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To the memory of Vladimir Igorevich Arnold

Preface

Numerous unsuccessful attempts to solve certain algebraic and differential equations "in finite terms" (i.e., "explicitly") led mathematicians to the belief that explicit solutions of such equations simply do not exist. This book is devoted to the question of the unsolvability of equations in finite terms, and in particular to the topological obstructions to solvability. This question has a rich history.

The first proofs of the unsolvability of algebraic equations by radicals were given by Abel and Galois. While thinking about the problem of explicit indefinite integration of an algebraic differential form, Abel laid the foundations for the theory of algebraic curves. Liouville continued Abel's work and proved that indefinite integrals of many algebraic and elementary differential forms are not elementary functions. Liouville was also the first to prove the unsolvability by quadratures of many linear differential equations.

It was Galois who first saw that the question of solvability by radicals is related to the properties of a certain finite group (now called the Galois group of an algebraic equation). Indeed, the notion of a finite group as introduced by Galois was motivated exactly by this question. Sophus Lie introduced the notion of a continuous transformation group while trying to solve differential equations explicitly by reducing them to a simpler form. To each linear differential equation, Picard associated its Galois group, which is a Lie group (and moreover, a linear algebraic group). Picard and Vessiot then showed that this particular group is responsible for the solvability of equations by quadratures. Next, Kolchin elaborated the theory of algebraic groups, completed the development of Picard–Vessiot theory, and generalized it to the case of holonomic systems of linear partial differential equations.

Vladimir Igorevich Arnold discovered that many classical questions in mathematics are unsolvable for topological reasons. In particular, he showed that a generic algebraic equation of degree 5 or higher is unsolvable by radicals precisely for topological reasons. Developing Arnold's approach, I constructed in the early 1970s a one-dimensional version of topological Galois theory. According to this theory, the way the Riemann surface of an analytic function covers the plane of complex numbers can obstruct the representability of this function by explicit formulas. The strongest known results on the unexpressibility of functions by explicit formulas have been obtained in this way. I had always been under the impression that a full-fledged multidimensional version of this theory was impossible. Then in spring 1999, I suddenly realized that, in fact, one can generalize the one-dimensional version of topological Galois theory to the multivariable case.

This book covers topological Galois theory. First, a complete and detailed exposition of the one-dimensional version is given, followed by a more schematic exposition of the multidimensional version. The topological theory is closely related to usual (algebraic) Galois theory as well as to differential Galois theory.

Algebraic Galois theory is simple, and its main ideas are connected with topological Galois theory. In the "permissive" part of topological Galois theory, not only is linear algebra used, but also results from Galois theory. In this book, Galois theory and its applications to the solvability of algebraic equations by radicals are presented with complete proofs. Apart from the problem of solvability by radicals, other closely related problems are also considered, including the problem of solvability of an equation with the help of radicals and auxiliary equations of degree at most k.

The main theorems of Picard–Vessiot theory are stated without proof, and the similarity with Galois theory is emphasized. We shall explain why, at least in principle, Picard–Vessiot theory answers the questions of solvability of linear differential equations in explicit form. The "permissive" part of topological Galois theory (which proves, in particular, that linear Fuchsian equations with solvable monodromy group are solvable by quadratures) uses only the simple, linear-algebraic, part of Picard–Vessiot theory. This linear algebra is covered in the book. The "prohibitive" part of topological Galois theory (which says, in particular, that linear differential equations with unsolvable monodromy group are not solvable by quadratures) will be explained in full detail. It is stronger than the "prohibitive" part of Picard–Vessiot theory.

This book also discusses beautiful constructions, due to Liouville, of the class of elementary functions, the class of functions expressible by quadratures, and so on, and his theory of elementary functions, which had a strong impact on all subsequent work in this area.

We will discuss three versions of Galois theory—algebraic, differential, and topological. These versions are unified by the same group-theoretic approach to the problems of solvability and unsolvability of equations. However, it is not true that all results on solvability and unsolvability are related to group theory. A number of brilliant results based on a different approach are contained in the theory of Liouville. To give a flavor of Liouville's theory, we provide a complete proof of his theorem stating that certain indefinite integrals are not elementary functions (this includes indefinite integrals of nonzero holomorphic differential forms on algebraic curves of higher genus).

We do not always follow the historical sequence of events. For example, the Picard–Vessiot theorem on the solvability of linear differential equations by quadratures was proved before the main theorem of differential Galois theory.

However, the Picard–Vessiot theorem is a direct corollary of this fundamental theorem, and it is presented here in this way.

A few words about the bibliography: The first modern book on integration in finite terms was written by Ritt [86]. Bronstein's book [16] contains a modern treatment of the subject together with many algorithms and includes much of what is in Sects. 1.6–1.9. Algebraic Galois theory is explained well in many textbooks; see, for example, [24, 25]. A clear and concise exposition of differential Galois theory is contained in Kaplansky's book [43]. For a more detailed and modern treatment, see the book [96] by van den Put and Singer. Kolchin's theory is covered in [64–67]. An interesting survey of work on the solvability and unsolvability of equations together with an extensive bibliography can be found in [93, 94].

My first results in topological Galois theory appeared in the early 1970s, when I was Arnold's student, to whom I am greatly indebted. Unfortunately, I did not publish my results in a timely manner: At first, I was unable to reconstruct the complicated history of the subject, and then I became interested in a totally different kind of mathematics. Much later, Andrei Bolibrukh convinced me to revisit the subject. My wife, Tatiana Belokrintskaya, prepared the Russian edition of this book for publication.

In this English-language edition, extra material has been added (Appendices A–D), the last two of which were written jointly with Yuri Burda. Vladlen Timorin and Valentina Kirichenko translated the Russian text into English. Michael Singer read the book and made many useful remarks and suggestions. David Kramer performed a careful editing of the book. I am grateful to all of them.

Toronto, Canada Askold Khovanskii

Contents

Chapter 1
Construction of Liouvillian Classes of Functions and Liouville's Theory

Some algebraic and differential equations are explicitly solvable. What does this mean? If an explicit solution is presented, the question answers itself. However, in most cases, every attempt to solve an equation explicitly is doomed to failure. We are then tempted to prove that certain equations have no explicit solutions. It is now necessary to define exactly what we mean by explicit solutions (otherwise, it is unclear what we are trying to prove).

From the modern viewpoint, the classical works on the subject lack rigorous definitions and statements of theorems. Nonetheless, it is clear that Liouville understood exactly what he was proving. He not only stated the problems on solvability of equations by elementary functions and by quadratures, but he also algebraized them. His work made it possible to define all such notions over an arbitrary differential field. But the standards of mathematical rigor were different in the time of Liouville. Indeed, according to Kolchin [64], even Picard failed to give accurate, unambiguous definitions. Kolchin's work satisfies modern standards, but his definitions are given for abstract differential fields from the very beginning.

However, the indefinite integral of an elementary function and the solution of a linear differential equation are functions rather than elements of an abstract differential field. In function spaces, for example, apart from differentiation and algebraic operations, an absolutely nonalgebraic operation is defined, namely composition. Anyhow, function spaces provide greater means for writing "explicit formulas" than abstract differential fields. Moreover, we should take into account that functions can be multivalued, can have singularities, and so on.

In function spaces, it is not hard to formalize the problem of unsolvability of equations in explicit form, and in this book, we are interested in this particular problem. One can proceed as follows: fix a class of functions and say that an equation is solvable explicitly if its solution belongs to this class. Different classes of functions correspond to different notions of solvability.

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1.1 Defining Classes of Functions by Lists of Basic Functions and Admissible Operations

A class of functions can be introduced by specifying a list of *basic functions* and a list of *admissible operations*. Given these two lists, the class of *functions* is defined as the set of all functions that can be obtained from the basic functions by repeated application of admissible operations. In Sect. 1.2, we define Liouvillian classes of functions in exactly this way.

Liouvillian classes of functions, which appear in problems of solvability in finite terms, contain multivalued functions. Thus the basic terminology should be made clear. In this section, we work with multivalued functions "globally," which leads to a more general understanding of classes of functions defined by lists of basic functions and admissible operations. In this global version, a multivalued function is regarded as a single entity, and we can define *operations on multivalued functions*.

The result of such an operation is a set of multivalued functions; every element of this set is referred to as a function obtained from the given functions by the given operation. The *class of functions* is defined as the set of all (multivalued) functions that can be obtained from the basic functions by repeated application of admissible operations.[1]

Let us define, for example, the sum of two multivalued functions of one variable.

Definition 1.1 Take an arbitrary point a on the complex line, a germ f_a of an analytic function f at the point a, and a germ g_a of an analytic function g at the same point a. We say that the multivalued function φ generated by the germ $\varphi_a = f_a + g_a$ is *representable as the sum of the functions f and g*.

For example, it is easy to see that exactly two functions are representable in the form $\sqrt{x} + \sqrt{x}$, namely, $f_1 = 2\sqrt{x}$ and $f_2 \equiv 0$. Other operations on multivalued functions are defined in exactly the same way. *For a class of multivalued functions, being stable under addition means that together with any pair of its functions, this class contains all functions representable as their sum.* The same applies to all other operations on multivalued functions understood in the same sense as above.

In the definition given above, it is not only the operation of addition that plays a key role but also the operation of analytic continuation hidden in the notion

[1] If f and g are multivalued functions and $\wedge$ is, say, a binary operation, then $f \wedge g$ is a set of multivalued functions. The class defined by a list $\{f_1, \dots, f_n\}$ of basic functions and a list $\{\wedge_1, \dots, \wedge_m\}$ of admissible binary operations is, by definition, the minimal set $\mathscr{C}$ of functions such that all $f_i \in \mathscr{C}$ and $f \wedge_j g \subseteq \mathscr{C}$ whenever $f, g \in \mathscr{C}$. An obvious modification can be made to include infinite sets of basic functions and admissible functions, such as unary, ternary, etc., operations.

of multivalued function. Indeed, consider the following example. Let f_1 be an analytic function defined on an open subset U of the complex line $\mathbb{C}^1$ and admitting no analytic continuation outside of U, and let f_2 be an analytic function on U given by the formula $f_2 = -f_1$. According to our definition, the zero function is representable in the form $f_1 + f_2$ *on the entire complex line*. From the commonly accepted viewpoint, the equality $f_1 + f_2 = 0$ holds inside the region U but not outside.

In working with multivalued functions globally, we do not insist on the existence of a *common region* where all necessary operations would be performed on single-valued branches of multivalued functions. A first operation can be performed in a first region, then a second operation can be performed in a second, different, region on analytic continuations of functions obtained in the first step. In essence, this more general understanding of operations is equivalent to including analytic continuation in the list of admissible operations on analytic germs. For functions of a single variable, it is possible to obtain topological obstructions even with this more general understanding of operations on multivalued analytic functions.

In the sequel, in considering topological obstructions to the membership of an analytic function of a single variable in a certain class, we will always mean this global definition of the function class via lists of basic functions and admissible operations.

For functions of several variables, things do not work in this general setting, and we are forced to adopt a more restrictive formulation (see Sect. 7.1.1) dealing with germs of functions. It is, however, no less natural, and perhaps even more so. The only place in the book where we use this more restrictive formulation is Chap. 7, in which we deal with multivariable functions.

1.2 Liouvillian Classes of Functions of a Single Variable

In this section, we define Liouvillian classes of functions of a single variable (for the multivariable case, the corresponding definitions are given in Chap. 7). We will describe these classes by lists of basic functions and admissible operations.

1.2.1 Functions of One Variable Representable by Radicals

List of basic functions:

- All complex constants
- An independent variable x

List of admissible operations:

- Arithmetic operations
- The operation of taking the nth root $\sqrt[n]{f}$, $n = 2, 3, \dots$, of a given function f

The function $g(x) = \sqrt[3]{5x + 2\sqrt[2]{x}} + \sqrt[7]{x^3 + 3}$ is an example of a function representable by radicals.

The famous problem of solvability of equations by radicals is related to this class. Consider the algebraic equation

$$y^n + r_1 y^{n-1} + \cdots + r_n = 0,$$

in which the r_i are rational functions of one variable. A complete answer to the question of solvability of such equations by radicals is given by Galois theory (see Chap. 2).

To define other classes, we will need the list of *basic elementary functions*. In essence, this list contains functions that are studied in high-school and college precalculus courses. They are the functions frequently found on pocket calculators.

List of basic elementary functions:

1. All complex constants and an independent variable x.
2. The exponential, the logarithm, and the power x^α, where α is any complex constant.
3. The trigonometric functions sine, cosine, tangent, cotangent.
4. The inverse trigonometric functions arcsine, arccosine, arctangent, arccotangent.

Let us now proceed with the list of *classical operations* on functions. We begin the list here. It will be continued in the following section.

List of classical operations:

1. The operation *composition* takes functions f, g to the function $f \circ g$.
2. The *arithmetic operations* take functions f and g to the functions $f + g$, $f - g$, fg, and f/g.
3. The operation *differentiation* takes a function f to the function f'.
4. The operation *integration* takes a function f to its indefinite integral y (i.e., to any function y such that $y' = f$; the function y is determined by the function f up to an additive constant).
5. The operation *solving an algebraic equation* takes functions $f_1, \dots, f_n$ to the function y such that $y^n + f_1 y^{n-1} + \cdots + f_n = 0$ (the function y is not quite uniquely determined by the functions $f_1, \dots, f_n$, since an algebraic equation of degree n can have n solutions).

We can now return to the definition of Liouvillian classes of functions of a single variable.

1.2.2 Elementary Functions of One Variable

List of basic functions:

- Basic elementary functions.

List of admissible operations:

- Compositions
- Arithmetic operations
- Differentiation

All elementary functions are given by formulas such as the following:

$$f(x) = \arctan(\exp(\sin x) + \cos x).$$

1.2.3 Functions of One Variable Representable by Quadrature

List of basic functions

- Basic elementary functions

List of admissible operations:

- Composition
- Arithmetic operations
- Differentiation
- Integration

For example, the elliptic integral

$$f(x) = \int_{x_0}^{x} \frac{dt}{\sqrt{P(t)}},$$

where P is a cubic polynomial, is representable by quadratures. However, Liouville showed that if the polynomial P has no multiple roots, then the function f is not elementary.

Generalized elementary functions of one variable This class of functions is defined in the same way as the class of elementary functions. We only need to add the operation of solving algebraic equations to the list of admissible operations.

Functions of one variable representable by generalized quadratures This class of functions is defined in the same way as the class of functions representable by quadratures. We only need to add the operation of solving algebraic equations to the list of admissible operations. Let us now define two more classes of functions similar to Liouvillian classes.

Functions of one variable representable by k-radicals This class of functions is defined in the same way as the class of functions representable by radicals. We only need to add the operation of solving algebraic equations of degree $\leq k$ to the list of admissible operations.

Functions of one variable representable by k-quadratures This class of functions is defined in the same way as the class of functions representable by quadratures. We only need to add the operation of solving algebraic equations of degree at most k to the list of admissible operations.

1.3 A Bit of History

The first rigorous proofs of unsolvability of some equations by quadratures and by elementary functions were obtained in the middle of the nineteenth century by Liouville (see [75–77, 86]). Later work of Chebyshev, Mordukhai-Boltovski, Ostrovskii, Ritt, Risch, Rosenlicht, Davenport, Singer, and Bronstein have elaborated on Liouville's results. A bibliography on this subject can be found in [93].

According to Liouville's theory of elementary functions, "sufficiently simple" equations have either "sufficiently simple" solutions or no explicit solutions at all. In some cases, the results go all the way to algorithms that either provide a proof of unsolvability of an equation in explicit form or construct an explicit solution.

Liouville's theory answers questions such as the following:

1. Under what conditions is an indefinite integral of an elementary function also an elementary function?
2. Under what conditions are all solutions of a linear differential equation all of whose coefficients are rational functions representable by generalized quadratures?

To demonstrate Liouville's method, we will give a proof of his theorem about integrals (see Sect. 1.6) and consider several applications of this theorem. Let $\alpha = R(z, u)\, dz$ be a 1-form, where R is a rational function of two variables, z is a complex variable, and u is a function of z. In Sect. 1.7, we consider the case that u is the natural logarithm of a rational function f of z, that is, $u = \log f(z)$. A procedure will be explained that allows us either to find an indefinite integral of α explicitly or to prove that it is not a generalized elementary function. In Sect. 1.8, a similar result is described in the case that u is the exponential of a rational function f of z, $u = \exp f(z)$. The case of an abelian 1-form α, where u is an algebraic function of z, is considered in Sect. 1.9. Necessary and sufficient conditions for the elementarity of an abelian integral are described. These conditions are hard to verify. In this sense, the algebraic case is more complicated than the logarithmic and the exponential cases. Sections 1.6–1.9 are not necessary for understanding the remainder of the book and can be omitted. To avoid references to these sections, we repeat, in Chap. 3, simple and short computations related to adjoining an integral, an exponential of an integral, and a root of an algebraic equation to a differential field.

In Sect. 1.4, we give significantly simpler definitions of Liouvillian classes of functions, due to Liouville (for example, that of the class of elementary functions). We explain how exactly Liouville succeeded in algebraizing the questions of solvability of equations by elementary functions or by other Liouvillian classes of functions. Liouville extensions of functional differential fields are constructed in Sect. 1.5.

In Sect. 1.10, we state some results from Liouville's theory concerning questions of solvability of linear differential equations. A more complete answer to this question is given by differential Galois theory (see Chap. 3).

1.4 New Definitions of Liouvillian Classes of Functions

Liouville algebraized the problem of solvability by elementary functions and by quadratures. The main obstacle in the algebraization is the absolutely nonalgebraic operation of composition. Liouville circumvented this obstacle in the following way: He associated to every function g from the list of basic functions the operation of postcomposition with this function. This operation takes a function f to the function $g \circ f$. Liouville noted that all basic elementary functions can be reduced to the logarithm and the exponential (see Lemma 1.2 below). The compositions $y = \exp f$ and $z = \log f$ can be regarded as solutions of the equations $y' = f'y$ and $z' = f'/f$. Thus, within Liouvillian classes of functions, it suffices to consider operations of solving some simple differential equations. After that, the solvability problem for Liouvillian classes of functions becomes differential-algebraic, and carries over to abstract differential fields. Let us proceed with the realization of this plan.

We will now continue the list of classical operations (the beginning of the list is given in the previous section).

List of classical operations (continued):

6. The operation *exponentiation* takes a function f to the function $\exp f$.
7. The operation of taking the logarithm, which we shall call *logarithmation*, takes a function f to the function $\log f$.

We will now give new definitions for transcendental Liouvillian classes of functions.

1.4.1 Elementary Functions of One Variable

List of basic functions:

- All complex constants
- An independent variable x

List of admissible operations:

- Exponentiation
- Logarithmation
- Arithmetic operations
- Differentiation

1.4.2 Functions of One Variable Representable by Quadratures

List of basic functions

- All complex constants

List of admissible operations:

- Exponentiation
- Arithmetic operations
- Differentiation
- Integration

1.4.3 Generalized Elementary Functions of One Variable and Functions of One Variable Representable by Generalized Quadratures and k-Quadratures

These functions are defined in the same way as the corresponding nongeneralized classes of functions; we have only to add the operation of solving algebraic equations or the operation of solving algebraic equations of degree $\leq k$ to the list of admissible operations.

Lemma 1.2 *Basic elementary functions can be expressed through exponentials and logarithms with the help of complex constants, arithmetic operations, and compositions.*

Proof For a power function x^α, the required expression is given by the equality $x^\alpha = \exp(\alpha \log x)$. For the trigonometric functions, the required expressions follow from Euler's formula $e^{a+bi} = e^a(\cos b + i \sin b)$. For real values of x, we have

$$\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right) \quad \text{and} \quad \cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right).$$

By analyticity, the same formulas remain true for all complex values of x. The tangent and the cotangent functions are expressed through the sine and the cosine. Let us now show that for all real x, the equality

$$\arctan x = \frac{1}{2i} \log z$$

holds, where

$$z = \frac{1+ix}{1-ix}.$$

Obviously,

$$|z| = 1, \quad \arg z = 2\arg(1+ix), \quad \tan(\arg(1+ix)) = x,$$

which proves the desired equality. By analyticity, the same equality also holds for all complex values of x. The remaining inverse trigonometric functions can be expressed through the arctangent. Namely,

$$\operatorname{arccot} x = \frac{\pi}{2} - \arctan x, \quad \arcsin x = \arctan \frac{x}{\sqrt{1-x^2}}, \quad \arccos = \frac{\pi}{2} - \arcsin x.$$

The square root that appears in the expression for the function arcsin can be expressed through the exponential and the logarithm: $x^{1/2} = \exp\left(\frac{1}{2}\log x\right)$. The lemma is proved. □

Theorem 1.3 *For every transcendental Liouvillian class of functions, the definitions in this section and those in Sect. 1.2 are equivalent.*

Proof In one direction, the theorem is obvious: it is clear that every function belonging to some Liouvillian class of functions in the sense of the new definition belongs to the same class in the sense of the old definition.

Let us prove the converse. By Lemma 1.2, the basic elementary functions lie in the class of elementary functions and in the class of generalized elementary functions in the sense of the new definition. It follows from the same lemma that the classes of functions representable by quadratures, generalized quadratures, and k-quadratures in the sense of the new definition also contain the basic elementary functions. Indeed, the independent variable x belongs to these classes, since it can be obtained as the integral of the constant function 1, since $x' = 1$. Instead of taking the logarithm, which is not among the admissible operations in these classes, one can use integration, since $(\log f)' = f'/f$.

It remains to show that the Liouvillian classes of functions in the sense of the new definition are stable under composition. The reason that they are is the following: composition commutes with all other operations that appear in the new definition of function classes, except for differentiation and integration. Thus, for example, the result of the operation exp applied to the composition $g \circ f$ coincides with the composition of the functions $\exp g$ and f, i.e., $\exp(g \circ f) = (\exp g) \circ f$. Similarly,

$$\log(g \circ f) = (\log g) \circ f,$$

$$(g_1 \pm g_2) \circ f = (g_1 \circ f) \pm (g_2 \circ f),$$
$$(g_1 g_2) \circ f = (g_1 \circ f)(g_2 \circ f),$$
$$(g_1/g_2) \circ f = (g_1 \circ f)/(g_2 \circ f).$$

If a function y satisfies an equation of the form $y^n + g_1 y^{n-1} + \cdots + g_n = 0$, then the function $(y \circ f)$ satisfies the equation $(y \circ f)^n + (g_1 \circ f)(y \circ f)^{n-1} + \cdots + (g_n \circ f) = 0$.

For differentiation and integration, we have the following simple commutation relations with the operation of composition: $(g)' \circ f = (g \circ f)'(f')^{-1}$ (if a function f is constant, then the function $(g)' \circ f$ is also constant), and if y is an indefinite integral of a function g, then $y \circ f$ is an indefinite integral of the function $(g \circ f)f'$ (in other words, composing the integral of a function g with a function f corresponds to the integration of the function $g \circ f$ multiplied by the function f').

This implies that the Liouvillian classes in the sense of the new definition are stable under composition. Indeed, if a function g is obtained from constants (or from constants and the independent variable) by operations discussed above, then the function $g \circ f$ is obtained by applying the same operations, or almost the same as in the case of integration and differentiation, to the function f. The theorem is proved. □

Remark 1.4 It is easy to see that differentiation can also be excluded from the lists of admissible operations for the Liouvillian classes of functions. To prove this, it suffices to use the explicit computation for the derivatives of the exponential and the logarithmic functions and the rules for differentiating formulas containing compositions and arithmetic operations. However, the exclusion of differentiation does not help in the problem of solvability of equations in finite terms (sometimes, the exclusion of differentiation makes it possible to state a result in a more invariant form; see the second formulation of Liouville's theorem on abelian integrals from Sect. 1.9).

1.5 Liouville Extensions of Abstract and Functional Differential Fields

A field K is said to be a *differential field* if an additive map $a \mapsto a'$ is defined that satisfies the Leibniz rule $(ab)' = a'b + ab'$. Such a map $a \mapsto a'$ is called a *derivation*. If a particular derivation is fixed, the element a' is sometimes called the *derivative* of a. The operation of taking derivatives is called *differentiation*.

An element y of a differential field K is called a *constant* if $y' = 0$. All constants in a differential field form a subfield, which is called the *field of constants*. In all cases that are of interest to us, the field of constants is the field of complex numbers. We shall always assume in the sequel that the differential field has characteristic zero and an algebraically closed field of constants.

An element y of a differential field is said to be

- An *exponential* of an element a if $y' = a'y$
- An *exponential of integral* of an element a if $y' = ay$ (we use "exponential of integral" as an indivisible term)
- A *logarithm* of an element a if $y' = a'/a$
- An *integral* of an element a if $y' = a$

In each of these cases, y is defined only up to an additive or multiplicative constant.

Suppose that a differential field K and a set M lie in some differential field F. The *adjunction* of the set M to the differential field K is the minimal differential field $K\langle M\rangle$ containing both the field K and the set M. We will refer to the transition from K to $K\langle M\rangle$ as *adjoining* the set M to the field K.

A differential field F containing a differential field K and having the same field of constants is said to be an *elementary extension* of the field K if there exists a chain of differential fields $K = F_1 \subseteq \cdots \subseteq F_n = F$ such that for every $i = 1, \dots, n-1$, the field $F_{i+1} = F_i\langle x_i\rangle$ is obtained by adjoining an element x_i to the field F_i, and x_i is an exponential or a logarithm of some element a_i from the field F_i. An element $a \in F$ is said to be *elementary* over K, $K \subset F$, if it is contained in some elementary extension of the field K.

A *generalized elementary extension*, a *Liouville extension*, a *generalized Liouville extension*, and a *k-Liouville extension* of a field K are defined in a similar way. In the construction of generalized elementary extensions, one is allowed to adjoin exponentials and logarithms and to take algebraic extensions. In the construction of Liouville extensions, one is allowed to adjoin integrals and exponentials of integrals. In generalized Liouville extensions and k-Liouville extensions, one is also allowed to take algebraic extensions and to adjoin solutions of algebraic equations of degree at most k. An element $a \in F$ is said to be *generalized elementary* (*representable by quadratures, by generalized quadratures, by k-quadratures*) over K, $K \subset F$, if a is contained in some generalized elementary extension (Liouville extension, generalized Liouville extension, k-Liouville extension) of the field K.

Remark 1.5 The equation for an exponential of integral is simpler than the equation for an exponential. That is why in the definition of Liouville extensions, etc., we adjoin exponentials of integrals. Instead, we could adjoin exponentials and integrals separately.

Let us now turn to functional differential fields. We will be dealing with this particular type of field in this book (although some results can be easily extended to abstract differential fields).

Let K be a subfield in the field of all meromorphic functions on a connected domain U of the Riemann sphere. Suppose that K contains all complex constants and is stable under differentiation (i.e., if $f \in K$, then $f' \in K$). Then K provides an example of a functional differential field. Let us now give a general definition. Let V, v be a pair consisting of a connected Riemann surface V and a meromorphic vector field v defined on it. The Lie derivative L_v along the vector field v acts on the field F of all meromorphic functions on the surface V and defines the derivation