Logic, Epistemology, and the Unity of Science 37

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Functions and Generality of Logic

Reflections on Dedekind's and Frege's Logicisms

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Logic, Epistemology, and the Unity of Science

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Preface

The present book results from three papers. Each of them has been written independently of the others by one of us, but they share a common philosophical and historical background. All of them question a too easy reading of the origins of logicism, one which assimilates different views and purposes, both with one another and with more modern (but not necessarily more appropriate) conceptions. The common aim is to emphasise nuances and peculiarities among different ways of pursuing a program which only very broadly could be described as the reduction of (a part of) mathematics to logic. Though mainly devoted to discuss (some of) Dedekind's and Frege's views, they also deal with other conceptions somehow connected with these, in particular some endorsed by Lagrange, Cauchy, Weierstrass, Hilbert, Russell, Ramsey and Carnap.

The papers, or some of their previous versions, have somehow circulated within the scientific community, but have all remained unpublished up to now. We decided to put them together in a single volume, both because of their dealing with a common topic and because of their complementarity. They stem from shared standpoints and conceptions concerning the particular subject of enquiry, as well as on matters of philosophical and historical methodology, and their final versions, which we present here, ensue from many exchanges among us. But they pursue different specific aims. We hope they could jointly contribute to a better and more detailed picture of a crucial event in the development of philosophy of mathematics and logic. The common questions which our papers deal with and their different intents have been described in a newly written, coauthored introduction.

The *Institute d'Histoire et Philosophie des Sciences et des Techniques* in Paris (IHPST) has been the common context of our research. It is an intellectual home for all of us. Though written independently of each other, our papers have been prompted by a number of discussions we had among us, and with a large number of colleagues at the IHPST, at its seminars and workshops, but also, and possibly above all, during the everyday life at the Institute. To put it in another way: our book is the outcome of the rich intellectual dynamic made possible within the IHPST.

But it owes a great deal also to other influences, suggestions and comments. The list of all those who variously contributed to this and would deserve our acknowledgement would be too long. Let us thank some of them, as representative of all the others, namely: Andrew Arana, Mark van Atten, Michael Beaney, Jean-Pierre Belna, Francesca Boccuni, Méven Cadet, Stefania Centrone, Annalisa Coliva, Sorin Costreie, Michael Detlefsen, Jean Dhombres, Jacques Dubucs, Giovanni Ferraro, José Ferreirós, Sébastien Gandon, Jean Gayon, Jeremy Gray, Niccolò Guicciardini, Brice Halimi, Raclavsky Jiri, Joseph Johnson, Gregory Landini, Paolo Mancosu, Sebastiano Moruzzi, Alberto Naibo, Fabrice Pataut, Carlo Penco, Eva Picardi, Dag Prawitz, Shahid Rahman, Philippe de Rouilhan, Andrea Sereni, Stewart Shapiro, Dirk Schlimm, François Schmitz, Wilfried Sieg, Göran Sundholm, Jamie Tappenden, Luca Tranchini, Gabriele Usberti and Pierre Wagner.

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Introduction

Logicism is usually presented as "the thesis that mathematics is reducible to logic", and is, then, "nothing but a part of logic". This is, at least, the way Carnap describes it in his influential 1931 paper ([41], p. 91, [10], p. 41). Though ascribing to Russell the role of "chief proponent" of it, Carnap also adds that "Frege was the first to espouse this view" (*ibid*).

Still, strictly speaking, Frege never argued for such a thesis. At most, he argued that arithmetic and real analysis are part of logic. But, also if it is so restricted, this thesis renders his view only very roughly. For what makes Frege's view distinctive is the way the inclusion relation between these mathematical theories and logic is conceived. And once this way is made clear, it also becomes clear that this relation does not depend, for him, on the mere possibility of a reduction of the former to the latter. Frege's point is, indeed, less that of showing how coming back from arithmetic and real analysis to logic, than that of developing logic enough so as to find natural and real numbers within it, and then show that what arithmetic and real analysis deal with is logical in nature.

Let us begin with arithmetic. In the *Vorwort* of *Grundgesetze*, he mentions the claim that "arithmetic is merely further developed logic [*weiter entwickelte Logik*]" ([97], *Vorwort,* p. VII, [110], p. VII_1), as the claim which he aims to argue for. This is only a rephrasing of the claim that Frege had taken himself to have established, though only informally, some years earlier, in the *Grundlagen*, namely that "Arithmetic is nothing but further pursued logic [*weiter ausgebildete Logik*], and every arithmetical statement is a law of logic, albeit a derived one" ([93], Sect. 87, [103], p. 99).[1] Frege's point seems, then, that "arithmetic is a branch [*Zweig*] of pure logic" ([97], *Einleitung*, pp. 1 and 3, [110], pp. 1_1 and 3_1) because the former results from an appropriate development (but not an extension), of the latter, that is,

[1]Here and later, from time to time, both in the present introduction and in the three following chapters, we feel free to slightly modify the English translations we quote, for sake of faithfulness to the original.

because "the simplest laws of cardinal number [*Anzahl*][2] [...][are] derived by logical means alone" ([97], *Einleitung*, p. 1, [110], p. 1_1).

The crucial point of Frege's arithmetical logicism consists in fixing these logical means. This, in turn, entails identifying appropriate logical laws (or axioms, in modern terminology), deductive rules, and definitions from and according to which arithmetical truths follow. The purpose of the first and second parts of *Grundgesetze* ([97], Sects. I.1–II.54)[3] is precisely that of fixing these means and using them for deriving these truths.

In Frege's mind, what ensures the logical nature of these laws, rules and definitions is that the laws and rules are appropriately general, while the definitions are explicit and have recourse, in their *definiendum*, only to linguistic tools already introduced by previous analogous definitions, or directly belonging to the language in which the laws are stated. The appropriate generality of the laws and rules is, in turn, ensured by the fact that all they concern is (the values of) a small number of basic functions, defined by merely appealing to two basic objects, the True and the False (whose existence is taken for granted), and to the totalities of objects and of first- and second-level one-argument functions (so as to avoid any appeal to each of these totalities of functions for defining a function belonging to it). In other terms, these laws and rules are general because they merely pertain to (the values of) some basic functions, which are defined by relying on no device used for selecting some specific portions or elements of these totalities other than the True and the False. Now, defining these basic functions results in fixing a language to be used to form either names of values of these functions or of whatever other functions resulting from appropriately composing them,[4] or general marks apt to "indeterminately indicate" [*unbestimmt andeuten*]" ([97], Sects. I.1, I.8, I.17, [110], pp. 5_1, 11_1, 31–32_1) these values. It follows, that, for Frege, the boundaries of logic are established by fixing a functional formal language, a small number of basic

[2]We agree with Ebert and Rossberg in translating Frege's term 'Anzahl', when used in a technical context, with 'cardinal number', by conserving 'number' for his term 'Zahl' (cf. [110], "Translators' Introduction", p. xvi). A reason for using 'cardinal number', rather than 'natural number' is that Frege explicitly distinguishes (both in *Grundlagen* and in *Grundgesetze*) *endlich Anzahlen* from *unendliche* ones, namely finite cardinal numbers from infinite ones, among the latter of which he pays particular attention to the *Anzahl Endlos*, the cardinal number belonging to the concept ⌜*endliche Anzahl* ⌝ (cf. [93], Sects. 84–86 and [97], *Vorwort*, p. 5, and Sects. I.122–157). Notice, moreover, that, in *Grundlagen*, Frege also uses twice (Sects. 19 and 43) the term 'natürliche Zahl' (to be mandatorily translated with 'natural number')—in the latter case, merely in a quote from Schröder, but in the former by speaking on his own behalf—and many times (Sects. 76–79, 81–84, 104, and 108) the term 'natürlichen Zahlenreihe' (to be mandatorily translated with 'series of natural numbers' or 'natural numbers series'). Though in *Grundgesetze* (Sects. I.43–46, I.66, I.88, I.100, I.104, etc.), this last term is replaced with 'Anzahlenreihe' (to be translated with 'series of cardinal numbers' or 'cardinal numbers series'), it seems, then, that Frege takes a natural number to be a finite cardinal one.

[3]We shall come back later on the third part.

[4]To be more precise, Frege does not admit a direct composition of functions. According to him, functions are rather composed indirectly, so to say, by composing the names of their values (cf. [37], pp. 29-30). We shall avoid here to insist on this subtleties.

truths stated in this language, and a small number of rules used to draw truths stated in this language from other such truths. To put it briefly, when he speaks of logic, Frege is referring to a well-identified and (in his mind) appropriately established formal system, and when he claims that arithmetic is a branch of logic, he is implying that arithmetical truths are nothing but theorems of this system.

As we shall see pretty soon, this is not as trivial as it may appear at first glance. But it is still compatible with a conception of arithmetical logicism as a reductionistic program. To see what makes Frege's arithmetical logicism much more than that, one has to consider another distinctive and essential aspect of it. This depends on Frege's considering that values of functions (of whatever level) are objects, and that "objects stand opposed to functions", to the effect that "everything that is not a function" is an object ([97], Sects. I.2, [110], p. 7_1). Insofar as functions are, for him, unsaturated, this entails that cardinal and, *a fortiori*, natural numbers could not but be objects, for him. It follows that Frege's arithmetical logicism involves the thesis that natural numbers are objects, namely logical objects—objects whose intrinsic nature is made manifest by explicit definitions stated in the language of logic—and arithmetical truths are truths about these objects. But, insofar as it seems quite clear that natural numbers cannot be the True and the False, arguing for this thesis requires admitting that the language of logic is enough for defining some objects other than the True and the False.

The problem arises, then: how can such other objects be defined through this language, provided that it merely results from defining the basic functions of logic, and this is done by merely appealing to the True, the False and to the totalities of objects and first- and second-level one-argument functions? The answer depends (and could not but depend) on Frege's countenance, among his basic functions, of a function having values other than the True and the False. This is the case of the value-ranges function: a second-level one-argument function taking first-level one-argument functions (without any restriction), and giving value-ranges. Still, given the defining on the way basic functions are defined, taking such a function as a basic one entails renouncing restrictions mentioned above it explicitly, and, then, admitting of an implicit definition for it. Frege's infamous Basic Law V provides such a non-explicit definition: it implicitly defines value-ranges by stating an identity condition for value-ranges of first-level one-argument functions, that is, by asserting, as it is well known, that the value-range of a first-level one-argument function $\Phi(\xi)$ is the same as that of a first-level one-argument function $\Psi(\xi)$ if and only if the value of $\Phi(\xi)$ is the same as that of $\Psi(\xi)$ for whatever argument, which in Frege's formal language is expressed thus: $(\grave{\varepsilon} f(\varepsilon) = \grave{\alpha} g(\alpha)) = (\underset{\mathfrak{a}}{\smile} f(\mathfrak{a}) = g(\mathfrak{a}))$, where '$f$' and '$g$' are marks used to indeterminately indicate first-level one-argument functions.

Frege was perfectly aware that, by admitting of such an implicit definition, he was derogating from the strict criterion of logicality that any other ingredient of his system meets. In the *Vorwort* of the *Grundgesetze* he recognises, indeed, that "a dispute" concerning the logical nature of this system "can arise […] only concerning […] Basic Law of value-ranges (V)" ([97], *Vorwort,* p. VII, [110], p. VII_1). Still, according to Frege, without this Law, and without value-ranges, there could not be other logical objects but the True and the False, and arithmetical logicism would, then, not be viable. This is what he openly claims in his tentative reply to Russell's paradox: "[…] even now I do not see how arithmetic can be founded scientifically, how the numbers can be apprehended as logical objects and brought under consideration, if it is not—at least conditionally—permissible to pass from a concept to its extension" ([97], *Nachwort,* p. 253, [110], p. 253_2).[5] Hence, for Frege, calling Basic Law V into question was not just calling into question his "approach to a foundation in particular, but rather the very possibility of any logical foundation of arithmetic" (*ibid.*). For, Frege seems to argue, if the value-range function is to be dismissed, what other logical function having other values than the True and the False is permissible? And if no such function may be permissible, how can natural numbers be logical objects? And if natural numbers are not logical objects, how can arithmetic be a branch of logic?

We know today that an alternative route for arithmetical logicism—allegedly understood, if not in the same way, at least in a way close to Frege's—has been suggested ([205], [118]). Still, it is clear that this route also depends on the admission of a basic function, namely the cardinal-number function, which, while being taken to be a logical function, is required to have as its possible values some particular objects whose existence is not a necessary condition for the admissibility of the relevant system of logic.

This is, in Frege's original terminology, a second-level one-argument function, like Frege's value-range one. And it is, like this latter function again, defined by a principle, namely Hume's principle, working as an axiom of the relevant system, and taken as an implicit definition. But, differently from Frege's value-range function, the cardinal-number function is not second-level insofar as its arguments are taken to be first-level functions. These arguments are rather taken to be concepts no more intended as functions from the totality of objects to the True and the False, but rather as the items designated by monadic first-order predicates.[6] The cardinal-number function is, thus, a total function, like the value-range one, only insofar as a previous restriction is, so to say, incorporated in the logical system its definition depends on: a restriction that makes the predicate variables of this system range only over concepts, rather than over items so generally conceived as to render the larger variety of Frege's first-level one-argument functions. This goes together with the fact that the values of the cardinal-number function are *ipso facto* cardinal numbers, rather than more general items among which cardinal, and, more specifically, natural numbers, are selected with the help of appropriate explicit definitions (which might suggest that this function is not general enough to count as logical in Frege's sense).

It is not our purpose, here, to discuss neologicism. Touching upon it is only meant to emphasise the main difficulty with Frege's logicism, by showing that,

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[5]Remember that a concept is, in Frege's terminology, a one-argument function whose values are either the True or the False, and its extension is nothing but its value-range.

[6]This entails that taking the cardinal-number function as a second-level function is imprecise, strictly speaking: this is, rather, a second-order function.

mutatis mutandis, it is still a crucial difficulty for its modern consistent version. This is the difficulty of fixing objects to be identified with natural numbers by having recourse only to means recognised as logical.

The way we have presented this difficulty hides a decisive aspect of it, however. This aspect only appears when it is made clear that, for Frege (as well as for the neologicists), nothing could be taken to be an object if it were not also taken to exist (in the only rightful sense in which anything can be taken to exist, both for him and for them), and no statement could be taken to be a truth if the singular terms and the first-order quantified variables included in it (if any) were not respectively taken to be names of, or to vary over existing individuals. The difficulty does not only consist, then, in defining natural numbers by having recourse only to means recognised as logical, but in doing it so as to ensure that these numbers exist, that is, that the (non-atomic) term that provides the *definiendum* of the explicit definition of each of them denotes an existing individual, and the (non-atomic) formula that provides the *definiendum* of the explicit definition of the property of being a natural number is satisfied by some (namely a countable infinity) of existing individuals (which means, in Frege's formalism, that the explicit definition of the first-level concept ⌜natural number⌝ designates a function whose value is the True for some, namely for countably many, arguments).

There is no room here for discussing the reason for neologicists to claim that their definitions comply with this condition. What is relevant is that, for Frege, no independent existence proof is needed for this purpose, since, for him, the relevant explicit definitions are so shaped as to ensure by themselves that this condition obtains. In other words, according to Frege, his explicit definition of each natural number directly exhibits an object to be identified with this number, while his definition of the property of being a natural number directly manifests that there are these numbers and which objects they are. This means that, according to Frege, these explicit definitions directly manifest that "there are logical objects" and that "the objects of arithmetic [i.e. the natural numbers] are such" ([97], Sect. II.147, [110], p. 149_2).

For real numbers, Frege does not seem to have thought that something like this would have been achievable, instead. Since, though he closed his informal exposition of the way he was planning to define these numbers by claiming that in this way he would have succeeded "in defining the real number purely arithmetically or logically as a ratio of magnitudes that are demonstrably there" ([97], Sect. II.164, [110], p. 162_2), his plan explicitly calls for an existence proof of domains of magnitudes going far beyond the simple inspection of the definition of these domains, and consisting, rather, in the independent exhibition of a particular domain of magnitudes generated from natural numbers. Frege actually fulfilled only a part of his plan: in the third part of *Grundgesetze* ([97], Sects. II.55–II.245), after having discussed and questioned several (informal) definitions of real numbers (*ibid*. II.55–II.155) and having exposed his plan (*ibid*. II.155–II.164), he proceeds to formally defining domains of magnitudes (*ibid*. II.165–II.245) and to prove some crucial properties of them, by leaving to a never appeared third volume of his treatise the accomplishment of the remaining part of the plan, including the existence proof of such domains, and the definition of real numbers as ratios over them.

Let $\mathrm{D}(\xi)$ be the first-level concept of domains of magnitudes, namely the concept under which an object falls if and only if it is a domain of magnitudes, which means that $\mathrm{D}(s)$ is the True if and only if s is such a domain. Frege's formal definition of domains of magnitudes consists in stating an identity like '$\mathrm{D}(s) = \mathcal{D}(s)$', where '$\mathcal{D}(s)$' stands for an appropriate (non-atomic) formula (*ibid.* II.173–174 and II.197).[7] In modern terminology, this means that domains of magnitudes are explicitly defined as the objects that satisfy this formula. And this formula is such that s satisfies it (which means, in Frege's terminology, that $\mathcal{D}(s)$ is the True) if and only if s is the extension of another first-level concept $\mathcal{M}(\xi)$, under which an object falls, in turn, if and only if it is the extension of a first-level binary relation that, if taken together with all the other extensions of a first-level binary relation that fall under this very concept, forms a certain structure.[8] This means that domains of magnitudes are explicitly defined as extensions of first-level concepts under which fall the extensions of some first-level binary relations that form, when taken all together, a certain structure.

This definition is stated within the same functional formal language in which natural numbers are defined. Still, Frege openly claims (*ibid.* II.164) that it does not ensure that there are objects that stand to each other in some binary relations whose extensions, when taken all together, meet the relevant structural condition, and are many enough for the ratios over them to be identified with the real numbers. And, he argues, if there were no such objects, real numbers could not be defined as ratios over domains of magnitudes. The existence proof of domains of magnitudes envisaged by Frege should have consisted in showing how, by starting from natural numbers and by appropriately operating on them, one can get enough—i.e., continuous many—other suitable objects. It is not necessary to enter the details of the way Frege planned to conduct this proof, in order to understand that he could not have imagined that the relevant objects could be directly exhibited by explicit definitions, as he held to have done for natural numbers. This, together with the fact that he held that his definition of domains of magnitudes does not secure, by itself, the existence of appropriate such domains, is enough for concluding that real numbers could not have been taken by Frege as logical objects in the same sense as natural numbers. Hence, his logicism about real numbers, once completely expounded in agreement with his plan, could not have appeared similar in nature to his arithmetical logicism.

These short and quite general remarks should be enough to make clear that Frege's logicism is quite complex a thesis, or better that it consists of two distinct quite complex theses, respectively, pertaining to natural and real numbers that are

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[7]As a matter of fact, this formula is not openly written by Frege, but it is easily deducible by other formulas which he openly writes.

[8]Remember that for Frege a binary first-level relation is a first-level two-arguments function whose values are either the True or the False.