left(x(t) - \frac{1}{a(t)}\right)^2 = -2\gamma V(x(t), t).\end{aligned}$$

Since $V(x(t), t) \geqslant 0$, then $\dot{V}(x(t), t) = -2\gamma V(x(t), t) \leqslant 0$, which guarantees the (final) negative-definiteness of $\dot{V}(x(t), t)$.

Furthermore, from $\dot{V}(x(t), t) = -2\gamma V(x(t), t)$, we have

$$V(x(t), t) = V(x(0), 0)\exp(-2\gamma t).$$

That is,

$$\frac{1}{2}\left(x(t)-\frac{1}{a(t)}\right)^2=\frac{1}{2}\left(x(0)-\frac{1}{a(0)}\right)^2\exp(-2\gamma t).$$

Thus, we have

$$\left|x(t)-\frac{1}{a(t)}\right|=\left|x(0)-\frac{1}{a(0)}\right|\exp(-\gamma t),$$

where symbol $|\cdot|$ denotes the absolute value of a scalar. With $\alpha=|x(0)-1/a(0)|$, the above equation is further rewritten as

$$\left|x(t)-\frac{1}{a(t)}\right|=\alpha\exp(-\gamma t),$$

which means that $x(t)$ exponentially converges to $a^{-1}(t)$ with the convergence rate $\gamma>0$. That is, starting from randomly-generated initial state $x(0)\neq 0\in\mathbb{R}$ which has the same sign as $a(0)$, the neural state $x(t)$ of ZD model (1.7) exponentially converges to the theoretical time-varying reciprocal $x^*(t)=a^{-1}(t)$ of $a(t)$ involved in time-varying Eq. (1.1). The proof is thus complete. □

As for ZD models (1.8)–(1.10), we also have the following convergence results, with the related proofs being generalized from the proof of Proposition 1.1 (and being left to interested readers to complete as a topic of exercise).

Proposition 1.2 *Consider a smoothly time-varying scalar $a(t)\neq 0\in\mathbb{R}$ involved in time-varying reciprocal problem (1.1). Starting from randomly-generated initial state $x(0)\neq 0\in\mathbb{R}$ which has the same sign as $a(0)$, the neural state $x(t)$ of ZD model (1.8) derived from ZF (1.4) converges to the theoretical time-varying reciprocal $x^*(t)$ of $a(t)$ [i.e., $a^{-1}(t)$], with the error defined in ZF (1.4) exponentially convergent to zero.*

Proposition 1.3 *Consider a smoothly time-varying scalar $a(t)\neq 0\in\mathbb{R}$ involved in time-varying reciprocal problem (1.1). Starting from randomly-generated initial state $x(0)\neq 0\in\mathbb{R}$ which has the same sign as $a(0)$, the neural state $x(t)$ of ZD model (1.9) derived from ZF (1.5) converges to the theoretical time-varying reciprocal $x^*(t)$ of $a(t)$ [i.e., $a^{-1}(t)$], with the error defined in ZF (1.5) exponentially convergent to zero.*

Proposition 1.4 *Consider a smoothly time-varying scalar $a(t)\neq 0\in\mathbb{R}$ involved in time-varying reciprocal problem (1.1). Starting from randomly-generated initial state $x(0)\neq 0\in\mathbb{R}$ which has the same sign as $a(0)$, the neural state $x(t)$ of ZD model (1.10) derived from ZF (1.6) converges to the theoretical time-varying reciprocal $x^*(t)$ of $a(t)$ [i.e., $a^{-1}(t)$], with the error defined in ZF (1.6) exponentially convergent to zero.*