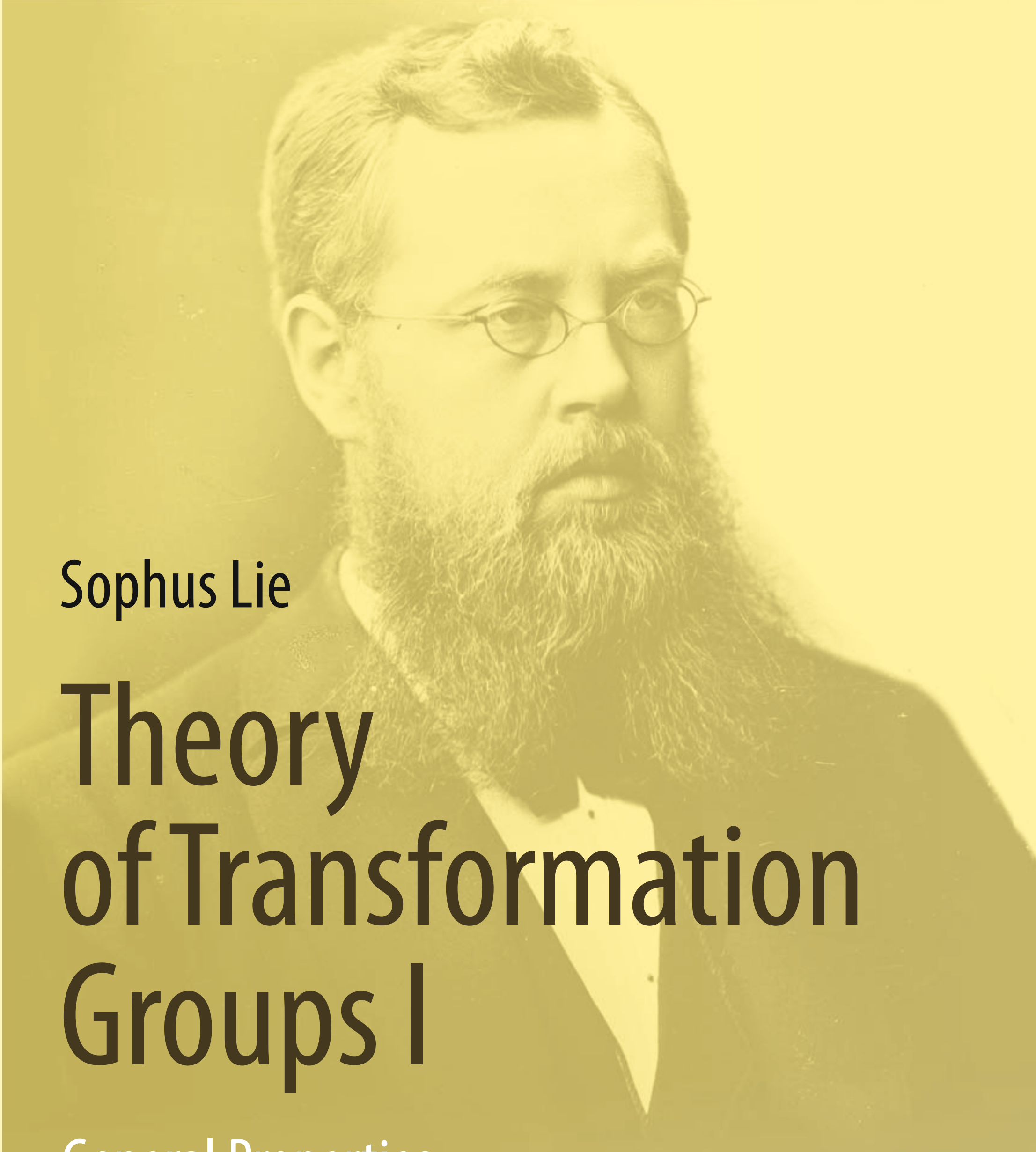

Sophus Lie

Theory of Transformation Groups I

General Properties of Continuous Transformation Groups. A Contemporary Approach and Translation

Editor and Translator:
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General Properties of Continuous Transformation Groups. A Contemporary Approach and Translation

With the collaboration of Friedrich Engel

Editor and Translator: Joël Merker

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THEORIE

DER

TRANSFORMATIONSGRUPPEN

ERSTER ABSCHNITT

UNTER MITWIRKUNG VON Prof. Dr. FRIEDRICH ENGEL

BEARBEITET VON

SOPHUS LIE,

WEIL. PROFESSOR DER GEOMETRIE AND DER UNIVERSITÄT LEIPZIG
UND PROFESSOR I TRANSFORMASJONSGRUPPENES TEORI AN DER
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UNVERÄNDERTER NEUDRUCK
MIT UNTERSTÜTZUNG DER KÖNIGLICHEN
FREDERIKS UNIVERSITÄT ZU OSLO

1930

**VERLAG UND DRUCK VON B.G. TEUBNER IN LEIPZIG
UND BERLIN**

Foreword

This modernized English translation grew out of my old simultaneous interest in the mathematics itself and in the metaphysical thoughts governing its continued development. I owe to the books of Robert Hermann, Peter Olver, Thomas Hawkins, and Olle Stormark my introduction to Lie's original vast field.

Up to the end of the 18$^{\text{th}}$ Century, the universal language of Science was Latin, until its centre of gravity shifted to German during the 19$^{\text{th}}$ Century, while nowadays — needless to say — English is widespread. Being intuitively convinced that Lie's original works contain much more than what has been modernized up to now, three years ago I started to learn German *from scratch* just in order to read Lie, with two main goals in mind:

□ to complete and modernize the Lie-Amaldi classification of finite-dimensional local Lie group holomorphic actions on spaces of complex dimensions 1, 2 and 3 for various applications in complex and Cauchy-Riemann geometry;

□ to better understand the roots of Élie Cartan's achievements.

Then it gradually appeared to me that *Lie's mathematical thought is universal and transhistorical*, hence it deserves *per se* to be translated. The present adapted English translation follows an earlier monograph[1] written in French and specially devoted to Engel and Lie's treatment of the so-called *Riemann-Helmholtz problem* in Volume III of the *Theorie der Transformationsgruppen*.

A few observations are in order concerning the chosen format. For several reasons, it was essentially impossible to directly translate the first few chapters in which Lie's intention was to set up the beginnings of the theory in the highest possible generality, especially in order to eliminate the axiom of inverse, an aspect never dealt with in modern treatises. As a result, I decided in the first four chapters to reorganize the material and to reprove the relevant statements, nevertheless retaining all of the embraced mathematical content. But starting with Chap. 5, Engel and Lie's exposition is so smooth, so rigorous, so understandable, so systematic, so astonishingly well organized — *so beautiful for thought* — that a pure translation is essential.

[1] Merker, J.: Sophus Lie, Friedrich Engel et le problème de Riemann-Helmholtz, Hermann Éditeur des Sciences et des Arts, Paris, xxiii+325 pp, 2010.

Lastly, the author is grateful to Gautam Bharali, Philip Boalch, Egmont Porten, and Masoud Sabzevari for a few fine suggestions concerning the language and for misprint chasing, but is of course solely responsible for the lack of idiomatic English.

Paris, École Normale Supérieure, *Joël Merker*
16 March 2010

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Part I
Modern Presentation

Chapter 1
Three Principles of Thought Governing the Theory of Lie

Let $x = (x_1, \dots, x_n)$ be coordinates on an n-dimensional real or complex euclidean space $\mathbb{C}^n$ or $\mathbb{R}^n$, considered as a source domain. The archetypal objects of Lie's Theory of Continuous Transformation Groups are *point transformation equations*:

$$x_i' = f_i(x_1, \dots x_n; a_1, \dots, a_r) \qquad {\scriptstyle (i=1\cdots n)},$$

parametrized by a finite number r of real or complex parameters $(a_1, \dots, a_r)$, namely each map $x' = f(x; a) =: f_a(x)$ is assumed to constitute a diffeomorphism from some domain[1] (Translator's note: By a *domain*, we will always mean a *connected*, nonempty open set.) in the source space into some domain in a target space of the same dimension equipped with coordinates $(x_1', \dots, x_n')$. Thus, the functional determinant:

$$\det \operatorname{Jac}(f) = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \cdots\cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots\cdots & \frac{\partial f_n}{\partial x_n} \end{vmatrix} = \sum_{\sigma \in \mathsf{Perm}_n} \operatorname{sign}(\sigma) \frac{\partial f_1}{\partial x_{\sigma(1)}} \frac{\partial f_2}{\partial x_{\sigma(2)}} \cdots \frac{\partial f_n}{\partial x_{\sigma(n)}}$$

does not vanish at any point of the source domain.

§ **15**. ([1], pp. 25–26) [The concepts of transformation $x' = f(x)$ and of transformation equations $x' = f(x; a)$ are of a purely analytic nature.] However, these concepts are given a graphical interpretation [ANSCHAULICH AUFFASUNG] when the concept of an n-times extended space [RAUM] is introduced.

If we interpret $x_1, \dots, x_n$ as the coordinates of the points [PUNKTCOORDINATEN] of such a space, then a transformation $x_i' = f_i(x_1, \dots, x_n)$ appears as a point transformation [PUNKTTRANSFORMATION]; consequently, this transformation can be interpreted as an *operation* such that every point x_i is transferred at the same time into the new position x_i'. One expresses this as follows: the

S. Lie, *Theory of Transformation Groups I*,
DOI 10.1007/978-3-662-46211-9_1

transformation in question is an operation via which the points of the space $x_1, \ldots, x_n$ are permuted with each other.

Before introducing the continuous group axioms in Chap. 3 below, the very first question to be settled is: how many different transformations $x_i' = f_i(x; a)$ correspond to the ∞^r different systems of values $(a_1, \ldots, a_r)$? Some parameters might indeed be superfluous, hence they should be removed from the outset, as will be achieved in Chap. 2. For this purpose, it is crucial to formulate explicitly, once and for all, *three principles of thought concerning the admission of hypotheses that hold throughout the theory of continuous groups developed by Lie*.

General Assumption of Analyticity

Curves, surfaces, manifolds, groups, subgroups, coefficients of infinitesimal transformations, etc., all mathematical objects of the theory will be assumed to be *analytic*, i.e. their representing functions will be assumed to be locally expandable in convergent, univalent power series defined in a certain domain of an appropriate $\mathbb{R}^m$.

Principle of Free Generic Relocalization

Consider a local mathematical object which is represented by functions that are analytic in some domain U_1, and suppose that a certain "generic" nice behavior holds on $U_1 \backslash \mathsf{D}_1$ outside a certain proper closed analytic subset $\mathsf{D}_1 \subset U_1$; for instance: the invertibility of a square matrix composed of analytic functions holds outside the zero-locus of its determinant. Then relocalize the considerations in some subdomain $U_2 \subset U_1 \backslash \mathsf{D}_1$.

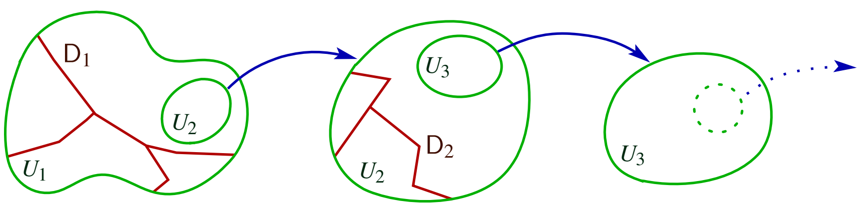

Fig. 1.1 Relocalizing finitely many times in neighborhoods of generic points

Then, in U_2, further reasoning may necessitate avoiding another proper closed analytic subset D_2, hence to relocalize the considerations in some subdomain $U_3 \subset U_2 \backslash \mathsf{D}_2$, and so on. Most proofs of the *Theorie der Transformationsgruppen*, and especially the classification theorems, allow such relocalizations a great number of times, often without mention, such an *act of thought* being considered as implicitly clear, and free relocalization being justified by the necessity of *initially studying generic objects*.

Giving no Name to Domains or Neighborhoods

Without providing a systematic notation, Lie and Engel commonly wrote *the* neighborhood [*der* UMGEBUNG] (of a point), similarly as one speaks of *the* neighborhood of a house, or of *the* surroundings of a town, whereas contemporary topology conceptualizes *a* (say, sufficiently small) given neighborhood amongst an infinity. Contrary to what the formalistic, twentieth-century mythology sometimes says, Lie and Engel did emphasize the local nature of the concept of transformation group in terms of narrowing down neighborhoods; we shall illustrate this especially when presenting Lie's attempt to economize the axiom of inverse. Certainly, it is true that most of Lie's results are stated without specifying domains of existence, but in fact, it is quite plausible that Lie soon realized that giving no name to neighborhoods, and avoiding superfluous denotation, is efficient and expeditious in order to perform far-reaching classification theorems.

Therefore, adopting the economical style of thought in Engel-Lie's treatise, our "modernization-translation" of the theory will, while nevertheless providing frequent reminders, presuppose that:

- mathematical objects are analytic;
- relocalization is freely allowed;
- open sets are often small, usually unnamed, and always *connected*.

Introduction ([1], pp. 1–8)

If the variables $x_1', \dots, x_n'$ are determined as functions of $x_1, \dots, x_n$ by n equations, solvable with respect to $x_1, \dots, x_n$:

$$x_i' = f_i(x_1, \dots, x_n) \qquad (i=1\cdots n),$$

then one says that these equations represent a transformation [TRANSFORMATION] between the variables x and x'. In the sequel, we will have to deal with such transformations; unless the contrary is expressly mentioned, we will restrict ourselves to the case where the f_i are *analytic* [ANALYTISCH] *functions* of their arguments. However, because a not negligible portion of our results are independent of this assumption, we will occasionally indicate how various developments take shape by taking into consideration functions of this sort.

When the functions $f_i(x_1, \dots, x_n)$ are analytic and are defined inside a common region [BEREICH], then according to the known studies of CAUCHY, WEIERSTRASS, BRIOT and BOUQUET, one can always delimit, in the manifold of all real and complex systems of values $x_1, \dots, x_n$, a region (x) such that all functions f_i are univalent in the complete extension [AUSDEHNUNG] of this region, and so that, in the neighborhood [UMGEBUNG] of every system of values $x_1^0, \dots, x_n^0$ belonging to the region (x), the functions behave regularly [REGULÄR VERHALTEN], that is to say, they can be expanded in ordinary power series with respect to $x_1 - x_1^0, \dots, x_n - x_n^0$ with only whole positive powers.

For the solvability of the equations $x_i' = f_i(x)$, a unique condition is necessary and sufficient, namely the condition that the functional determinant:

$$\sum \pm \frac{\partial f_1}{\partial x_1} \cdots \frac{\partial f_n}{\partial x_n}$$

should not vanish identically. If this condition is satisfied, then the region (x) defined above can specially be defined so that the functional determinant does not take the value zero for any system of values in (x). Under this assumption, if one lets the x gradually take all systems of values in the region (x), then the equations $x_i' = f_i(x)$ determine, in the domain [GEBIETE] of the x', a region of such a nature that $x_1, \dots, x_n$, in the neighborhood of every system of values ${x_1'}^0, \dots, {x_n'}^0$ in this new region, behave regularly as functions of $x_1', \dots, x_n'$, and hence can be expanded as ordinary power series of $x_1' - {x_1'}^0, \dots, x_n' - {x_n'}^0$. It is well known that from this, it does not follow that the x_i are univalent functions of $x_1', \dots, x_n'$ in the complete extension of the new region; but when necessary, it is possible to narrow down the region (x) defined above so that two different systems of values $x_1, \dots, x_n$ of the region (x) always produce two, also different, systems of values $x_1' = f_1(x), \dots, x_n' = f_n(x)$.

Thus, the equations $x_i' = f_i(x)$ establish a univalent invertible relationship [BEZIEHUNG] between regions in the domain of the x and regions in the domain of the x'; to every system of values in one region, they associate one and only one system of values in the other region, and conversely.

If the equations $x_i' = f_i(x)$ are solved with respect to the x, then in turn, the resulting equations:

$$x_k = F_k(x_1', \dots, x_n') \qquad (k = 1 \cdots n)$$

again represent a transformation. The relationship between this transformation and the initial one is evidently a reciprocal relationship; accordingly, one says: the two transformations are *inverse* to one another. From this definition, it visibly follows:

If one first executes the transformation:

$$x_i' = f_i(x_1, \dots, x_n) \qquad (i = 1 \cdots n)$$

and then the transformation inverse to it:

$$x_i'' = F_i(x_1', \ldots, x_n') \qquad (i=1\cdots n),$$

then one obtains the identity transformation:

$$x_i'' = x_i \qquad (i=1\cdots n).$$

Here lies the real definition of the concept [BEGRIFF] of two transformations inverse to each other.

In general, if one executes two arbitrary transformations:

$$x_i' = f_i(x_1, \ldots, x_n), \qquad x_i'' = g_i(x_1', \ldots, x_n') \qquad (i=1\cdots n)$$

one after the other, then one obtains a new transformation, namely the following one:

$$x_i'' = g_i\big(f_1(x), \ldots, f_n(x)\big) \qquad (i=1\cdots n).$$

In general, this new transformation naturally changes when one changes the order [REIHENFOLGE] of the two transformations; however, it can also happen that the order of the two transformations is indifferent. This case occurs when one has identically:

$$g_i\big(f_1(x), \ldots, f_n(x)\big) \equiv f_i\big(g_1(x), \ldots, g_n(x)\big) \qquad (i=1\cdots n);$$

we then say, as in the process of the Theory of Substitutions [VORGANG DER SUBSTITUTIONENTHEORIE]: *the two transformations:*

$$x_i' = f_i(x_1, \ldots, x_n) \qquad (i=1\cdots n)$$

and:

$$x_i' = g_i(x_1, \ldots, x_n) \qquad (i=1\cdots n)$$

are interchangeable [VERTAUSCHBAR] *with one another.* —

A finite or infinite family [SCHAAR] *of transformations between the x and the x' is called a* group of transformations *or a* transformation group *when any two transformations of the family executed one after the other give a transformation which again belongs to the family.*[1]

A transformation group is called *discontinuous* when it consists of a discrete number of transformations, and this number can be finite or infinite. Two transformations of such a group are finitely different from each other. The discontinuous groups belong to the domain of the *Theory of Substitutions*, so in the sequel, they will remain out of consideration.

[1] Sophus LIE, Gesellschaft der Wissenschaften zu Christiania 1871, p. 243. KLEIN, Vergleichende Betrachtungen über neuere geometrische Forschungen, Erlangen 1872. LIE, Göttinger Nachrichten 1873, 3. Decemb.

The discontinuous groups stand in opposition to the *continuous* transformation groups, which always contain infinitely many transformations. A transformation group is called *continuous* when it is possible, for every transformation belonging to the group, to indicate certain other transformations which differ only infinitely little from the transformation in question, and when by contrast, it is not possible to reduce the complete totality [INBEGRIFF] of transformations contained in the group to a single discrete family.

Now, amongst the continuous transformation groups, we again consider two separate categories [KATEGORIEN] which, in the nomenclature [BENENNUNG], are distinguished as *finite continuous* groups and as *infinite continuous* groups. To begin with, we can only give provisional definitions of the two categories, and these definitions will be apprehended precisely later.

A *finite continuous transformation group* will be represented by *one* system of n equations:

$$x_i' = f_i(x_1, \dots, x_n, a_1, \dots, a_r) \qquad (i=1\cdots n),$$

where the f_i denote analytic functions of the variables $x_1, \dots, x_n$ and of the arbitrary parameters $a_1, \dots, a_r$. Since we have to deal with a group, two transformations:

$$x_i' = f_i(x_1, \dots, x_n, a_1, \dots, a_r)$$
$$x_i'' = f_i(x_1', \dots, x_n', a_1, \dots, a_r),$$

when executed one after the other, must produce a transformation which belongs to the group, hence which has the form:

$$x_i'' = f_i\big(f_1(x,a), \dots, f_n(x,a), b_1, \dots, b_r\big) = f_i(x_1, \dots, x_n, c_1, \dots, c_r).$$

Here, the c_k are naturally independent of the x and so, are functions of only the a and the b.

Example. A known group of this sort is the following:

$$x' = \frac{x + a_1}{a_2\, x + a_3},$$

which contains the three parameters a_1, a_2, a_3. If one executes the two transformations:

$$x' = \frac{x + a_1}{a_2\, x + a_3}, \qquad x'' = \frac{x' + b_1}{b_2\, x' + b_3}$$

one after the other, then one obtains:

$$x'' = \frac{x + c_1}{c_2\, x + c_3},$$

where c_1, c_2, c_3 are defined as functions of the a and the b by the relations:

$$c_1 = \frac{a_1 + b_1\, a_3}{1 + b_1\, a_2}, \qquad c_2 = \frac{b_2 + a_2\, b_3}{1 + b_1\, a_2}, \qquad c_3 = \frac{b_2\, a_1 + b_3\, a_3}{1 + b_1\, a_2}.$$

The following group with n^2 parameters a_{ik} is also well known:

$$x_i' = \sum_{k=1}^{n} a_{ik}\, x_k \qquad (i=1\cdots n).$$

Here, if one sets:

$$x_\nu'' = \sum_{i=1}^{n} b_{\nu i}\, x_i' \qquad (i=1\cdots n),$$

then we have:

$$x_\nu'' = \sum_{i,\,k}^{1\cdots n} b_{\nu i}\, a_{ik}\, x_k = \sum_{k=1}^{n} c_{\nu k}\, x_k,$$

where the $c_{\nu k}$ are determined by the equations:

$$c_{\nu k} = \sum_{i=1}^{n} b_{\nu i}\, a_{ik} \qquad (\nu,\, k=1\cdots n). \text{ —}$$

In order to arrive at a usable definition of a finite continuous group, we first want to somehow reshape the definition of finite continuous groups. On the occasion, we shall use a proposition from the theory of differential equations which we will return to later in a more comprehensive way (cf. Chap. 10).

Let the equations:

$$x_i' = f_i(x_1, \dots, x_n,\, a_1, \dots, a_r) \qquad (i=1\cdots n)$$

represent an arbitrary continuous group. According to the proposition in question, it is then possible to define the functions f_i through a system of differential equations, insofar as they depend upon the x. To this end, one only has to differentiate the equations $x_i' = f_i(x,a)$ with respect to $x_1, \dots, x_n$ sufficiently often and then to set up all equations that may be obtained by elimination of $a_1, \dots, a_r$. If one has gone sufficiently far by differentiation, then by elimination of the a, one obtains a system of differential equations for $x_1', \dots, x_n'$, whose most general system of solutions is represented by the initial equations $x_i' = f_i(x,a)$ with the r arbitrary parameters. Now, since by assumption the equations $x_i' = f_i(x,a)$ define a group, it follows that the concerned system of differential equations possesses the following remarkable property: if $x_i' = f_i(x_1, \dots, x_n,\, b_1, \dots, b_r)$ is a system of solutions of it, and if $x_i' = f_i(x,a)$ is a second system of solutions, then:

$$x_i' = f_i\big(f_1(x,a), \dots, f_n(x,a),\, b_1, \dots, b_r\big) \qquad (i=1\cdots n)$$

is also a system of solutions.

From this, we see that the equations of an arbitrary finite continuous transformation group can be defined by a system of differential equations which possesses certain specific properties. Firstly, from two systems of solutions of the concerned differential equations one can always derive, in the way indicated above, a third system of solutions: it is precisely in this that we have to deal with a group. Secondly, the most general system of solutions of the concerned differential equations depends only upon a finite number of arbitrary constants: this circumstance expresses that our group is finite.

Now, we assume that there is a family of transformations $x_i' = f_i(x_1,\dots,x_n)$ which is defined by a system of differential equations of the form:

$$W_k\left(x_1',\dots,x_n',\,\frac{\partial x_1'}{\partial x_1},\,\dots,\,\frac{\partial^2 x_1'}{\partial x_1^2},\,\dots\right)=0 \qquad (k=1,2\cdots).$$

Moreover, we assume that this system of differential equations possesses the first of the two mentioned properties, but not the second; therefore, if $x_i' = f_i(x_1,\dots,x_n)$ and $x_i' = g_i(x_1,\dots,x_n)$, then $x_i' = g_i\big(f_1(x),\dots,f_n(x)\big)$ is also a system of solutions of these differential equations, and the most general system of solutions of them does not only depend upon a finite number of arbitrary constants, but also upon higher sorts of elements, as for example, upon arbitrary functions. Then the totality of all transformations which satisfy the concerned differential equations evidently again forms a group, and in general, a continuous group, though no more a finite one, but one which we call *infinite continuous*.

Straightaway, we give a few simple examples of infinite continuous transformation groups.

When the differential equations which define the concerned infinite group reduce to the identity equation $0 = 0$, then the transformations of the group read:

$$x_i' = \Pi_i(x_1,\dots,x_n) \qquad (i=1\cdots n)$$

where the Π_i denote arbitrary analytic functions of their arguments.

The equations:

$$\frac{\partial x_i'}{\partial x_k} = 0 \qquad (i\neq k;\; i,k=1\cdots n)$$

also define an infinite group, namely the following one:

$$x_i' = \Pi_i(x_i) \qquad (i=1\cdots n),$$

where again the Π_i are absolutely arbitrary.

Furthermore, it is to be observed that the concept of an infinite continuous group can be understood more generally than what has been said here. Actually, one could call infinite continuous any continuous group which is not finite. However, this definition does not coincide with the one given above.

For instance, the equations:

$$x' = F(x), \qquad y' = F(y)$$

in which F, for the two cases, denotes the same function of its arguments, represent a group. This group is continuous, since all its transformations are represented by a single system of equations; in addition, it is obviously not finite. Consequently, it would be an infinite continuous group if we interpreted this concept in the more general sense indicated above. But on the other hand, it is not possible to define the family of the transformations:

$$x' = F(x), \qquad y' = F(y)$$

by differential equations that are free of arbitrary elements. Consequently, the definition stated first for an infinite continuous group does not apply to this case. Nevertheless, we find it suitable to consider only the infinite continuous groups which can be defined by differential equations and hence, we always set as fundamental our first, tight definition.

We wish to emphasize that the concept of "transformation group" is still not exhausted by the difference between discontinuous and continuous groups. Rather, there are transformation groups which are subordinate to neither of these two classes but which have something in common with each of them. In the sequel, we must at least occasionally treat this sort of group. Provisionally, two examples will suffice.

The totality of all coordinate transformations of a plane by which one transfers an ordinary right-angled system of coordinates to another forms a group which is neither continuous nor discontinuous. Indeed, the group in question contains two separate categories of transformations between which a continuous transition is not possible: firstly, the transformations by which the old and the new systems of coordinates are congruent, and secondly, the transformations by which these two systems are not congruent.

The first transformations have the form:

$$x' - a = x\cos\alpha - y\sin\alpha, \qquad y' - b = x\sin\alpha + y\cos\alpha,$$

while the analytic expression of the second transformations reads:

$$x' - a = x\cos\alpha + y\sin\alpha, \qquad y' - b = x\sin\alpha - y\cos\alpha.$$

Each of these systems of equations represents a continuous family of transformations, hence the group is not discontinuous; but it is also not continuous, because both systems of equations taken together provide all transformations of the group; thus, the transformations of the group decompose into two discrete families. If one imagines the x, y plane in ordinary space and if one adds

z as a third right-angled coordinate, then one can imagine the totality of coordinate transformations of the plane $z = 0$ is a totality of certain movements [BEWEGUNGEN] of the space, namely the movements for which the plane $z = 0$ keeps its position. Correspondingly, these movements separate into two classes, namely the class which only translates the plane into itself and the class which rotates the plane.

As a second example of such a group, one can take the totality of all projective and dualistic transformations of the plane.

According to these general remarks on the concept of a transformation group, we turn ourselves to the consideration of the finite continuous transformation groups which constitute the object of the following studies. These studies are divided into three volumes [ABSCHNITTE]. The *first* volume treats finite continuous groups in general. The *second* volume treats the finite continuous groups whose transformations are so-called *contact transformations* [BERÜHRUNGSTRANSFORMATIONEN]. Lastly, in the *third* volume, certain general problems of group theory will be carried out in great detail for a small number of variables.

References

1. Engel, F., Lie, S.: Theorie der Transformationsgruppen. Erster Abschnitt. Unter Mitwirkung von Prof. Dr. Friedrich Engel, bearbeitet von Sophus Lie, Verlag und Druck von B.G. Teubner, Leipzig und Berlin, xii+638 pp. (1888). Reprinted by Chelsea Publishing Co., New York, N.Y. (1970)

Chapter 2
Local Transformation Equations and Essential Parameters

Abstract Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, throughout. As said in Chap. 1, transformation equations $x_i' = f_i(x; a_1, \dots, a_r)$, $i = 1, \dots, n$, which are local, analytic diffeomorphisms of $\mathbb{K}^n$ parametrized by a finite number r of real or complex numbers $a_1, \dots, a_r$, constitute the archetypal objects of Lie's theory. The preliminary question is to decide whether the f_i really depend upon *all* parameters, and also, to get rid of superfluous parameters, if there are any.

Locally in a neighborhood of a fixed x_0, one expands $f_i(x; a) = \sum_{\alpha \in \mathbb{N}^n} \mathscr{U}_\alpha^i(a)\,(x - x_0)^\alpha$ in power series and one looks at the *infinite coefficient mapping* $\mathsf{U}_\infty : a \longmapsto \big(\mathscr{U}_\alpha^i(a)\big)_{\alpha \in \mathbb{N}^n}^{1 \leqslant i \leqslant n}$ from $\mathbb{K}^r$ to $\mathbb{K}^\infty$, which is expected to faithfully describe the dependence with respect to a in question. If ρ_∞ denotes the maximal, generic and locally constant rank of this map, with of course $0 \leqslant \rho_\infty \leqslant r$, then the answer says that locally in a neighborhood of a generic a_0, there exist both a local change of parameters $a \mapsto \big(\mathsf{u}_1(a), \dots, \mathsf{u}_{\rho_\infty}(a)\big) =: \mathsf{u}$ decreasing the number of parameters from r down to ρ_∞, and new transformation equations:

$$x_i' = g_i\big(x; \mathsf{u}_1, \dots, \mathsf{u}_{\rho_\infty}\big) \qquad (i = 1 \cdots n)$$

depending *only* upon ρ_∞ parameters which give again the old ones:

$$g_i\big(x; \mathsf{u}(a)\big) \equiv f_i(x; a) \qquad (i = 1 \cdots n).$$

At the end of this brief chapter, before giving a precise introduction to the local Lie group axioms, we present an example due to Engel which shows that the axiom of inverse cannot be deduced from the axiom of composition, contrary to one of Lie's *Idées fixes*.

2.1 Generic Rank of the Infinite Coefficient Mapping

Thus, we consider local transformation equations:

S. Lie, *Theory of Transformation Groups I*,
DOI 10.1007/978-3-662-46211-9_2

$$x_i' = f_i(x_1, \ldots, x_n; a_1, \ldots, a_r) \qquad (i=1\cdots n).$$

We want to illustrate how the principle of free generic relocalization described above on p. 4 helps to get rid of superfluous parameters a_k. We assume that the f_i are defined and analytic for x belonging to a certain (unnamed, connected) domain of $\mathbb{K}^n$ and for a belonging to some domain of $\mathbb{K}^r$.

Expanding the f_i of $x_i' = f_i(x; a)$ in power series with respect to $x - x_0$ in some neighborhood of a point x_0:

$$f_i(x; a) = \sum_{\alpha \in \mathbb{N}^n} \mathscr{U}_\alpha^i(a)\, (x - x_0)^\alpha,$$

we get an infinite number of analytic functions $\mathscr{U}_\alpha^i = \mathscr{U}_\alpha^i(a)$ of the parameters that are defined in some uniform domain of $\mathbb{K}^r$. Intuitively, this infinite collection of coefficient functions $\mathscr{U}_\alpha^i(a)$ should show how $f(x; a)$ depends on a.

To make this claim precise, we thus consider the map:

$$\mathsf{U}_\infty : \quad \mathbb{K}^r \ni a \longmapsto \big(\mathscr{U}_\alpha^i(a)\big)_{\alpha \in \mathbb{N}^n}^{1 \leqslant i \leqslant n} \in \mathbb{K}^\infty.$$

For the convenience of applying standard differential calculus in finite dimensions, we simultaneously consider all of its κ-th truncations:

$$\mathsf{U}_\kappa : \quad \mathbb{K}^r \ni a \longmapsto \big(\mathscr{U}_\alpha^i(a)\big)_{|\alpha| \leqslant \kappa}^{1 \leqslant i \leqslant n} \in \mathbb{K}^{n \frac{(n+\kappa)!}{n!\,\kappa!}},$$

where $\frac{(n+\kappa)!}{n!\,\kappa!}$ is the number of multiindices $\alpha \in \mathbb{N}^n$ whose length $|\alpha| := \alpha_1 + \cdots + \alpha_n$ satisfies the upper bound $|\alpha| \leqslant \kappa$. We call U_κ, U_∞ the *(in)finite coefficient mapping(s)* of $x_i' = f_i(x; a)$.

The *Jacobian matrix* of U_κ is the $r \times \big(n \frac{(n+\kappa)!}{n!\,\kappa!}\big)$ matrix:

$$\left(\frac{\partial \mathscr{U}_\alpha^i}{\partial a_j}(a)\right)_{1 \leqslant j \leqslant r}^{|\alpha| \leqslant \kappa,\, 1 \leqslant i \leqslant n},$$

its r rows being indexed by the partial derivatives. The *generic rank* of U_κ is the largest integer $\rho_\kappa \leqslant r$ such that there is a $\rho_\kappa \times \rho_\kappa$ minor of $\mathrm{Jac}\,\mathsf{U}_\kappa$ which does not vanish identically, but all $(\rho_\kappa + 1) \times (\rho_\kappa + 1)$ minors do vanish identically. The uniqueness principle for analytic functions then insures that the common zero-set of all $\rho_\kappa \times \rho_\kappa$ minors is a *proper* closed analytic subset D_κ (of the unnamed domain where the $\mathscr{U}_\alpha^i$ are defined), so it is stratified by a finite number of submanifolds of codimension $\geqslant 1$ ([8, 2, 3, 5]), and in particular, it has empty interior, hence it is intuitively "thin".

So the set of parameters a at which there is a least one $\rho_\kappa \times \rho_\kappa$ minor of $\mathrm{Jac}\,\mathsf{U}_\kappa$ which does not vanish is open and *dense*. Consequently, "for a generic point a", the map U_κ is of rank $\geqslant \rho_\kappa$ at every point a' sufficiently close to a (since the corresponding $\rho_\kappa \times \rho_\kappa$ minor does not vanish in a neighborhood of a), and because all $(\rho_\kappa + 1) \times (\rho_\kappa + 1)$ minors of $\mathrm{Jac}\,\mathsf{U}_\kappa$ were assumed to vanish identically, the map

U_κ happens to be in fact of *constant* rank U_κ in a (small) neighborhood of every such generic a.

Insuring constancy of a rank is one important instance of why free relocalization is useful: a majority of theorems of the differential calculus and of the classical theory of (partial) differential equations hold under specific local constancy assumptions.

As κ increases, the number of columns of $\operatorname{Jac} U_\kappa$ increases, hence $\rho_{\kappa_1} \leqslant \rho_{\kappa_2}$ for $\kappa_1 \leqslant \kappa_2$. Since $\rho_\kappa \leqslant r$ is bounded, the generic rank of U_κ becomes constant for all $\kappa \geqslant \kappa_0$ bigger than some sufficiently large κ_0. Thus, let $\rho_\infty \leqslant r$ denote this maximal possible generic rank.

Definition 2.1. The parameters $(a_1, \dots, a_r)$ of given point transformation equations $x_i' = f_i(x;a)$ are called *essential* if, after expanding $f_i(x;a) = \sum_{\alpha \in \mathbb{N}^n} \mathscr{U}_\alpha^i(a)\,(x-x_0)^\alpha$ in power series at some x_0, the generic rank ρ_∞ of the coefficient mapping $a \longmapsto \big(\mathscr{U}_\alpha^i(a)\big)_{\alpha\in\mathbb{N}^n}^{1\leqslant i\leqslant n}$ is maximal, equal to the number r of parameters: $\rho_\infty = r$.

Without entering into technical details, we make a remark. It is a consequence of the principle of analytic continuation and of some reasonings with power series that the *same* maximal rank ρ_∞ is enjoyed by the coefficient mapping $a \mapsto \big(\mathscr{U}'^i_\alpha(a)\big)_{\alpha\in\mathbb{N}^n}^{1\leqslant i\leqslant n}$ for the expansion of $f_i(x;a) = \sum_{\alpha\in\mathbb{N}^n} \mathscr{U}'^i_\alpha(a)\,(x-x_0')^\alpha$ at another, arbitrary point x_0'. Also, one can prove that ρ_∞ is independent of the choice of coordinates x_i and of parameters a_k. These two facts will not be needed, and the interested reader is referred to [9] for proofs of quite similar statements holding true in the context of *Cauchy-Riemann geometry*.

2.2 Quantitative Criterion for the Number of Superfluous Parameters

It is not very practical to compute the generic rank of the infinite Jacobian matrix $\operatorname{Jac} \mathsf{U}_\infty$. To check essentiality of parameters in concrete situations, a helpful criterion due to Lie is **(iii)** below.

Theorem 2.1. *The following three conditions are equivalent:*

(i) *In the transformation equations*

$$x_i' = f_i(x_1,\dots,x_n;\, a_1,\dots,a_r) = \sum_{\alpha\in\mathbb{N}^n} \mathscr{U}_\alpha^i(a)\,(x-x_0)^\alpha \qquad (i=1\cdots n),$$

the parameters $a_1,\dots,a_r$ are not *essential.*

(ii) *(By definition) The generic rank ρ_∞ of the infinite Jacobian matrix:*

$$\operatorname{Jac}\mathsf{U}_\infty(a) = \Big(\frac{\partial \mathscr{U}_\alpha^i}{\partial a_j}(a)\Big)_{1\leqslant j\leqslant r}^{\alpha\in\mathbb{N}^n,\,1\leqslant i\leqslant n}$$

is strictly less than r.

(iii) *Locally in a neighborhood of* every (x_0, a_0), *there exists a not identically zero analytic vector field on the parameter space:*

$$\mathscr{T} = \sum_{k=1}^{n} \tau_k(a)\, \frac{\partial}{\partial a_k}$$

which annihilates all the $f_i(x;a)$*:*

$$0 \equiv \mathscr{T} f_i = \sum_{k=1}^{n} \tau_k \frac{\partial f_i}{\partial a_k} = \sum_{\alpha \in \mathbb{N}^n} \sum_{k=1}^{r} \tau_k(a)\, \frac{\partial \mathscr{U}_\alpha^i}{\partial a_k}(a)\,(x-x_0)^\alpha \qquad (i=1\cdots n).$$

More generally, if ρ_∞ *denotes the generic rank of the infinite coefficient mapping:*

$$\mathsf{U}_\infty : \qquad a \longmapsto \big(\mathscr{U}_\alpha^i(a)\big)_{\alpha\in\mathbb{N}^n}^{1\leqslant i\leqslant n},$$

then locally in a neighborhood of every (x_0, a_0), *there exist exactly* $r-\rho_\infty$, *and no more, analytic vector fields:*

$$\mathscr{T}_\mu = \sum_{k=1}^{n} \tau_{\mu k}(a)\, \frac{\partial}{\partial a_k} \qquad (\mu=1\cdots r-\rho_\infty),$$

with the property that the dimension of $\mathrm{Span}\big(\mathscr{T}_1\big|_a, \ldots, \mathscr{T}_{r-\rho_\infty}\big|_a\big)$ *is equal to* $r-\rho_\infty$ *at every parameter a at which the rank of* U_∞ *is maximal, equal to* ρ_∞, *such that the derivations* $\mathscr{T}_\mu$ *all annihilate the* $f_i(x;a)$*:*

$$0 \equiv \mathscr{T}_\mu f_i = \sum_{k=1}^{r} \tau_{\mu k}(a)\, \frac{\partial f_i}{\partial a_k}(x;a) \qquad (i=1\cdots n;\ \mu=1\cdots r-\rho_\infty).$$

Proof. Just by the chosen definition, we have **(i)** $\Longleftrightarrow$ **(ii)**. Next, suppose that condition **(iii)** holds, in which the coefficients $\tau_k(a)$ of the concerned nonzero derivation $\mathscr{T}$ are locally defined. Recalling that the Jacobian matrix $\mathrm{Jac}\,\mathsf{U}_\infty$ has r rows and an infinite number of columns, we then see that the n annihilation equations $0 \equiv \mathscr{T} f_i$, when rewritten in matrix form as:

$$0 \equiv \big(\tau_1(a), \ldots, \tau_r(a)\big) \left(\frac{\partial \mathscr{U}_\alpha^i}{\partial a_j}(a)\right)_{1\leqslant j\leqslant r}^{\alpha\in\mathbb{N}^n,\,1\leqslant i\leqslant n}$$

just say that the transpose of $\mathrm{Jac}\,\mathsf{U}_\infty(a)$ has nonzero kernel at each a where the vector $\mathscr{T}\big|_a = \big(\tau_1(a), \ldots, \tau_r(a)\big)$ is nonzero. Consequently, $\mathrm{Jac}\,\mathsf{U}_\infty$ has rank strictly less than r locally in a neighborhood of every a_0, hence in the whole a-domain. So **(iii)** $\Rightarrow$ **(ii)**.

Conversely, assume that the generic rank ρ_∞ of $\mathrm{Jac}\,\mathsf{U}_\infty$ is $< r$. Then there exist $\rho_\infty < r$ "basic" coefficient functions $\mathscr{U}_{\alpha(1)}^{i(1)}, \ldots, \mathscr{U}_{\alpha(\rho_\infty)}^{i(\rho_\infty)}$ (there can be several choices) such that the generic rank of the extracted map $a \mapsto \big(\mathscr{U}_{\alpha(l)}^{i(l)}\big)_{1\leqslant l\leqslant\rho_\infty}$ equals ρ_∞ already. We abbreviate:

$$\mathsf{u}_l(a) := \mathscr{U}^{i(l)}_{\alpha(l)}(a) \qquad (l=1\cdots\rho_\infty).$$

The goal is to find vectorial local analytic solutions $\big(\tau_1(a),\dots,\tau_r(a)\big)$ to the infinite number of linear equations:

$$0 \equiv \tau_1(a)\,\frac{\partial \mathscr{U}^i_\alpha(a)}{\partial a_1}(a) + \cdots + \tau_r(a)\,\frac{\partial \mathscr{U}^i_\alpha(a)}{\partial a_r}(a) \qquad (i=1\cdots n;\ \alpha\in\mathbb{N}^n).$$

To begin with, we look for solutions of the finite, extracted linear system of ρ_∞ equations with the r unknowns $\tau_k(a)$:

$$\begin{cases} 0 \equiv \tau_1(a)\,\dfrac{\partial \mathsf{u}_1}{\partial a_1}(a) + \cdots + \tau_{\rho_\infty}(a)\,\dfrac{\partial \mathsf{u}_1}{\partial a_{\rho_\infty}}(a) + \cdots + \tau_r(a)\,\dfrac{\partial \mathsf{u}_1}{\partial a_r}(a) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 \equiv \tau_1(a)\,\dfrac{\partial \mathsf{u}_{\rho_\infty}}{\partial a_1}(a) + \cdots + \tau_{\rho_\infty}(a)\,\dfrac{\partial \mathsf{u}_{\rho_\infty}}{\partial a_{\rho_\infty}}(a) + \cdots + \tau_r(a)\,\dfrac{\partial \mathsf{u}_{\rho_\infty}}{\partial a_r}(a). \end{cases}$$

After possibly renumbering the variables $(a_1,\dots,a_r)$, we can assume that the left $\rho_\infty\times\rho_\infty$ minor of this system:

$$\Delta(a) := \det\left(\frac{\partial \mathsf{u}_l}{\partial a_m}(a)\right)^{1\leqslant l\leqslant\rho_\infty}_{1\leqslant m\leqslant\rho_\infty}$$

does not vanish identically. However, it can vanish at some points, and while endeavoring to solve the above linear system by an application of the classical Cramer rule, the necessary division by the determinant $\Delta(a)$ introduces poles that are undesirable, for we want the $\tau_k(a)$ to be analytic. So, for any μ with $1\leqslant\mu\leqslant r-\rho_\infty$, we look for a solution (rewritten as a derivation) in the specific form:

$$\mathscr{T}_\mu := -\Delta(a)\,\frac{\partial}{\partial a_{\rho_\infty+\mu}} + \sum_{1\leqslant k\leqslant\rho_\infty}\tau_{\mu k}(a)\,\frac{\partial}{\partial a_k} \qquad (\mu=1\cdots r-\rho_\infty),$$

in which we introduce in advance a factor $\Delta(a)$ designed to compensate the unavoidable division by $\Delta(a)$. Indeed, such a $\mathscr{T}_\mu$ will annihilate the u_l:

$$0 \equiv \mathscr{T}_\mu\,\mathsf{u}_1 \equiv \cdots \equiv \mathscr{T}_\mu\,\mathsf{u}_{\rho_\infty}$$

if and only its coefficients are solutions of the linear system: