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Preface

Vortical flows are flows with *vortices* as their skeleton structures. Vortices are seen everywhere in our universe and on the earth: from spiral galaxies, atmospheric and oceanic circulations to hurricanes and typhoons, tornadoes to bath stub vortices; from volcanoes' erupted smoke rings and mushroom clouds of nuclear explosions to vortex rings ejected from the mouth of dolphin and smoker, or formed in a heart downstream of the mitral valve that separates the left atrium and left ventricle; from tip vortices of aircraft, rotor blade, and turbo fan to complicated ring-like structures in the wake of birds, insects and fishes; from well organized laminar vortices to coherent turbulent structures.

This book provides a systematic introduction to the physical theory of vortical flows at graduate level. It grew from our monograph *Vorticity and Vortex Dynamics* (Springer 2006), but has been thoroughly rewritten. Some advanced topics in the monograph have been removed, and more basic topics have been added. Recent advances since 2006 in the field of fundamental interest are included. Nevertheless, two basic characteristics of the monograph are inherited and further enhanced, which make both the monograph and the present book differ from other existing books on the subject:

(1) We consider the theory of vortical flows as a branch of fluid dynamics focusing on *shearing process* in fluid motion, measured by *vorticity*. A vortex is defined as a fluid body with high vorticity concentration. The evolution of vorticity field is governed by *vorticity dynamics*. Coexisting with this process is the compression–expansion process (*compressing process* for short) measured by dilatation, pressure, or other thermodynamic variables, of which the main structure is shock waves where *entropy process* is naturally involved. The three fundamental processes in fluid motion are coupled with each other both inside the flow field and at solid boundary. We believe that only on the basis of this broad background can the physics of vortical flows be fully understood.

(2) We study vortical flows according to their natural evolution stages, from being generated to dissipated. As preparation, the first three chapters of the book provide background knowledge for entering vortical flows. Due to the coupling of shearing process with other processes, this knowledge appears wider and more

profound than common books on vortical flows. Chapter 1 reviews standard fundamental kinematics and dynamics of generic viscous and compressible flow, including some elementary results of process identification and decomposition. The whole of Chap. 2 is devoted to the basic theory of fundamental processes in fluid motion, their splitting and coupling. Chapter 3 discusses general theory and physics of vorticity dynamics. Although later chapters will be mainly confined to incompressible flow, Chaps. 1–3 cover much broader materials with a hope to facilitate future exploration of more complicated compressible vortical flows.

The rest of the book deals with vortices and vortical flows. Of various vortices the primary form is *layer-like vortices* or *shear layers*, and secondary but stronger form is *axial vortices* mainly formed by the rolling up of shear layers. Thus, Chap. 4 is on attached shear layer (namely *boundary layer*) and free shear layers. As Reynolds number approaches infinity, these layers become asymptotically attached and free vortex sheets, which are the subject of Chap. 5. This chapter ends with vortex-sheet rolling up and initial formation of axial vortices, so it is naturally followed by Chaps. 6 and 7 on typical solutions of columnar vortices and vortex rings, respectively. Chapter 8 studies flow separation first, which is a key localized dynamic process turning a simple attached flow to complex, namely to become global separated flow with concentrated vortices that is studied next. Chapter 9 is an introduction to total force and moment acting to a body moving through the fluid, in terms of various vortical structures.

Chapters 10 and 11 discuss the instability and breakdown of axial vortices, and vortical structures in transitional and turbulent shear flows, respectively. Both chapters require some elementary knowledge of flow instability and turbulence, which are placed (somewhat artificially) in the beginnings of Chaps. 10 and 11, respectively. Finally, A general theory of vector and tensor field is presented in the Appendix for readers' convenience.

Problems are given at the end of each chapter and Appendix, some for helping to understand the basic theories, and some involving specific applications; but the emphasis of both is always on physical thinking. Problems with asterisk may need more effort.

The reader of this book is assumed to have learned undergraduate fluid mechanics or aerodynamics in majors of mechanics, aerospace and mechanical engineering, and be familiar with physics, advanced calculus and differential equations. Better background of these fields will make it easier to understand the present book. Most part of the book materials has been used as Lecture Notes and were taught by J.Z. at Peking University over the past 15 years as a one-semester graduate course of advanced fluid dynamics. The course has been proved acceptable by most students with warm and inspiring feedback.

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The writing of the book has been proceeded through intensive interactions with many of our former and current graduate students. Prof. Hui Zhao, Drs. Yan-Tao Yang, Ri-Kui Zhang, Li-Jun Xuan and Feng Mao, as well as Mrs. Jin-Yang Zhu, Luo-Qin Liu, Shu-Fan Zou, An-Kang Gao, and Lin-Lin Kang have provided extensive support in reviewing the drafts, designing and answering problems, and editing texts. Ms. Feng-Rong Zhu has provided excellent support to figure drawing. We owe a lot to all of them.

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Contents

Chapter 1
Fundamentals of Fluid Dynamics

1.1 Basic Fluid Kinematics

1.1.1 Description and Visualization of Fluid Motion

Fluid dynamics studies the motion of continuous media with fluidity. A fluid then has a dual feature and can be described in two ways. On the one hand, the fluid consists of continuously distributed *material elements* or *particles*, each of which retains its identity all the time so that one can trace the fluid motion and evolution by tracking each element. We may define the identity or "label" of a fluid particle by a known particle's location $\boldsymbol{x}_0$ at $t = t_0$, say $\boldsymbol{x}_0 = \boldsymbol{X}$. As it moves smoothly as t increases, its new position $\boldsymbol{x}$ at $t > t_0$ is a differentiable function of $\boldsymbol{X}$ and t:

$$\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{X}, t). \tag{1.1.1a}$$

Note that as the label of a fixed particle, $\boldsymbol{X}$ does not change with time; namely there must be $\partial \boldsymbol{X}/\partial t = \boldsymbol{0}$. Inversely, we may write

$$\boldsymbol{X} = \boldsymbol{F}(\boldsymbol{x}, t). \tag{1.1.1b}$$

Initially separated particles cannot merge to a single particle at later time, even though they could be tightly squeezed together. Meanwhile, a single particle cannot be split into two or more particles at later time. This implies that the mapping between $\boldsymbol{x}$ and $\boldsymbol{X}$ is one-to-one. Their transformation Jacobians are never zero nor infinity. Figure 1.1 sketches the mapping of a material fluid body from the $\boldsymbol{X}$-space (a reference space) to the $\boldsymbol{x}$-space (the physical space) and its reverse.

Now, following the movement of a particle $\boldsymbol{X}$, its position $\boldsymbol{x}$ becomes a function of t, so that the particle has a velocity

$$\boldsymbol{u} = \frac{\partial \boldsymbol{x}}{\partial t}(\boldsymbol{X}, t) \quad \text{or} \quad u_i = \frac{\partial x_i}{\partial t}, \quad i = 1, 2, 3, \tag{1.1.2}$$

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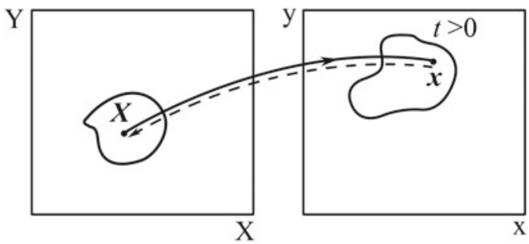

Fig. 1.1 One-to-one mapping between the $\boldsymbol{X}$-space and $\boldsymbol{x}$-space for a material fluid body

with X_i being parameters. This way of description, known as *particle or Lagrangian description*, is a direct extension of Newton's particle kinematics.

On the other hand, fluid motion can be treated by a *field theory* where, like in an electromagnetic field, the spatial position $\boldsymbol{x}$ and time t are independent variables. The fields of velocity $\boldsymbol{u}$, pressure p, and other derived physical quantities are all functions of $(\boldsymbol{x}, t)$ and will be assumed sufficiently smooth except on certain surfaces of discontinuity. If the fluid is unbounded, except otherwise stated it is assumed to be at rest at infinity or, by a Galilean transformation, have uniform motion. If the flow has a boundary, the boundary is assumed to be piecewise smooth. The collection of these fields constitutes a *flow field*. This way of description is known as *field or Eulerian description*. The two descriptions enrich each other.

To see this duality of material and field in fluid motion, consider an elementary manifestation of the fluidity: the velocity difference at two neighboring points $\boldsymbol{x}_0$ and $\boldsymbol{x} = \boldsymbol{x}_0 + \delta\boldsymbol{x}$ with $|\delta\boldsymbol{x}| = \delta r \to 0$. In the field description, a use of Taylor expansion gives

$$\delta\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}) - \boldsymbol{u}(\boldsymbol{x}_0) = \delta\boldsymbol{x} \cdot \nabla\boldsymbol{u}(\boldsymbol{x}_0) + O(\delta r^2). \tag{1.1.3}$$

In contrast, in the material description, after δt the particles at the two ends of $\delta\boldsymbol{x}$, $\boldsymbol{x}_0$ and $\boldsymbol{x}$, will move to

$$\begin{aligned} \boldsymbol{x}_0 &\to \boldsymbol{x}_0 + \boldsymbol{u}(\boldsymbol{x}_0)\delta t, \\ \boldsymbol{x} &\to \boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x})\delta t = \boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x}_0)\delta t + \delta\boldsymbol{x} \cdot \nabla\boldsymbol{u}(\boldsymbol{x}_0)\delta t + O(\delta r^2). \end{aligned}$$

Hence, there is

$$\delta\boldsymbol{x}(\boldsymbol{x}, t + \delta t) = \delta\boldsymbol{x}(\boldsymbol{x}, t) + \delta\boldsymbol{u}(\boldsymbol{x}, t)\delta t, \quad \delta\boldsymbol{u}(\boldsymbol{x}, t) = \delta\boldsymbol{x} \cdot \nabla\boldsymbol{u}(\boldsymbol{x}_0, t),$$

so that

$$\frac{d\delta\boldsymbol{x}}{dt} = \delta\boldsymbol{u} = \delta\boldsymbol{x} \cdot \nabla\boldsymbol{u} \equiv \frac{D\delta\boldsymbol{x}}{Dt}, \tag{1.1.4}$$

which is the same as (1.1.3) but enriches the latter's implication by comparing the velocities at neighboring points and identifying $\delta\boldsymbol{u}$ as the rate of change of a *material* *line element* $\delta \boldsymbol{x}$. The operator D/Dt is used to emphasize that the derivative is taken by following the particle, and is called *material derivative*.

In general, the field description and material description are not fully equivalent. The former does not care which specific particle is moving through a field point $\boldsymbol{x}$ at time t, but the latter does and has to ensure the identification of all infinitely many particles at all time. In other words, the two descriptions will become fully equivalent only if to the field description we add a vectorial constraint $\partial \boldsymbol{X}/\partial t = \boldsymbol{0}$ or

$$\frac{D\boldsymbol{X}}{Dt} = \boldsymbol{0}. \tag{1.1.5}$$

For most of fluid dynamics problems it suffices to stay on the simpler field description without constraint (1.1.5), as we do in the major portion of the book.[1] But, as will be seen in Sect. 1.2, in developing the formulation of fluid dynamics in terms of the field description, tracking material fluid elements is still necessary because the link between the flow field evolution and its causes, i.e., the forces acting on the fluid, is provided by the Newton mechanics which is formulated for material fluid body and particles.

It is appropriate here to introduce four different types of lines in a flow field, defined based on the above two descriptions. First, a curve tangent to the velocity $\boldsymbol{u}(\boldsymbol{x}, t)$ everywhere at a time t is a *streamline* at this time. Let it be represented as $\boldsymbol{x}(s)$ in terms of parameter s. Then its equation follows from eliminating dt in the component form of (1.1.2):

$$\frac{d\boldsymbol{x}}{ds} = \boldsymbol{u}(\boldsymbol{x}, t) \quad \text{or} \quad \frac{dx_1}{u_1(\boldsymbol{x}, t)} = \frac{dx_2}{u_2(\boldsymbol{x}, t)} = \frac{dx_3}{u_3(\boldsymbol{x}, t)}, \tag{1.1.6}$$

of which the solution passing a given $\boldsymbol{x}(s_0)$ is the required streamline. The concept of streamlines does not distinguish different particles and belongs to the field description. In an experiment, if we spread the tracer particles in the flow and take a photo with *very short time exposure*, then we see a set of short line segments, of which a smooth connection can represent a family of instantaneous streamlines.

Next, a *particle-path line* or *pathline* is the curve created by the motion of a particle $\boldsymbol{X}$ as time goes on, which can be obtained by solving the ordinary differential equation (1.1.2) in time:

$$\frac{\partial \boldsymbol{x}}{\partial t} = \boldsymbol{u}(\boldsymbol{x}, t), \quad \boldsymbol{x}(\boldsymbol{X}, t_0) = \boldsymbol{X}. \tag{1.1.7}$$

The concept of pathlines belongs to the material description. In flow visualization, if we introduce a tracer particle into the fluid and photograph its motion by a *long time exposure*, we obtain a pathline.

[1] Later in Chap. 3 we shall see that for a special class of flows the two descriptions become equivalent, and the constraint (1.1.5) can be dropped.

To visualize a flow, instead of taking a long-time exposure to trace a single particle, it is more informative to introduce some dyed fluid continuously at a fixed point $\boldsymbol{x}_0$ and take a *snapshot* at a later time t_0. The photo shows a curve consisting of the spatial positions at t_0 of *all* fluid particles which have passed $\boldsymbol{x}_0$ at any time $\tau \leq t_0$ and continue to move ahead, where $\tau \in (-\infty, t_0]$ is a parameter for identifying different particles. Such a curve is called a *streakline*, the third type of lines. Mathematically, the positions of these particles at t_0 can be obtained by applying (1.1.7) not to a single particle but all particles $\boldsymbol{X}(\tau)$:

$$\frac{\partial \boldsymbol{x}}{\partial t} = \boldsymbol{u}(\boldsymbol{x}, t), \quad \boldsymbol{x}(\boldsymbol{X}(\tau), \tau) = \boldsymbol{x}_0, \quad -\infty < \tau \leq t_0. \tag{1.1.8}$$

Changing t_0 continuously will yield an animation of the streak-line evolution.

While a streakline involves a time-sequence of particles $\boldsymbol{X}(\tau)$ passing a single $\boldsymbol{x}_0$, the visualization can be extended to releasing dyed particles from different points of a line $\boldsymbol{X}(s)$, say, and then take a snapshot. Furthermore, one may insert a thin metal wire $\boldsymbol{X}(s)$ in a moving fluid (e.g. water) and introduce a pulsating current of frequency f through it. The wire will electrolyze the water and release hydrogen bubbles on-and-off at discrete time $\tau_n = nT = n/f$ with $n = 0, 1, 2, \ldots,$ which can be illuminated. Thus, each current-on action produces a bright-dark strip or column of short pathlines of the hydrogen bubbles. The strips are initially parallel to the wire but then advected by local flow velocity, and hence exhibit approximately the local velocity profiles including vortical structures. These pulsating strips and short bubble traces therein are called *time lines*, the fourth type of lines. Photos or animations of the velocity profiles may clearly exhibit the flow pattern and its evolution. They are defined by $\boldsymbol{x}_n(\boldsymbol{X}(s, \tau_n), t)$ that are governed by

$$\frac{\partial \boldsymbol{x}_n}{\partial t} = \boldsymbol{u}(\boldsymbol{x}_n, t), \quad \boldsymbol{x}_n(\boldsymbol{X}(s, \tau_n), \tau_n) = \boldsymbol{X}(s). \tag{1.1.9}$$

As a simple example for the behavior of these lines, consider a two-dimensional (2D) and incompressible unsteady velocity field $\boldsymbol{u} = (u, v)$ in the (x, y)-plane:

$$u = -x, \quad v = y + t.$$

Then by (1.1.6)–(1.1.8), the parametric expressions for the streamline, pathline, and streakline are found to be:

$$\begin{aligned}
&\text{Streamline}: \quad x = x_0 e^{-(s-s_0)}, \quad y = (y_0 + t)e^{(s-s_0)} - t,\\
&\text{pathline}: \quad x = x_0 e^{-(t-t_0)}, \quad y = (y_0 + t_0 + 1)e^{t-t_0} - t - 1,\\
&\text{streakline}: \quad x = x_0 e^{-(t-\tau)}, \quad y = (y_0 + \tau + 1)e^{t-\tau} - t - 1, \quad \tau \in (-\infty, t_0).
\end{aligned}$$

It is easily verified that at a given t_0, the streamline passing $\boldsymbol{x}_0$, the pathline of a particle locating at $\boldsymbol{x}_0$, and the streakline passing $\boldsymbol{x}_0$ have a common tangent vector at $\boldsymbol{x}_0$. Thus, over a very short time interval streaklines, pathlines, and instantaneous

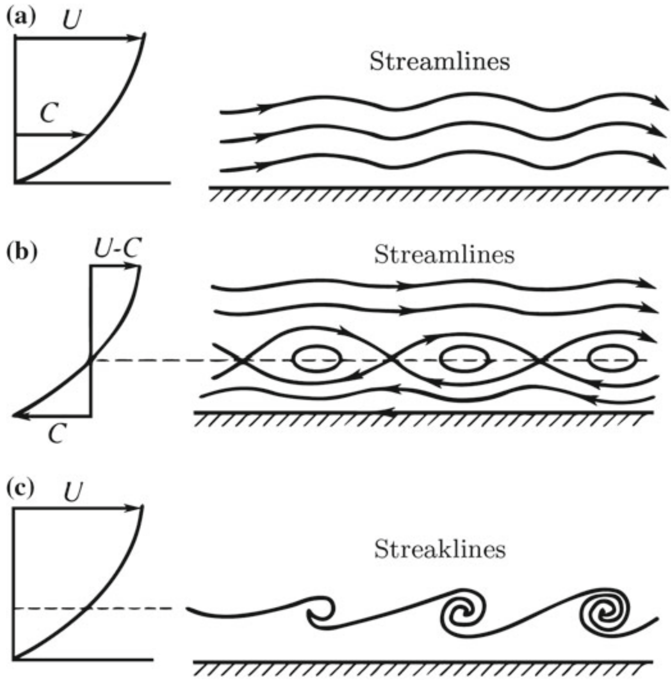

Fig. 1.2 Schematic streamlines (viewed in different frames) and streaklines in a boundary layer with travelling instability waves. C is the wave speed. Reproduced from Taneda (1985)

streamlines are identical. When the flow is steady, i.e., in (1.1.2) $\boldsymbol{u}$ is independent of t, the three curves coincide at any time. For more general unsteady flows, however, the three curves are entirely different. A pathline or a streakline can intersect itself, but a streamline cannot. The behavior of streamlines and pathlines vary drastically as the observer changes from a fixed frame of reference to a moving one, but the streaklines will remain the same.

In the past, most experimental visualizations of vortical flows exhibited streaklines, and most numerical visualizations plot exhibited streamlines. It has now been realized that observing more kinds of lines can lead to clearer understanding of a complex flow field, but their interpretation needs great care since vortical flows are inherently more or less unsteady. Figure 1.2 sketches the unsteady streamlines and streaklines due to the instability travelling waves in a flat-plate boundary layer, viewed from different frames of reference. Note that the streamlines in the frame moving with the wave exhibit some vortex-like structure (so-called "cat-eyes"), but whether or not these cat-eyes can be classified as vortices should be judged by the degree of concentration of the vorticity rather than merely by the frame-dependent streamlines.

Figure 1.3 shows both streamlines and streaklines due to the unsteady vortex shedding from a circular cylinder, where their difference is obvious. However, while

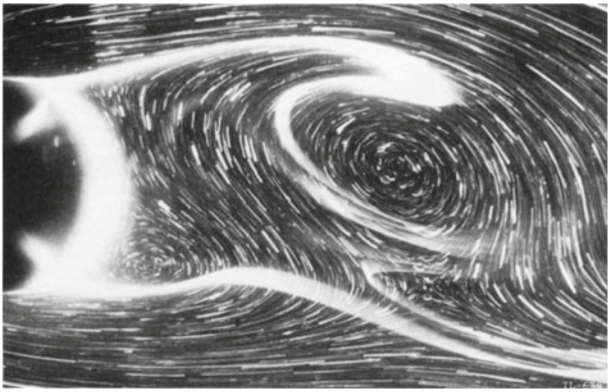

Fig. 1.3 Streamlines and streaklines in unsteady vortex shedding from a *circular cylinder*. From Taneda (1985)

streaklines can tell where the vorticity resides in a flow, it tells very little about the surrounding fluid and the entrainment process. In this regard instantaneous streamlines in an unsteady flow are still useful. A correct understanding of some highly time-dependent vortical flows, of which the evolution is essentially a material process, should use jointly all three kinds of lines to avoid misinterpretation that could happen if only streaklines are visualized (cf. Kurosaka and Sundaram 1986). Meanwhile, Lagrangian formulation may be of crucial importance. These issues will be further addressed in Sect. 8.5 when we study *unsteady flow separation*. For example, streamlines, pathlines, and streak lines of the same unsteady flow are simultaneously displayed in Fig. 8.28, along with a discussion of their respective roles.

Figure 1.4 is a time-line photo of the flow over a circular cylinder. The metal wire is a vertical straight line at the far left of the photo.

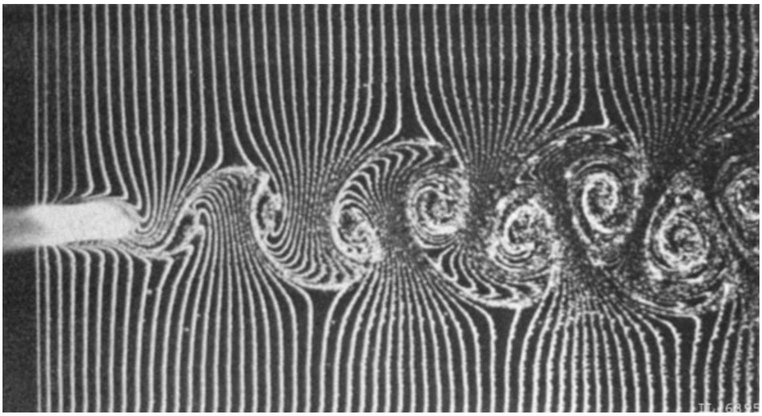

Fig. 1.4 Time lines behind a *circular cylinder* at Reynolds number 152. From Taneda (1985)

1.1.2 Dilatation and Vorticity

While the velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ discussed in the preceding subsection is the most primary vector field in any flows, our major concern is various *flow structures* like those seen in Figs. 1.3 and 1.4, which are highly localized and occupy only a very small portion of the flow domain, but play a role as the "organizers" of the entire flow. Structures come from the *variation* of the velocity in space and time rather than the velocity itself. To understand these structures, we have to consider the spatial derivatives of the velocity field and their temporal evolution.

In a flow domain, the structures are locally characterized by various products of the gradient operator ∇ with $\boldsymbol{u}$, because the vector ∇ measures both the direction along which the variation is steepest and the magnitude of variation in that direction per unit length. The mathematical foundation of our study is the general vector-field theory, of which a systematic introduction is given in Appendix,[2] of which the results will be simply cited in the main text.

The most primary derived fields that describe the local spatial variation of a velocity field $\boldsymbol{u}$ are its divergence, a scalar field called *dilatation*, and its curl, an axial vector called *vorticity*:

$$\vartheta \equiv \nabla \cdot \boldsymbol{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}, \tag{1.1.12a}$$

$$\boldsymbol{\omega} \equiv \nabla \times \boldsymbol{u} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right). \tag{1.1.12b}$$

Intuitively, ϑ measures the isotropic expansion or compression of the fluid, while $\boldsymbol{\omega}$ measures the rotation of fluid particles. Their physical meanings can be more clearly understood by considering their volume integrals. By the generalized Gauss theorem (A.2.8), we obtain

$$\vartheta = \lim_{V \to 0} \frac{1}{V} \int_{\partial V} \boldsymbol{n} \cdot \boldsymbol{u} dS, \tag{1.1.13a}$$

$$\boldsymbol{\omega} = \lim_{V \to 0} \frac{1}{V} \int_{\partial V} \boldsymbol{n} \times \boldsymbol{u} dS, \tag{1.1.13b}$$

indicating that their net contributions are solely from the *normal* and *tangential* components of $\boldsymbol{u}$ on the boundary, respectively, as sketched in Fig. 1.5 with V being a small sphere. Clearly, the dilatation represents a net isotropic *outflow* where only the velocity normal to sphere's surface plays a role. This is an outcome of the compressing process. In contrast, the vorticity represents a non-isotropic "*curl up*" property where only the velocity tangent to sphere's surface is involved. This is an outcome of the shearing process. Note that the velocity curling-up associated with vorticity can be

[2] This theory is also applicable to all other vector fields to be encountered in our study. Before moving on, the reader is strongly recommended to get a full familiarity of the materials in Appendix as a necessary preparation.

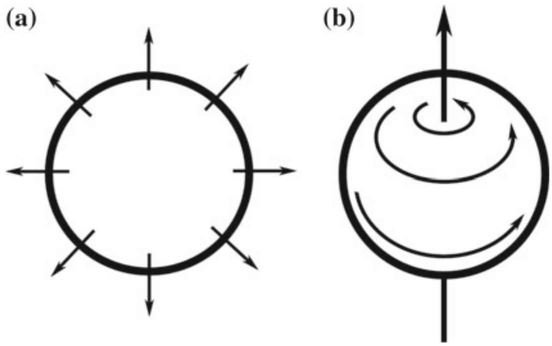

Fig. 1.5 The velocities associated with dilatation (**a**) and vorticity (**b**) in a small sphere

further revealed by the *circulation* of $\boldsymbol{u}$ along a closed loop C, which by the Stokes theorem gives

$$\oint_C \boldsymbol{u} \cdot d\boldsymbol{x} = \int_S \boldsymbol{\omega} \cdot \boldsymbol{n} dS, \tag{1.1.14}$$

where S is any directional surface spanned by C. A further taste of the behavior of dilatation and vorticity can be felt by considering a simple plane-wave fluctuating velocity field $\boldsymbol{u} = \hat{\boldsymbol{u}}(t)e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-nt)}$, where $\boldsymbol{k} = k\boldsymbol{e}_k$ is the wave vector with $k = 2\pi/\lambda$ being the wave number of $\boldsymbol{u}$ and $\boldsymbol{e}_k$ the unit vector in the wave propagation direction, $n = 2\pi f$ is the circular frequency, and $\hat{\boldsymbol{u}}$ is a uniform amplitude. For this wave the differentiation operation by ∇ is reduced to the multiplication by $i\boldsymbol{k}$, so that

$$\nabla \cdot \boldsymbol{u} = i\boldsymbol{k} \cdot \boldsymbol{u}, \quad \nabla \times \boldsymbol{u} = i\boldsymbol{k} \times \boldsymbol{u}. \tag{1.1.15}$$

Thus, the ϑ-wave is a *longitudinal* wave propagating along the velocity direction, while the $\boldsymbol{\omega}$-wave is a *transverse* wave propagating perpendicular to the velocity direction. This is in consistency with (1.1.13), with $\boldsymbol{n}$ there being analogous to $i\boldsymbol{k}$ here. Note that in an arbitrary bounded domain the geometric relations between wave oscillating and propagating directions may not be as simple as that given by (1.1.15). In general, we may split the velocity field into two parts, $\boldsymbol{u} = \boldsymbol{U}(\boldsymbol{x}) + \boldsymbol{u}'(\boldsymbol{x}, t)$, where $\boldsymbol{U}(\boldsymbol{x})$ is an arbitrary steady basic flow which can have both divergence and curl, and $\boldsymbol{u}'(\boldsymbol{x}, t)$ describes an unsteady velocity wave. Then, if $\nabla \times \boldsymbol{u}' = \boldsymbol{\omega}' = \boldsymbol{0}$, we say the wave is longitudinal; while if $\nabla \cdot \boldsymbol{u}' = \vartheta' = 0$, we say it is transverse. Later we shall see that the two waves are qualitatively different; their propagation speeds are determined by different dynamic mechanisms and have different values. Physics fields that have these wave behaviors are usually called *longitudinal field* and *transverse field*, respectively (cf. Morse and Feshbach 1953); we now see that in fluid motion the longitudinal and transverse fields are specified to compressing and shearing fields.[3]

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[3]In this book the two pair of names will be used alternatively.

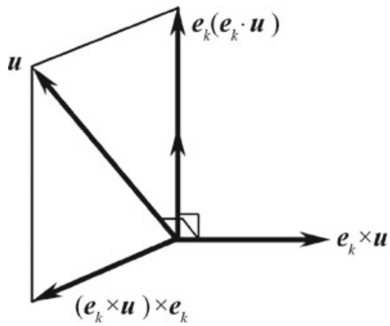

Fig. 1.6 Geometric orthogonal decomposition of $\boldsymbol{u}$ with respect to wave vector $\boldsymbol{k}$

If a $\boldsymbol{u}$-fluctuation emits simultaneously both longitudinal and transverse waves, the fluctuation $\boldsymbol{u}$-field as a whole must be a composite wave. For the above plane wave the composition is simple, since (ignoring factor i) the two waves in (1.1.15) are just orthogonal. Indeed, since $\boldsymbol{k} = \boldsymbol{e}_k k$, $\boldsymbol{e}_k(\boldsymbol{e}_k \cdot \boldsymbol{u})$ is a component vector of $\boldsymbol{u}$ along $\boldsymbol{k}$; while $\boldsymbol{e}_k \times \boldsymbol{u}$ differs from the perpendicular components of $\boldsymbol{u}$ (which is $\boldsymbol{u} - \boldsymbol{e}_k(\boldsymbol{e}_k \cdot \boldsymbol{u})$) by a 90° on the plane normal to $\boldsymbol{k}$. The former can be turned back to the latter simply by its one more cross product with $\boldsymbol{e}_k$, see Fig. 1.6. Therefore, we construct the composite $\boldsymbol{u}$ as

$$|\boldsymbol{k}|^2\boldsymbol{u} = \boldsymbol{k}(\boldsymbol{k} \cdot \boldsymbol{u}) - \boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{u}). \tag{1.1.16}$$

Then, recalling the rule $\nabla \to i\boldsymbol{k}$, from (1.1.16) we arrive at a differential identity

$$\nabla^2\boldsymbol{u} = \nabla(\nabla \cdot \boldsymbol{u}) - \nabla \times (\nabla \times \boldsymbol{u}) = \nabla\vartheta - \nabla \times \boldsymbol{\omega}, \tag{1.1.17}$$

which clearly reveals the functional relation between $\boldsymbol{u}$, ϑ, and $\boldsymbol{\omega}$. The above elementary observations suggest that the dilatation and vorticity fields are measures of two physically distinct processes, which are mutually independent of and complementary to each other in many aspects.

1.1.3 Velocity Gradient and Its Decompositions

We now move to the next product of the gradient operator ∇ and $\boldsymbol{u}$: the dyad $\nabla\boldsymbol{u}$ known as the *velocity gradient tensor*. Once again, our interest is the intrinsic decompositions of $\nabla\boldsymbol{u}$.[4]

Like any matrix, the second-order tensor $\nabla\boldsymbol{u}$ can be split into a symmetric part and an antisymmetric part:

[4]Whenever written in component form, throughout this book we use the convention that the (i, j)th component of $\nabla\boldsymbol{u}$ is $\nabla_i u_j = \partial_i u_j = u_{j,i}$.

$$\nabla \boldsymbol{u} = \mathbf{D} + \boldsymbol{\Omega}, \tag{1.1.18}$$

where, with the superscript T denoting transpose,

$$\mathbf{D} = \frac{1}{2}[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T] \tag{1.1.19a}$$

$$\boldsymbol{\Omega} = \frac{1}{2}[\nabla \boldsymbol{u} - (\nabla \boldsymbol{u})^T] \tag{1.1.19b}$$

are the symmetric *strain-rate tensor* and antisymmetric *vorticity tensor* (or *spin tensor*), respectively. These tensors are associated with dilatation and vorticity via the following relations:

$$\vartheta = \nabla \cdot \boldsymbol{u} = D_{ii}, \tag{1.1.20a}$$

$$\omega_i = \epsilon_{ijk}\Omega_{jk}, \quad \Omega_{jk} = \frac{1}{2}\epsilon_{ijk}\omega_i, \tag{1.1.20b}$$

where ϵ_{ijk} are the components of the permutation tensor (Appendix A.1.4). Thus, $\boldsymbol{\omega}$ is equivalent to $\boldsymbol{\Omega}$. Note that for any vector $\boldsymbol{b}$ there is identity

$$2\boldsymbol{b} \cdot \boldsymbol{\Omega} = \boldsymbol{\omega} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{\omega}, \tag{1.1.21a}$$

so there also is

$$2\nabla \cdot \boldsymbol{\Omega} = -\nabla \times \boldsymbol{\omega}. \tag{1.1.21b}$$

From (1.1.20) we see a remarkable difference of the two fundamental processes measured by ϑ and $\boldsymbol{\omega}$ or $\boldsymbol{\Omega}$: *the compressing process can exist in any of one-, two-, or three-dimensional flow, but the shearing process does not appear in one-dimensional flow at all, appears merely in part in two-dimensional flow like a scalar (its direction is always perpendicular to the flow plane), and exhibits its full behavior in and only in three-dimensional flow*. Moreover, as a divergence-free vector, $\boldsymbol{\omega}$ has just two independent components in three dimensions. This geometric difference of the two processes is transformed to a more intuitive difference in their integrals (1.1.13), where the value of ϑ and the two independent components of $\boldsymbol{\omega}$ produce the normal component and two tangential components of the velocity at boundary, respectively. Similar contrast appears in (1.1.15) as well, where the normal vector $\boldsymbol{n}$ is replaced by the wave vector $\boldsymbol{k}$.

As a very basic application of (1.1.18), let us revisit the velocity difference of two neighboring points $\boldsymbol{x}$ and $\boldsymbol{x}_0$ given by (1.1.4), which can now be decomposed to

$$\delta \boldsymbol{u} = \frac{D\delta \boldsymbol{x}}{Dt} = \delta \boldsymbol{x} \cdot \mathbf{D}_0 + \frac{1}{2}\boldsymbol{\omega}_0 \times \delta \boldsymbol{x}, \tag{1.1.22}$$

where $\delta \boldsymbol{x} = \boldsymbol{x} - \boldsymbol{x}_0$ is a material line element and suffix 0 denotes taking values at fixed $\boldsymbol{x}_0$. Thus, like rigid-body rotation, the role of $\boldsymbol{\omega}_0$ is to rotate the line element $\delta \boldsymbol{x}$ around $\boldsymbol{x}_0$ with angular velocity $\boldsymbol{\omega}_0/2$. Namely, *the vorticity can be understood as twice of the angular velocity of a fluid element*.

In contrast, the first term of (1.1.22) is unique to deformable body and deserves a detailed examination. Let $\delta s = |\delta \boldsymbol{x}|$, the inner product of (1.1.22) and $\delta \boldsymbol{x}$ yields[5]

$$\frac{1}{2}\frac{D}{Dt}(\delta s^2) = \delta \boldsymbol{x} \cdot \mathbf{D}_0 \cdot \delta \boldsymbol{x} \equiv 2\phi, \tag{1.1.23}$$

and hence the relative velocity solely "induced" by pure straining,

$$\delta \boldsymbol{u}_\phi \equiv \delta \boldsymbol{x} \cdot \mathbf{D}_0 = \nabla \phi, \tag{1.1.24}$$

is *irrotational* or has a *potential*. It is along the *normal* to surfaces $\phi =$ constant, which are *quadric center surfaces* (ellipsoid, hyperboloid, and paraboloid, etc. centered at $\boldsymbol{x}_0$), known as the *tensor surfaces* of $\mathbf{D}$ (Appendix A.1.3).

How the specific elements of D_{ij} affect $\delta \boldsymbol{u}_\phi$ can be further clarified.[6] First, consider the diagonal components of D_{ij} with $i = j$. Since

$$\phi_{,ii} = \delta x_{j,i} D_{0ji} = D_{0ii} = \vartheta_0,$$

the trace of $\mathbf{D}$ simply represents an isotropic expansion/compression, which vanishes if the flow is incompressible. Then, similar to the derivation of (1.1.23), the rate of change of δs reads

$$\frac{1}{\delta s}\frac{D}{Dt}(\delta s) = D_{0ij}\frac{dx_i}{ds}\frac{dx_j}{ds}. \tag{1.1.25}$$

In particular, if $\delta \boldsymbol{x}$ is along the x_1-axis, say, such that $dx_i/ds = \delta_{i1}$ and $dx_j/ds = \delta_{j1}$, (1.1.25) implies

$$\frac{1}{\delta s}\frac{D}{Dt}(\delta s) = D_{0ij}\delta_{i1}\delta_{j1} = D_{011}. \tag{1.1.26}$$

Thus, the diagonal elements of D_{ij} are responsible for the relative stretching rate of a line element parallel to x_i-axis. They are called *normal components of the strain-rate*.

---

[5] Since only material line element $\delta \boldsymbol{x}$ is involved, operator D/Dt can be replaced by d/dt, see (1.1.4).

[6] See, e.g. Aris (1962), Zhuang et al. (2009), and Panton (2013).

Next, for understanding the off-diagonal elements of D_{ij}, consider two material line elements $\delta\boldsymbol{x}$ and $\delta\boldsymbol{x}'$ initiated from $\boldsymbol{x}_0$ with angle θ, such that $\delta\boldsymbol{x}\cdot\delta\boldsymbol{x}' = \delta s\delta s' \cos\theta$. Then one finds

$$2D_{0ij}\frac{dx_i}{ds}\frac{dx'_j}{ds'} = \cos\theta\left[\frac{1}{\delta s}\frac{d}{dt}(\delta s) + \frac{1}{\delta s'}\frac{d}{dt}(\delta s')\right] - \sin\theta\frac{d\theta}{dt}. \tag{1.1.27}$$

In particular, when $\delta\boldsymbol{x}$ and $\delta\boldsymbol{x}'$ are along the x_1- and x_2-axes, respectively, we simply have

$$D_{012} = -\frac{1}{2}\frac{d\theta}{dt}. \tag{1.1.28}$$

Namely, the off-diagonal elements of D_{ij} measure half of the decrease rate of change of the angle of two material line elements originally along the ith and jth axes, respectively. These off-diagonal elements of D_{ij} are called *shearing components of the strain-rate*.

The strain-rate tensor **D** has three real principal values. **D** has a *principal-axis coordinate system* in which with $D_{ij} = 0$ for $i \neq j$ (Appendix A.1.3). Then the above results show that each instantaneous *principal axis* keeps straight. It experiences a stretching (shrinking) along its direction and rotating with angular velocity $\boldsymbol{\omega}/2$, but with no tilting.

In summary, the velocity at $\boldsymbol{x} = \boldsymbol{x}_0 + \delta\boldsymbol{x}$ is

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}_0 + \nabla\phi + \frac{1}{2}\boldsymbol{\omega}_0 \times \delta\boldsymbol{x}. \tag{1.1.29}$$

This result is known as the *Cauchy-Stokes theorem* or **fundamental theorem of deformation kinematics** (Truesdell 1954): *The instantaneous state of a fluid motion at every point is the superposition of a uniform translation, an irrotational stretching or shrinking along three orthogonal principal axes, and a rigid rotation around an axis*. Figure 1.7 illustrates this theorem schematically.

Example: Simple shear flow. Consider a unidirectional shear flow on the (x, y)-plane, $\boldsymbol{u} = (ky, 0, 0)$ with constant shear rate k. Its velocity-gradient tensor and strain-rate tensor are

$$\{u_{j,i}\} = \begin{pmatrix} 0 & 0 & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \{D_{ij}\} = \begin{pmatrix} 0 & k/2 & 0 \\ k/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

respectively, while by (1.1.12) there is $\vartheta = 0$ and $\boldsymbol{\omega} = (0, 0, -k)$. This is a simplified prototype of boundary layer and free shear layer to be addressed in Chap. 4, in which both the strain rate and vorticity are very strong. Here, D_{ij} has principal values $\pm k/2$ and can be reduced to diagonal form by rotating the axes counter-clockwise through an angle $\pi/4$. This strain rate represents a uniform elongation in one principle direction and a uniform foreshortening in the second one at right angle to the first,

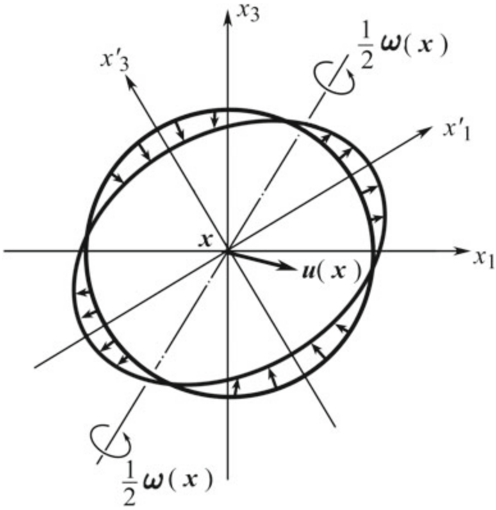

Fig. 1.7 The deformation of small fluid sphere. Only the pattern on the (x_1, x_3) plane is shown

so that the solenoidality condition $\vartheta = 0$ is ensured. Thus, a small sphere of radius ϵ will deform to an ellipsoid during a time interval dt, with the lengths of its semi-axes being

$$\epsilon(1 + kdt/2), \ \epsilon(1 - kdt/2), \ \epsilon.$$

However, to maintain the simple shear motion, the simultaneous rotation around the z axis with angular velocity $\omega/2 = -k/2$ just produces a clockwise turning of the principal axes back to their original directions. For example, after a dt time the rotation makes the actual angle between the first principal axis and the x-axis become $\pi/4 - kdt/2$. The situation is shown in Fig. 1.8, which shows that *in order to maintain the shear flow pattern it is essential for the strain rate to be accompanied by the vorticity*.

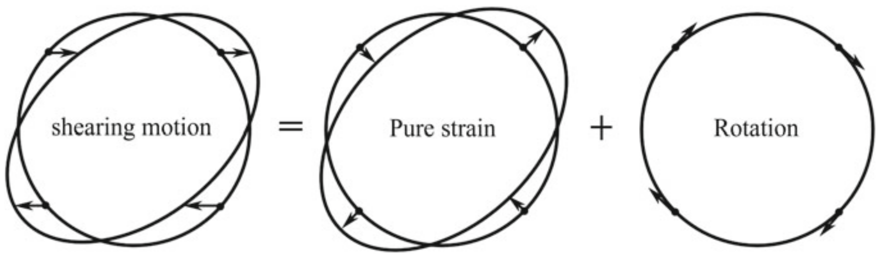

Fig. 1.8 Decomposition of a simple shear motion into a pure strain and a rotation. From Lighthill (1986)

Having seen the double decomposition (1.1.18) of $\nabla \boldsymbol{u}$, we recall that its divergence is the familiar $\nabla^2 \boldsymbol{u}$ that has decomposition (1.1.17), or

$$\nabla \cdot \nabla \boldsymbol{u} = \nabla \cdot (\vartheta \mathbf{I} + 2\boldsymbol{\Omega}),$$

indicating that while $\nabla \boldsymbol{u}$ has nine independent components in a three-dimensional space, its divergence has only three. This fact suggests that from $\nabla \boldsymbol{u}$ one may separate a divergence-free tensor to let the remaining part be solely expressed by ϑ and $\boldsymbol{\omega}$. Indeed, since $\mathbf{D} = \mathbf{D}^T$ and $\boldsymbol{\Omega} = -\boldsymbol{\Omega}^T$, we may write

$$(\nabla \boldsymbol{u})^T = \mathbf{D} - \boldsymbol{\Omega} + \vartheta \mathbf{I} - \vartheta \mathbf{I},$$

so that the strain-rate tensor and, by (1.1.18) the velocity gradient tensor, have *intrinsic triple decompositions*

$$\mathbf{D} = \vartheta \mathbf{I} + \boldsymbol{\Omega} - \mathbf{B}, \tag{1.1.30a}$$

$$\nabla \boldsymbol{u} = \vartheta \mathbf{I} + 2\boldsymbol{\Omega} - \mathbf{B}, \tag{1.1.30b}$$

where

$$\mathbf{B} \equiv \vartheta \mathbf{I} - (\nabla \boldsymbol{u})^T \quad \text{with} \quad \nabla \cdot \mathbf{B} = \mathbf{0} \tag{1.1.31}$$

is the divergence-free tensor we are seeking for, which is known as the *surface-deformation tensor* due to its physical implication to be explained later in Sect. 1.1.4. The divergence of (1.1.30b) recovers (1.1.17) at once, with no contribution from $\mathbf{B}$. This triple decomposition is very useful as one studies the velocity field near a material surface, and in dynamic problems for which the traceless property of $\mathbf{B}$ can bring considerable simplification.

Interestingly, similar to (1.1.30), a triple decomposition can be found for the double inner-product $\mathbf{D} : \mathbf{D}$, which as will be seen in Sect. 1.2.2 plays a key role in viscous flow dissipation into heat. In fact, $\nabla \boldsymbol{u} : \nabla \boldsymbol{u} = u_{j,i} u_{i,j}$ may be alternatively decomposed to

$$\begin{aligned} u_{j,i} u_{i,j} &= (u_{j,i} u_i)_{,j} - u_i \vartheta_{,i} = (u_{j,i} u_i - \delta_{ij} u_i \vartheta)_{,j} + \vartheta^2; \\ &= (D_{ij} + \Omega_{ij})(D_{ji} + \Omega_{ji}) = D_{ij} D_{ji} - \frac{1}{2}\omega^2. \end{aligned}$$

Then, a comparison of these two expressions yields the desired decomposition identity at once:

$$\mathbf{D} : \mathbf{D} = \vartheta^2 + \frac{1}{2}\omega^2 - \nabla \cdot (\mathbf{B} \cdot \boldsymbol{u}). \tag{1.1.32}$$

But since this is a nonlinear product, coupling among different constituents of the velocity must appear as can be seen in $\mathbf{B} \cdot \boldsymbol{u}$.

1.1.4 Local and Global Material Derivatives

In fluid kinematics we study not only the spatial relations of the flow quantities as we did in the preceding subsections, but also their temporal variation in a universal way, namely without concerning specific physical cause and effect of these temporal variation.[7] Of the temporal variations the most important kind is the time rate of change of flow quantities, which is examined now. We shall make a combined use of the material and field descriptions to derive the governing equations of fluid motion. Readers are assumed to have been familiar with the procedure and results, and the focus here is a neater formulation based on tensor analysis (Appendix A.1) and deeper physical understanding thereof.

Consider any field quantity $\mathcal{F}(\boldsymbol{x}, t)$ carried by a material element located at $\boldsymbol{x}$ at time t. Since the element motion makes $\boldsymbol{x} = \boldsymbol{x}(t)$ as in (1.1.1a), we have $\mathcal{F}(\boldsymbol{x}, t) = \mathcal{F}(\boldsymbol{x}(t), t)$; thus, the material rate of change of $\mathcal{F}$ is

$$\frac{D}{Dt}\mathcal{F}(\boldsymbol{x}(t), t) = \frac{\partial \mathcal{F}}{\partial t} + \frac{\partial \mathcal{F}}{\partial x_i}\frac{dx_i}{dt},$$

which by (1.1.2) implies

$$\frac{D\mathcal{F}}{Dt} = \frac{\partial \mathcal{F}}{\partial t} + \boldsymbol{u} \cdot \nabla \mathcal{F}. \tag{1.1.34}$$

Thus, when acting on any field quantities, the material-derivative operator D/Dt is split to a local time-variation $\partial/\partial t$ and a variation by advection $\boldsymbol{u} \cdot \nabla$. In particular, the acceleration $\boldsymbol{a}$ of a material element reads

$$\boldsymbol{a} = \frac{D\boldsymbol{u}}{Dt} = \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}. \tag{1.1.35}$$

Here again appears the velocity gradient tensor $\nabla \boldsymbol{u}$. In addition to producing the strain rate $\mathbf{D}$, the vorticity $\boldsymbol{\omega}$, and the dilatation ϑ, by various operations one can generate some other important quantities from this tensor. In fact, not only the inner products of $\nabla \boldsymbol{u}$ with $\delta \boldsymbol{x}$ and $\boldsymbol{u}$ *from left* yield the rate of change of $\delta \boldsymbol{x}$ and that of $\boldsymbol{u}$ by advection, respectively, as we have seen, but also its inner product with $\boldsymbol{u}$ *from right* is meaningful: $\nabla \boldsymbol{u} \cdot \boldsymbol{u} = \boldsymbol{u} \cdot (\nabla \boldsymbol{u})^T = \nabla(|\boldsymbol{u}|^2/2)$ is the gradient of kinetic energy.

Then, since $\nabla \boldsymbol{u} = \nabla \boldsymbol{u} - (\nabla \boldsymbol{u})^T + (\nabla \boldsymbol{u})^T$, by (1.1.18) and (1.1.21a) we can split $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ into two terms, and hence refine (1.1.35) to

$$\boldsymbol{a} = \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\omega} \times \boldsymbol{u} + \nabla \left(\frac{1}{2}q^2\right), \quad q = |\boldsymbol{u}|. \tag{1.1.36}$$

[7] Some authors use the term "kinematics" more restrictively, only to the spatial relations of the relevant quantities at a single time instance.

Namely, the advective acceleration has two causes: the gradient of kinetic energy and a vorticity cause in the direction perpendicular to both $\boldsymbol{u}$ and $\boldsymbol{\omega}$. Vector $\boldsymbol{\omega} \times \boldsymbol{u}$ is known as the *Lamb vector*, which implies a *transverse force* to the fluid motion, and as will be seen in Chap. 9 it is responsible for the lift of an aircraft.

So far we have traced the local variation of a field quantity by following a material fluid element. We may now similarly trace the global variation of a field quantity in an arbitrary domain $D(t)$, of which the boundary may move and change shape over time with velocity $\boldsymbol{v_b}$. To this end we recall the Newton-Leibniz formula in elementary calculus,

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)dx = \int_a^b \frac{\partial f}{\partial t}dx + \frac{db}{dt}f(b,t) - \frac{da}{dt}f(a,t). \tag{1.1.37}$$

This formula can be extended to multi-dimensional space, where we replace the speed of moving bounds, da/dt and db/dt, by the normal velocity $\boldsymbol{n} \cdot \boldsymbol{v_b}$ of the moving boundary surface $\partial D(t)$. This yields

$$\frac{d}{dt}\int_{D(t)} \mathcal{F}(\boldsymbol{x},t)dV = \int_{D(t)} \frac{\partial \mathcal{F}}{\partial t}dV + \int_{\partial D(t)} \boldsymbol{n}\cdot\boldsymbol{v}_b \mathcal{F} dS. \tag{1.1.38}$$

In particular, we want to trace the global variation of $\mathcal{F}$ in a *material volume* $\mathcal{V}$, which consists of the same fluid body and whose boundary velocity is the flow velocity $\boldsymbol{u}$. Let $D(t) = \mathcal{V}$, and notice that the time derivative on the right-hand side of (1.1.38) has been shifted inside the volume integral, we may replace $\mathcal{V}$ by a fixed *control volume* V that is instantaneously coincide with $\mathcal{V}$. Thus, (1.1.38) yields the *material derivative* of the integral:

$$\frac{d}{dt}\int_{\mathcal{V}} \mathcal{F}dv = \int_V \frac{\partial \mathcal{F}}{\partial t}dV + \int_{\partial V} \boldsymbol{n}\cdot\boldsymbol{u}\mathcal{F}dS \tag{1.1.39a}$$

$$= \int_V \left[\frac{\partial \mathcal{F}}{\partial t} + \nabla\cdot(\boldsymbol{u}\mathcal{F})\right]dV \tag{1.1.39b}$$

$$= \int_V \left(\frac{D\mathcal{F}}{Dt} + \vartheta\mathcal{F}\right)dV. \tag{1.1.39c}$$

These formulas give alternative expressions of the time rate of the material-volume integral of the $\mathcal{F}$-field. Equation (1.1.39a) is known as the **Reynolds transport theorem**. It indicates that the time variation of $\mathcal{F}$ in $\mathcal{V}$ has two parts: one is due to the local time derivative of $\mathcal{F}$ in V, and the other due to the moving of $\mathcal{V}$ which brings some $\mathcal{F}$ across the boundary ∂V with the rate $u_n\mathcal{F}$ per unit area, see Fig. 1.9. Equation (1.1.39b) is from (1.1.39a) by the Gauss theorem (Appendix A.2.1), which can in turn be cast to (1.1.39c) by using the fact $\nabla\cdot(\boldsymbol{u}\mathcal{F}) = \boldsymbol{u}\cdot\nabla\mathcal{F} + \vartheta\mathcal{F}$ and (1.1.34).

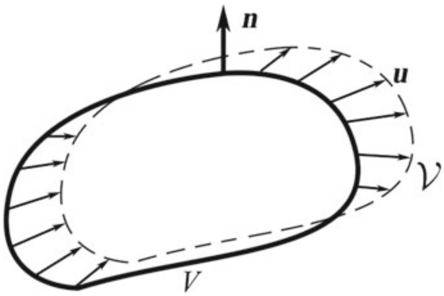

Fig. 1.9 Fluid motion in a control volume V that leads to the Reynolds transport theorem

The local and global variations of any quantity $\mathcal{F}$ in a flow field, (1.1.34) and (1.1.39), are the basis of deriving basic equations of fluid dynamics in the next section.

In addition to the above formulas, we shall also deal with the rate of change of integrals over material lines and surfaces. Since the fluid particles forming a material line, surface, or volume do not change as time, the operator d/dt in front of any of such integrals can be shifted into the integration symbol to become D/Dt therein. But line element $d\boldsymbol{x}$ and surface element[8] $d\boldsymbol{S} = \boldsymbol{n}dS$ in these integrals all vary as time. Thus we first need to list their material rates of change (the rate of change of dv is also included for completeness):

$$\frac{D}{Dt}(d\boldsymbol{x}) = d\boldsymbol{x} \cdot \nabla \boldsymbol{u}, \tag{1.1.40}$$

$$\frac{D}{Dt}(d\boldsymbol{S}) = d\boldsymbol{S} \cdot \mathbf{B}, \tag{1.1.41}$$

$$\frac{D}{Dt}(dv) = \vartheta dv, \tag{1.1.42}$$

where $\mathbf{B}$ is the divergence-free surface-deformation tensor introduced by (1.1.31). Equation (1.1.40) follows directly from (1.1.4) by setting $d\boldsymbol{r} = d\boldsymbol{x}$, while (1.1.42) follows from (1.1.39c) by setting $\mathcal{F} = 1$. To derive (1.1.41), we construct a volume element $dv = d\boldsymbol{x} \cdot d\boldsymbol{S}$ and then use (1.1.40) and (1.1.42) (Problem 1.7). Therefore, in addition to (1.1.39c), we can write down the general rules of the rate of change of the integrals of any quantity $\mathcal{F}$ over material line $\mathcal{C}$ and surface $\mathcal{S}$:

$$\frac{d}{dt}\int_{\mathcal{C}} d\boldsymbol{x} \circ \mathcal{F} = \int_{\mathcal{C}} \left\{ d\boldsymbol{x} \circ \frac{D\mathcal{F}}{Dt} + (d\boldsymbol{x} \cdot \nabla \boldsymbol{u}) \circ \mathcal{F} \right\}, \tag{1.1.43}$$

$$\frac{d}{dt}\int_{\mathcal{S}} d\boldsymbol{S} \circ \mathcal{F} = \int_{\mathcal{S}} \left\{ d\boldsymbol{S} \circ \frac{D\mathcal{F}}{Dt} + (d\boldsymbol{S} \cdot \mathbf{B}) \circ \mathcal{F} \right\}, \tag{1.1.44}$$

[8] A surface element is a vector consisting of its normal direction $\boldsymbol{n}$ and area dS.