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Preface

The theory of finite-dimensional division algebras witnessed several breakthroughs in the latter decades of the twentieth century. Important advances, such as Amitsur's construction of noncrossed product division algebras and Platonov's solution of the Tannaka–Artin problem, relied on an inventive use of valuation theory, applied in the context of noncommutative rings. The subsequent development of valuation theory for finite-dimensional division algebras led to significant simplifications of the initial results and to a host of new constructions of division algebras satisfying various conditions, which shed much light on the structure of these algebras. In this research area, valuation theory has become a standard tool, for which this book is intended to provide a useful reference.

The theory of valuations and valuation rings has been extended to division rings in several different ways. We treat here only the most stringent of these extensions, which is the one that has turned out to be most useful in applications. Thus, our valuations on division algebras are defined by the same axioms as the (Krull) valuations on fields; hence they restrict to a valuation in the classical sense on the center of the division ring. Yet, noncommutative valuation theory has some significant features that give it a different flavor from the commutative theory. Notably, there are many fewer valuations on division algebras than on fields: A valuation always extends from a field F to any field containing F; often there are many such extensions. But if D is a division algebra with center F and finite-dimensional over F, a valuation on F extends to D if and only if it has a *unique* extension to every field between F and D. Thus, very often it has no extension to D at all. But if it does extend, then the extension is unique. Consequently, the presence of a valuation on a division algebra D is a rather special phenomenon. When this occurs, it often gives a great deal of information about D and its subalgebras that can be virtually inaccessible for most division algebras. For this reason, valuation theory has had some of its greatest success in the construction of examples, such as noncrossed product algebras and division algebras with nontrivial reduced Whitehead group SK_1.

Henselian valuations on fields and Henselizations play a role for general valuations analogous to that of complete valuations and completions for rank 1 valuations. Henselian valuations on the center are even more important in the noncommutative theory because a Henselian valuation on a field F has a unique extension to each field algebraic over F. Consequently, it extends (uniquely) to each division algebra finite-dimensional over F. Much of the work on valued division algebra has thus focussed on algebras over Henselian fields. Also, notable results on arbitrary valued division algebras have been obtained by first proving the Henselian case (e.g., "Ostrowski's Theorem" on the defect of valued division algebras).

Another distinctive feature of valuation theory on division algebras is a greater complexity of the residue structure, and some notable interaction between the residue algebra and the value group: There is a canonical action of the value group of a valued division algebra on the center of its residue division algebra. This provides an important piece of information even in the most classical cases studied by Hasse in the 1930s, as it is related to the local invariant of division algebras over local fields.

When division algebras are being investigated, simple algebras with zero divisors frequently arise, e.g., as tensor products or scalar extensions of division algebras. Therefore, it has been a drawback for noncommutative valuation theory that valuations make sense only for division algebras: The basic axiom that

$$v(ab) \;=\; v(a) + v(b) \qquad (*)$$

breaks down if $ab = 0$ for nonzero a and b. A few years ago the authors found a way to address this difficulty by defining a more general notion of value function that we call a *gauge*, which can exist on a (finite-dimensional) semisimple algebra A over a field F, with respect to a valuation on F. For a function α on A to be a gauge, we replace the multiplicative condition $(*)$ for a valuation with the following *surmultiplicativity* condition:

$$\alpha(ab) \;\geq\; \alpha(a) + \alpha(b) \quad \text{for all } a, b \in A.$$

The filtration of A induced by α yields an associated graded algebra $\mathsf{gr}(A)$, and gauges are distinguished among surmultiplicative value functions by a condition on $\mathsf{gr}(A)$: This graded algebra must be graded semisimple, which means that it has no nilpotent homogeneous ideals.

Gauges work remarkably well. They show good behavior with respect to tensor products of algebras and scalar extensions. Moreover, there are natural constructions of gauges on many symbol algebras, cyclic algebras, and crossed product algebras. Additionally, over a Henselian base field, the gauges on the endomorphism algebra of a vector space are exactly the operator norms that are familiar in functional analysis.

Even for valuations on division algebras associated graded structures prove particularly useful. They encapsulate all the information about the residue algebra, the value group, and the canonical action of the value group on the center of the residue algebra. They should be regarded as a substantially enhanced analogue of the residue algebra. Usually, when one passes from a ring to an associated graded ring one obtains a simplified structure, but at the price of significant loss of information about the original ring. With valued division algebras, the graded ring is definitely simpler and easier to work with than the original valued algebra, with surprisingly little lost in the transition to the graded setting. Indeed, if the valuation on the center is Henselian, we will see that under mild tameness conditions (which hold automatically whenever the characteristic of the residue field is 0 or prime to the degree of the division algebra) the graded algebra $\mathsf{gr}(D)$ associated to a division algebra D determines D up to isomorphism; moreover, the graded subalgebras of $\mathsf{gr}(D)$ then classify the subalgebras of D.

Graded structures are thus central to our approach of valuation theory. Our general strategy is to prove results first in the graded setting, where the arguments are often easier and more transparent. With gauges at our disposal, the passage to the corresponding results for valued division algebras is often very quick. To take full advantage of this method, we build a solid foundation on graded algebras with grade set lying in a torsion-free abelian group. It is worth pointing out that, in contrast with the classical theory, which mostly deals with valuations with value group $\mathbb{Z}$, our valuations take their values in arbitrary totally ordered abelian groups. Valuations of higher rank (i.e., with value group not embeddable in $\mathbb{R}$) allow a greater richness in the possible structure of the value group and of the residue algebra. Moreover, new phenomena occur, such as totally ramified division algebras and algebras with noncyclic center of the residue—these have been particularly important in the construction of significant examples.

The material in this book can be roughly divided into three parts, which we briefly outline below, referring to the introduction of each chapter for additional information.

The first part consists of Chapters 1–4. They lay the groundwork for the theory of valuations on finite-dimensional division algebras and its extension to the theory of gauges on finite-dimensional semisimple algebras. The first chapter introduces the fundamental notions associated with valuations on division algebras and provides assorted examples. We view a valuation on the algebra as an extension of a known valuation on its center. In Chapter 2, the focus shifts to graded structures with a torsion-free abelian grade group. Graded rings in which the nonzero homogeneous elements are invertible are called graded division rings, because they display properties that are strikingly similar to those of the usual division rings. We are thus led to introduce graded vector spaces, and we develop a graded analogue of the Wedderburn and Noether theory of simple algebras. In Chapter 3, we return to the theme of valuations, which we extend to vector spaces and algebras over valued fields in order to define gauges on semisimple algebras. This first part of the book culminates in Chapter 4 with a determination of the necessary and sufficient condition for the existence of gauges. This condition involves the division algebras Brauer-equivalent to the simple components of the semisimple algebra after scalar extension to a Henselization of the base field: These division algebras must each be defectless, which means that their dimension over their center must be the product of the residue degree and the ramification index. In particular, gauges always exist when the residue characteristic is zero.

The second part, comprising Chapters 5–7, addresses various topics related to the Brauer group of valued fields. We first discuss graded field extensions in Chapter 5, and review properties of valued field extensions from the perspective of their associated graded field extensions. Brauer groups of graded fields and of valued fields form the subject of Chapter 6. Valuation-theoretic properties define an ascending sequence of three subgroups of the Brauer group $Br(F)$ of a valued field: the inertial part $Br_{in}(F)$, the inertially split part $Br_{is}(F)$, and the tamely ramified part $Br_{tr}(F)$. We use gauges to relate these subgroups to corresponding subgroups of the Brauer group $\mathsf{Br}(\mathsf{gr}(F))$ of the associated graded field $\mathsf{gr}(F)$. The main result of this part of the book yields for a Henselian field F a canonical index-preserving isomorphism $Br_{tr}(F) \xrightarrow{\sim} \mathsf{Br}(\mathsf{gr}(F))$ mapping the Brauer class of a tame division algebra D to the Brauer class of $\mathsf{gr}(D)$. We can then easily read off information about the pieces of $Br_{tr}(F)$ from the corresponding data about $\mathsf{Br}(\mathsf{gr}(F))$. The inertial, or unramified part of the Brauer group is canonically isomorphic to the Brauer group of the residue field: $Br_{in}(F) \cong Br(\overline{F})$. The inertially split part $Br_{is}(F)$ consists of the classes of division algebras split by the maximal inertial (= unramified) extension field of F. We give a generalization of Witt's classical description of the Brauer group of a complete discretely-valued field, in the form of a "ramification" isomorphism from the quotient $Br_{is}(F)/\,Br_{in}(F)$ to a group of characters of the absolute Galois group of the residue field $\overline{F}$. The next quotient $Br_{tr}(F)/\,Br_{is}(F)$ is described in Chapter 7, where division algebras totally ramified over their centers are thoroughly investigated. When the base field is Henselian, the properties of such algebras can be read off from the extension of value groups, with the help of a canonical alternating pairing with values in the group of roots of unity of the residue field. Since totally ramified division algebras arise only when the value group has rank at least 2, such algebras have been relatively less studied in the literature; yet their structure is very simple and explicit.

In the third part of the book, Chapters 8–12, we apply the preceding results to investigate the structure of division algebras over Henselian fields, and we present several applications. Following the same methodology as in previous chapters, in Chapter 8 we first consider the structure of graded division algebras; we then derive corresponding structure theorems for division algebras over Henselian fields by relating the algebra to its associated graded algebra. We thereby recover easily several results that have been previously established by much more complicated methods. Historically, a primary application of valuation theory has been in the construction of significant examples. Our last four chapters are devoted to the presentation of such examples. In Chapter 9 we obtain information on the maximal subfields and splitting fields of valued division algebras, and construct noncyclic division algebras with pure maximal subfields, noncyclic p-algebras, and noncrossed product algebras. Examples of division algebras that do not decompose into tensor products of proper subalgebras are given in Chapter 10, and Chapter 11 discusses reduced Whitehead group computations: We show that if D is a division algebra tamely ramified over a Henselian field then $SK_1(D) \cong SK_1(\mathsf{gr}(D))$. This leads to quick proofs of many formulas for $SK_1(D)$. Finally, we give in Chapter 12 a modified version of recent results of Merkurjev and Baek–Merkurjev using valuation theory to obtain lower bounds on the essential dimension of central simple algebras of given degree and exponent.

The assumed background for this book is acquaintance with the classical theory of central simple algebras, together with a basic knowledge of the valuation theory of fields, as given for example in Bourbaki, *Algèbre Commutative*, Ch. VI. For the convenience of the reader, we have included an appendix covering some of the more technical facts we need in commutative valuation theory, especially concerning Henselian valuations and Henselizations. The theoretical aspects developed throughout the book are illustrated by many examples, which are listed by chapter in another appendix.

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A note on notation

As pointed out above, we compare throughout most of the book algebras over valued fields and graded algebras. As a visual aid to help the reader determine whether a given statement lies in the context of graded algebras, we use sans serif letters (A, F, V, ...) to designate graded structures and associated constructions. Thus, for instance $\mathsf{End}_{\mathsf{D}}(\mathsf{V})$ denotes the graded algebra of endomorphisms of the graded vector space V over the graded division algebra D, and we write $\mathsf{gr}(D)$ for the graded algebra associated to the valued division algebra D, and $\mathsf{Br}(\mathsf{F})$ for the Brauer group of a graded field F.

The blackboard bold symbols $\mathbb{C}$, $\mathbb{F}_q$, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{Q}_p$, $\mathbb{R}$, and $\mathbb{Z}$ have their customary meanings: the complex numbers, the finite field of cardinality q, the nonnegative integers, the rational numbers, the p-adic completion of $\mathbb{Q}$, the real numbers, and the integers.

Louvain-la-Neuve, La Jolla
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Contents

Chapter 1
Valuations on Division Rings

In this chapter we introduce the central object of study in this book: valuations on division algebras D finite-dimensional over their centers. In §1.1 we define valuations (not assuming finite-dimensionality of D) and describe the associated structures familiar from commutative valuation theory: the valuation ring $\mathcal{O}_D$, its unique maximal left and maximal right ideal $\mathfrak{m}_D$, the residue division algebra $\overline{D}$, and the value group Γ_D. While the residue field $\overline{Z(D)}$ of the center of D lies in the center $Z(\overline{D})$ of $\overline{D}$, the inclusion is often strict. We describe an important and distinctively noncommutative feature, namely a canonical homomorphism θ_D from Γ_D to the automorphism group $Aut\big(Z(\overline{D})\big/\,\overline{Z(D)}\,\big)$; θ_D is induced by conjugation by elements of $D^\times$.

In §1.2 we focus on a division algebra D finite-dimensional over its center F. We prove the "Fundamental Inequality" for valued division algebras. We then look at valuations on D from the perspective of F. We show that a valuation on F has at most one extension to D, and prove a criterion for when such an extension exists. When this occurs, we show that $Z(\overline{D})$ is a finite-dimensional normal field extension of $\overline{F}$ and that θ_D is surjective. We also describe the technical adjustments needed to apply the classical method of "composition" of valuations to division algebras.

The filtration on D induced by a valuation leads to an associated graded ring $\mathsf{gr}(D)$, which we describe in §1.3. Throughout the book we emphasize use of $\mathsf{gr}(D)$ to help understand the valuation on D. We give many examples of division algebras with valuations throughout the chapter.

1.1 Basic definitions and examples

The fundamental structures associated to valuations on division rings are defined in §1.1.1, and illustrated in §1.1.2–§1.1.4 for several examples obtained by various kinds of series constructions.

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1.1.1 Valuations and associated structures

The valuations we consider on a division ring D are functions

$$v \colon D \longrightarrow \Gamma \cup \{\infty\}$$

where Γ is a totally ordered additive abelian group and ∞ is a symbol such that $\gamma < \infty$ and $\gamma + \infty = \infty + \infty = \infty$ for all $\gamma \in \Gamma$, subject to the following conditions: for all $x, y \in D$,

(i) $v(x) = \infty$ if and only if $x = 0$;
(ii) $v(x+y) \geq min\big(v(x), v(y)\big)$;
(iii) $v(xy) \ = \ v(x) + v(y)$.

Thus, the restriction of v to the multiplicative group of units $D^\times$ is a group homomorphism $D^\times \to \Gamma$. It readily follows that

$$v(1) \ = \ 0 \qquad \text{and} \qquad v(x^{-1}) \ = \ -v(x) \quad \text{for all } x \in D^\times.$$

Also, since Γ has no torsion, we have

$$v(-1) \ = \ 0 \qquad \text{hence} \qquad v(-x) \ = \ v(x) \quad \text{for all } x \in D.$$

By writing $x = (x+y)-y$, it follows from (ii) that $v(x) \geq min\big(v(x+y), v(y)\big)$; hence, $v(x+y) = v(x)$ if $v(y) > v(x)$. Therefore, we have for all $x, y \in D$

$$v(x+y) \ = \ min\big(v(x), v(y)\big) \qquad \text{if } v(x) \neq v(y).$$

Associated to the valuation v there are the following structures:

- $\Gamma_D = v(D^\times)$, the *value group* of v, which is a subgroup of Γ;
- $\mathcal{O}_D = \{x \in D \mid v(x) \geq 0\}$, which is a subring of D called the *valuation ring* of D;
- $\mathfrak{m}_D = \{x \in D \mid v(x) > 0\}$, which is a two-sided ideal that is the unique maximal left and maximal right ideal of $\mathcal{O}_D$ since the group of units in $\mathcal{O}_D$ is $\mathcal{O}_D^\times = \{x \in D \mid v(x) = 0\} = \mathcal{O}_D \setminus \mathfrak{m}_D$;
- $\overline{D} = \mathcal{O}_D/\mathfrak{m}_D$, the residue division ring.

If we need to specify the valuation on D, we will write $\Gamma_{D,v}$, $\mathcal{O}_{D,v}$, $\mathfrak{m}_{D,v}$, and $\overline{D}^v$. But most of the time we will be considering only one valuation on D, and the simpler notation will suffice.

For $x \in \mathcal{O}_D$ we let $\overline{x}$ be the image of x in $\overline{D}$,

$$\overline{x} \ = \ x + \mathfrak{m}_D \ \in \ \overline{D}.$$

A distinctive feature of noncommutative valuation theory is the interaction between the value group and the residue division ring, which takes the following form: any valuation v on a division ring D restricts to a valuation on its center $Z(D)$, and we may consider $\overline{Z(D)} \subseteq \overline{D}$. Clearly, we have

$$\overline{Z(D)} \ \subseteq \ Z(\overline{D}).$$

Certain automorphisms of this field extension are associated to elements in the value group: any $d \in D^\times$ determines the inner automorphism $int(d): D \to D$ given by $x \mapsto dxd^{-1}$. Since $v\big(int(d)(x)\big) = v(x)$, we have

$$int(d)(\mathcal{O}_D) = \mathcal{O}_D \qquad \text{and} \qquad int(d)(\mathfrak{m}_D) = \mathfrak{m}_D,$$

so $int(d)$ induces an automorphism $\overline{int(d)}$ of $\overline{D}$, hence by restriction an automorphism of $Z(\overline{D})$ fixing every element of $\overline{Z(D)}$. If $v(d) = 0$, then $\overline{int(d)} = int(\overline{d})$, and the restriction of $int(\overline{d})$ to $Z(\overline{D})$ is the identity. Thus, conjugation induces a group homomorphism mapping $D^\times/\mathcal{O}_D^\times$ to $Aut\big(Z(\overline{D})/\,\overline{Z(D)}\,\big)$. On the other hand, v induces an isomorphism $D^\times/\mathcal{O}_D^\times \xrightarrow{\sim} \Gamma_D$, so there is a well-defined group homomorphism

$$\theta_D\colon \Gamma_D \longrightarrow Aut\big(Z(\overline{D})/\,\overline{Z(D)}\,\big), \tag{1.1}$$

which can be described as follows: for any $\gamma \in \Gamma_D$ and any $d \in D^\times$ with $v(d) = \gamma$ and any $x \in \mathcal{O}_D$ with $\overline{x} \in Z(\overline{D})$,

$$\theta_D(\gamma)(\overline{x}) = \overline{dxd^{-1}}.$$

We call θ_D the *canonical homomorphism* of the valuation v on D. It is clear that $\Gamma_{Z(D)} \subseteq ker\,\theta_D$, so we may also consider the induced homomorphism

$$\overline{\theta}_D\colon \Gamma_D/\Gamma_{Z(D)} \longrightarrow Aut\big(Z(\overline{D})/\,\overline{Z(D)}\,\big).$$

1.1.2 Examples: twisted Laurent series

Let A be a division ring, and let σ be an automorphism of A. The *twisted Laurent series ring* $A((x;\sigma))$ is defined as the set of formal series

$$\sum_{i=k}^{\infty} a_i x^i, \qquad \text{where } k \in \mathbb{Z} \text{ and } a_i \in A \text{ for all } i.$$

The addition is defined as usual, and multiplication is given by

$$\sum_i a_i x^i \cdot \sum_j b_j x^j = \sum_{i,j} a_i \sigma^i(b_j) x^{i+j} \qquad \text{for } a_i,\, b_j \in A.$$

Let $D = A((x;\sigma))$. For $d = \sum_{i=k}^{\infty} a_i x^i \in D$, let

$$supp(d) = \{i \in \mathbb{Z} \mid a_i \neq 0\} \quad \text{and} \quad v_x(d) = min\big(supp(d)\big) \ \ (\text{so } v_x(0) = \infty).$$

If $v_x(d) > 0$, then the element $1 + d + d^2 + \ldots$ is defined in D, and

$$(1-d)(1 + d + d^2 + \ldots) = 1 = (1 + d + d^2 + \ldots)(1-d),$$

so $1 - d$ is invertible. It is then easy to see that D is a division ring: for an arbitrary nonzero element $d = \sum_{i=k}^{\infty} a_i x^i$ with $a_k \neq 0$, we have

$$\sigma^{-k}(a_k^{-1})x^{-k}d = x^{-k}a_k^{-1}d = 1 - d_0$$

and

$$d\sigma^{-k}(a_k^{-1})x^{-k} = dx^{-k}a_k^{-1} = 1 - d_1$$

for some d_0, $d_1 \in D$ with $v_x(d_0)$, $v_x(d_1) > 0$. As $1 - d_0$ and $1 - d_1$ are invertible, it follows that d is invertible. Therefore, D is a division ring. It is easy to check that v_x is a valuation on D with $\Gamma_D = \mathbb{Z}$. It is known as the *x-adic valuation* on D. Clearly $\overline{D} = A$.

A series $\sum_{i=k}^{\infty} a_i x^i \in D$ lies in the center of D if and only if it commutes with x and centralizes A. Therefore,

$$Z(D) = \Big\{\sum_{i=k}^{\infty} a_i x^i \mid \sigma(a_i) = a_i \text{ and } ba_i = a_i\sigma^i(b) \text{ for all } i \text{ and all } b \in A\Big\}.$$

In particular, the residue field of $Z(D)$ is the subfield of $Z(A)$ fixed under σ,

$$\overline{Z(D)} = Z(A)^{\sigma}.$$

The homomorphism $\theta_D \colon \mathbb{Z} \to Aut(Z(A)/Z(A)^{\sigma})$ maps $1 \in \mathbb{Z}$ to $\sigma|_{Z(A)}$.

1.1.3 Examples: iterated Laurent series

The construction above can of course be iterated: if τ is an automorphism of $A((x;\sigma))$ we may consider the division ring $A((x;\sigma))((y;\tau))$. This division ring carries the y-adic valuation v_y with value group $\mathbb{Z}$, but it also has a composite valuation $v_x * v_y$ with value group $\mathbb{Z}^2$, as we will see shortly. We will make use of the following result:

Proposition 1.1. *Every automorphism τ of $A((x;\sigma))$ preserves the x-adic valuation v_x, i.e., $v_x \circ \tau = v_x$.*

Proof. We need to show that $v_x\big(\tau(d)\big) = v_x(d)$ for all $d \in A((x;\sigma))$. We proceed in four steps:

Step 1: If $v_x(d) > 0$, then $v_x\big(\tau(d)\big) \geq 0$. Let n be any positive integer prime to the characteristic of A. By substituting d for the variable X in the Taylor expansion of the function $\sqrt[n]{1+X}$, we obtain a series $s \in A((x;\sigma))$ such that $s^n = 1 + d$, hence also $\tau(s)^n = 1 + \tau(d)$. If $v_x\big(\tau(d)\big) < 0$, this equality shows that $v_x\big(\tau(d)\big) = n\, v_x\big(\tau(s)\big) \in n\mathbb{Z}$. This relation cannot hold for infinitely many integers n, hence it is impossible that $v_x\big(\tau(d)\big) < 0$.

Step 2: If $v_x(d) = 0$, then $v_x\big(\tau(d)\big) = 0$. Suppose instead that $v_x\big(\tau(d)\big) \neq 0$. We may then find an integer $z \in \mathbb{Z}$ such that $z\, v_x\big(\tau(d)\big) < -v_x\big(\tau(x)\big)$. Then $v_x\big(\tau(d^z x)\big) = z\, v_x\big(\tau(d)\big) + v_x\big(\tau(x)\big) < 0$ while $v_x(d^z x) = 1$, a contradiction to step 1.

Step 3: If $d \neq 0$, then $v_x\big(\tau(d)\big) = v_x(d) v_x\big(\tau(x)\big)$. Let $z = v_x(d)$. We have $v_x(d^{-1}x^z) = 0$; hence step 2 yields $v_x\big(\tau(d^{-1}x^z)\big) = 0$, and it follows that $v_x\big(\tau(d)\big) = z\, v_x\big(\tau(x)\big)$.

Step 4: $v_x\big(\tau(x)\big) = 1$. Step 1 shows that $v_x\big(\tau(x)\big) \geq 0$, and step 3 shows that $v_x\big(\tau(x)\big)$ divides $v_x\big(\tau(d)\big)$ for all nonzero $d \in A((x;\sigma))$. Since τ is onto, $v_x\big(\tau(x)\big)$ divides every integer, so $v_x\big(\tau(x)\big) = 1$.

The proposition follows from steps 3 and 4. □

One way to build an automorphism on $A((x,\sigma))$ is by extending an automorphism of A:

Lemma 1.2. *Suppose ρ is an automorphism of A, and suppose there is a $b \in A^\times$ with $int(b)\sigma\rho = \rho\sigma$. Then, there is an automorphism $\widehat{\rho}$ of $A((x;\sigma))$ with $\widehat{\rho}|_A = \rho$ and $\widehat{\rho}(x) = bx$.*

Proof. Define $\widehat{\rho}$ by $\widehat{\rho}\big(\sum_{i=k}^{\infty} a_i x^i\big) = \sum_{i=k}^{\infty} \rho(a_i)(bx)^i$. The hypothesis on b implies that $(bx)\rho(c) = \rho\sigma(c)(bx)$ and $(bx)^{-1}\rho(c) = \rho\sigma^{-1}(c)(bx)^{-1}$. Hence, by upward and downward induction on i,

$$(bx)^i\rho(c) \;=\; \rho\sigma^i(c)\,(bx)^i \qquad \text{for all } i \in \mathbb{Z},\ c \in A. \tag{1.2}$$

Clearly $\widehat{\rho}(1) = 1$ and $\widehat{\rho}(s+t) = \widehat{\rho}(s) + \widehat{\rho}(t)$ for all $s, t \in A((x;\sigma))$. To verify that $\widehat{\rho}(st) = \widehat{\rho}(s)\widehat{\rho}(t)$, it suffices to check this for monomials. Say $s = ax^i$ and $t = cx^j$ with $a, c \in A$, $i, j \in \mathbb{Z}$. Then, using (1.2),

$$\begin{aligned} \widehat{\rho}(st) &= \widehat{\rho}\big(a\sigma^i(c)x^{i+j}\big) \;=\; \rho(a)\rho\sigma^i(c)(bx)^{i+j} \\ &= \big[\rho(a)(bx)^i\big]\big[\rho(c)(bx)^j\big] \;=\; \widehat{\rho}(s)\,\widehat{\rho}(t), \end{aligned}$$

as desired. □

Now, fix some automorphism τ of $A((x;\sigma))$ and consider the division ring $E = A((x;\sigma))((y;\tau))$. Let v_x be the x-adic valuation on $A((x;\sigma))$ and v_y the y-adic valuation on E. For any nonzero series $s = \sum_{i=k}^{\infty} d_i y^i \in E$ with $d_i \in A((x;\sigma))$ define

$$v_{x,y}(s) \;=\; (v_x(d_k), k) \in \mathbb{Z}^2 \qquad \text{if } d_k \neq 0.$$

Let also $v_{x,y}(0) = \infty$. If $s = d_k y^k + \sum_{i>k} d_i y^i$ and $s' = d'_\ell y^\ell + \sum_{j>\ell} d'_j y^j$, then

$$ss' \;=\; d_k\tau^k(d'_\ell)y^{k+\ell} + \sum_{i+j>k+\ell} d_i\tau^i(d'_j)y^{i+j}.$$

Since $v_x\big(d_k\tau^k(d'_\ell)\big) = v_x(d_k) + v_x(d'_\ell)$ by Prop. 1.1, we have

$$v_{x,y}(ss') \;=\; v_{x,y}(s) + v_{x,y}(s').$$

It is also easy to see that $v_{x,y}(s+s') \geq min\big(v_{x,y}(s), v_{x,y}(s')\big)$ when $\mathbb{Z}^2$ is given the right-to-left lexicographic ordering: indeed we have $v_{x,y}(x+y) = (1,0)$, hence $(1,0) < (0,1)$. The map $v_{x,y}$ is thus a valuation on E with value group $\mathbb{Z}^2$. It is called the (x,y)*-adic valuation*, which is the composite $v_x * v_y$ of the y-adic valuation v_y on E and the x-adic valuation v_x on the residue division ring $\overline{E}^{\,v_y} = A((x;\sigma))$. See Exercise 1.2 for a general construction of composite valuations on division rings of Laurent series.

With a view toward the general construction of Mal'cev–Neumann series below, observe that the nonzero elements in E are formal series $s = \sum_{i,j} a_{i,j} x^i y^j$ (with $a_{i,j} \in A$) for which the set $S = \{(i,j) \in \mathbb{Z}^2 \mid a_{i,j} \neq 0\}$, called the *support* of s, has a characteristic property with respect to the right-to-left lexicographic ordering on $\mathbb{Z}^2$: it has a minimum, which is $v_{x,y}(s)$, but also every subset of S has a minimum because for $(i,j) \in S$ there are only finitely many $k \in \mathbb{Z}$ such that $k \leq i$ and $(k,j) \in S$. This property is expressed by saying that the set S is *well-ordered* for the right-to-left lexicographic ordering. Conversely, if $S \subset \mathbb{Z}^2$ is any nonempty well-ordered subset, then the series $\sum_{(i,j)\in S} a_{i,j} x^i y^j$ lies in E for any choice of the coefficients $a_{i,j} \in A$.

This construction can be used inductively to obtain iterated twisted Laurent series rings with a valuation with value group $\mathbb{Z}^n$, for any integer $n \geq 1$. As a typical example, consider a field M with a family $\boldsymbol{\sigma} = (\sigma_i)_{i=1}^n$ of pairwise commuting automorphisms, and fix a collection $\boldsymbol{u} = (u_{i,j})_{i,j=1}^n$ of elements in $M^\times$ subject to the following conditions:

$$u_{i,i} = 1, \quad u_{i,j} u_{j,i} = 1, \quad u_{i,j} u_{j,k} u_{k,i} = \sigma_k(u_{i,j}) \sigma_i(u_{j,k}) \sigma_j(u_{k,i}) \tag{1.3}$$

for all i, j, $k = 1, \ldots, n$. Consider the division ring of iterated Laurent series

$$\mathcal{L}((M; \boldsymbol{\sigma}, \boldsymbol{u})) = M((x_1; \sigma_1))((x_2; \widehat{\sigma_2})) \ldots ((x_n; \widehat{\sigma_n})), \tag{1.4}$$

where the automorphism $\widehat{\sigma_i}$ of $M((x_1; \sigma_1))((x_2; \widehat{\sigma_2})) \ldots ((x_{i-1}; \widehat{\sigma_{i-1}}))$ is defined for $i = 2, \ldots, n$ by

$$\widehat{\sigma_i}\big(\textstyle\sum m_{k_1 \ldots k_{i-1}} x_1^{k_1} \ldots x_{i-1}^{k_{i-1}}\big) = \textstyle\sum \sigma_i(m_{k_1 \ldots k_{i-1}}) (u_{i,1} x_1)^{k_1} \ldots (u_{i,i-1} x_{i-1})^{k_{i-1}}.$$

To see that $\widehat{\sigma_i}$ is an automorphism, assume inductively that $\widehat{\sigma_2}, \ldots, \widehat{\sigma_{i-1}}$ are automorphisms, and let

$$A_0 = M \quad \text{and} \quad A_j = M((x_1; \sigma_1))((x_2; \widehat{\sigma_2})) \ldots ((x_j; \widehat{\sigma_j}))$$

for $1 \leq j \leq i-1$, and let $\sigma_{i,j} = \widehat{\sigma_i}|_{A_j}$. Then, $\sigma_{i,0}$ is the automorphism σ_i on M. For ℓ with $1 \leq \ell \leq i-1$, assume $\sigma_{i,\ell-1}$ is an automorphism of $A_{\ell-1}$. We apply Lemma 1.2 to $A_\ell = A_{\ell-1}((x_\ell; \widehat{\sigma_\ell}))$ with $\rho = \sigma_{i,\ell-1}$ and $b = u_{i,\ell}$ to see that $\widehat{\rho} = \sigma_{i,\ell}$ is an automorphism of A_ℓ. For this we need

$$\mathit{int}(u_{i,\ell})\, \widehat{\sigma_\ell}\, \sigma_{i,\ell-1}(a) = \sigma_{i,\ell-1}\, \widehat{\sigma_\ell}(a) \qquad \text{for all } a \in A_{\ell-1}. \tag{1.5}$$

It suffices to check this equality for $a \in M$ and $a = x_j$ for $1 \leq j \leq \ell - 1$. It holds for $a \in M$ as M is commutative, $\widehat{\sigma_\ell}|_M = \sigma_\ell$, $\sigma_{i,\ell-1}|_M = \sigma_i$, and $\sigma_i \sigma_\ell = \sigma_\ell \sigma_i$. For $a = x_j$, (1.5) becomes

$$u_{i,\ell}\, \sigma_\ell(u_{i,j,})\, u_{\ell,j}\, x_j\, u_{i,\ell}^{-1} = \sigma_i(u_{\ell,j})\, u_{i,j}\, x_j,$$

which holds by (1.3) as $x_j u_{i,\ell}^{-1} = \sigma_j(u_{i,\ell}^{-1}) x_j$ and M is commutative. Thus, by induction on ℓ, $\widehat{\sigma_i} = \sigma_{i,i-1}$ is an automorphism of A_{i-1}. Note that the fundamental relations in $\mathcal{L}((M; \boldsymbol{\sigma}, \boldsymbol{u}))$ are

$$x_i m = \sigma_i(m) x_i \text{ and } x_i x_j = u_{i,j} x_j x_i \quad \text{for all } m \in M \text{ and } i,\ j = 1, \ldots, n.$$

For simplicity, let $D = \mathcal{L}((M; \boldsymbol{\sigma}, \boldsymbol{u}))$. The ring D carries the $(x_1, \ldots, x_n)$-adic valuation $v_{x_1,\ldots,x_n}$ with value group $\mathbb{Z}^n$ given the right-to-left lexicographic ordering. This is the total ordering in which

$$(r_1, \ldots, r_n) \,<\, (s_1, \ldots, s_n) \quad \text{just when} \quad \begin{cases} \text{there is an } \ell \text{ with } r_\ell < s_\ell \\ \text{and } r_j = s_j \text{ for } \ell + 1 \le j \le n. \end{cases}$$

Since $\mathbb{Z}^n$ with this ordering is discrete of rank n as an ordered abelian group, the valuation $v_{x_1,\ldots,x_n}$ is discrete of rank n. (See the review of ranks for valuations in §A.4 of Appendix A, and Remark A.34 on discrete valuations.) The residue ring of D is $\overline{D} = M$ $(= Z(\overline{D}))$. A series in $\mathcal{O}_D \cap Z(D)$ has constant term fixed under $\sigma_1, \ldots, \sigma_n$, hence $\overline{Z(D)}$ is the subfield of M fixed under $\sigma_1, \ldots, \sigma_n$. The map $\theta_D\colon \mathbb{Z}^n \to Aut(Z(\overline{D})/\,\overline{Z(D)}\,)$ carries $(i_1, \ldots, i_n) \in \mathbb{Z}^n$ to $\sigma_1^{i_1} \ldots \sigma_n^{i_n}$.

1.1.4 Examples: Mal'cev–Neumann series

All the examples above are particular cases of the following general construction due to Mal'cev and Neumann (see Cohn [55, §2.4] or Lam [121, §14]): let Γ be an arbitrary totally ordered abelian group and let D be an arbitrary division ring. Let $f\colon \Gamma \times \Gamma \to D^\times$ and $\omega\colon \Gamma \to Aut(D)$ be maps satisfying the following properties:

$$\omega_\gamma\big(f(\delta, \varepsilon)\big) f(\gamma, \delta + \varepsilon) \,=\, f(\gamma, \delta) f(\gamma + \delta, \varepsilon) \qquad \text{for all } \gamma,\, \delta,\, \varepsilon \in \Gamma, \tag{1.6}$$

$$\omega_\gamma \circ \omega_\delta(d) \,=\, f(\gamma, \delta)\omega_{\gamma+\delta}(d) f(\gamma, \delta)^{-1} \qquad \text{for all } \gamma,\, \delta \in \Gamma \text{ and } d \in D, \tag{1.7}$$

and moreover

$$\omega_0 \,=\, id_D, \qquad f(0, \gamma) \,=\, f(\gamma, 0) \,=\, 1 \qquad \text{for all } \gamma \in \Gamma. \tag{1.8}$$

Define the *support* of a map $\varphi\colon \Gamma \to D$ by

$$supp(\varphi) \,=\, \{\gamma \in \Gamma \mid \varphi(\gamma) \neq 0\}.$$

In the additive group $\mathcal{F}(\Gamma, D)$ of all maps $\Gamma \to D$, the set $\mathcal{F}_{\mathrm{wo}}(\Gamma, D)$ of maps with well-ordered support is a subgroup. The following modified convolution product is well-defined for $\varphi,\, \psi \in \mathcal{F}_{\mathrm{wo}}(\Gamma, D)$:

$$(\varphi * \psi)(\gamma) \,=\, \sum_{\delta \in \Gamma} \varphi(\delta)\omega_\delta\big(\psi(\gamma - \delta)\big) f(\delta, \gamma - \delta) \qquad \text{for } \gamma \in \Gamma,$$

because the sum on the right has only finitely many nonzero terms. For if the set $supp(\varphi) \cap (\gamma - supp(\psi))$ were infinite, it would contain a strictly increasing infinite sequence $\delta_1 < \delta_2 < \ldots$, as $supp(\varphi)$ is well-ordered; but then $supp(\psi)$ would contain the strictly descending infinite sequence $\gamma - \delta_1 > \gamma - \delta_2 > \ldots$, contradicting the well-ordering of $supp(\psi)$. Moreover, see Cohn [55, p. 75] or Lam [121, p. 243], $\varphi * \psi \in \mathcal{F}_{\mathrm{wo}}(\Gamma, D)$, and the sum and convolution product define a ring structure on $\mathcal{F}_{\mathrm{wo}}(\Gamma, D)$; we use the notation $D((\Gamma;\omega, f))$ for this ring to emphasize the dependence on the maps ω and f. Mapping each $d \in D$ to the map φ_d such that $\varphi_d(0) = d$ and $\varphi_d(\gamma) = 0$ for $\gamma \neq 0$ yields an identification of D with a subring of $D((\Gamma;\omega, f))$.

To make the definition of the multiplication in $D((\Gamma;\omega, f))$ more transparent, it is useful to change notation. For each $\gamma \in \Gamma$ we let z^γ denote an indeterminate. Each $\varphi \in \mathcal{F}(\Gamma, D)$ is identified with a formal series as follows:

$$\varphi \,=\, \sum_{\gamma\in\Gamma} \varphi(\gamma) z^\gamma \quad \Big(= \sum_{\gamma\in supp(\varphi)} \varphi(\gamma) z^\gamma\Big).$$

The multiplication in $D((\Gamma;\omega, f))$ is then

$$\Big(\sum_{\gamma\in\Gamma} \varphi(\gamma) z^\gamma\Big) \cdot \Big(\sum_{\delta\in\Gamma} \psi(\delta) z^\delta\Big) \,=\, \sum_{\gamma,\delta\in\Gamma} \varphi(\gamma)\omega_\gamma\big(\psi(\delta)\big) f(\gamma,\delta) z^{\gamma+\delta}.$$

We have $z^0 = 1$ in $D((\Gamma;\omega, f))$, and each element $d \in D$ is identified with $dz^0 \in D((\Gamma;\omega, f))$. Thus, we have

$$z^\gamma \cdot d \,=\, \omega_\gamma(d) \cdot z^\gamma \quad \text{and} \quad z^\gamma \cdot z^\delta \,=\, f(\gamma,\delta) z^{\gamma+\delta} \quad \text{for } \gamma,\, \delta \in \Gamma \text{ and } d \in D.$$

Define a map $v\colon D((\Gamma;\omega, f)) \to \Gamma \cup \{\infty\}$ by

$$v(\varphi) \,=\, min(supp\, \varphi) \qquad \text{for } \varphi \in D((\Gamma;\omega, f))$$

(and $v(0) = min(\varnothing) = \infty$). It is easy to see that this map satisfies all the conditions that define a valuation, so $D((\Gamma;\omega, f))$ has no zero divisors. If $\varphi \in D((\Gamma;\omega, f))$ satisfies $v(\varphi) > 0$, then with some effort one can prove that the set $\bigcup_{n\in\mathbb{N}} supp(\varphi^n)$ is well-ordered (see Cohn [55, Lemma 2.4.3, Th. 2.4.4, pp. 73–75] or Lam [121, Lemma 14.22(1), p. 244]); hence, $\sum_{n\in\mathbb{N}} \varphi^n \in D((\Gamma;\omega, f))$, and it is easy to check that $\sum_{n=0}^{\infty} \varphi^n = (1-\varphi)^{-1}$. Arguing as for twisted Laurent series at the beginning of this subsection, one can then see that $D((\Gamma;\omega, f))$ is a division ring, and v is a valuation on $D((\Gamma;\omega, f))$. Its value group is Γ and the residue division algebra is D. The map $\theta_{D((\Gamma;\omega,f))}$ carries each $\gamma \in \Gamma$ to $\omega_\gamma|_{Z(D)}$.

Let D be any division ring and let Γ be a totally ordered abelian group. Note that if we let $\omega\colon \Gamma \to Aut(D)$ be the trivial homomorphism, and define $\mathbf{1}\colon \mathbb{Z}\times\mathbb{Z} \to D^\times$ by $\mathbf{1}(\gamma,\delta) = 1$ for all γ, $\delta \in \Gamma$, then ω and $\mathbf{1}$ satisfy conditions (1.6)–(1.8). The resulting Mal'cev–Neumann ring $D((\Gamma, \omega, \mathbf{1}))$ has value group Γ. Thus, every totally ordered group Γ is the value group of some division algebra. If we take $D = F$, a field, and $\Gamma = \mathbb{Z}^n$ with right-to-left lexicographic ordering, then $F((\mathbb{Z}^n, \omega, \mathbf{1}))$ can be identified with the n-fold iterated Laurent series field:

$$F((\mathbb{Z}^n, \omega, \mathbf{1})) \,=\, F((x_1))\ldots((x_n)).$$

If τ is an automorphism of D and $\Gamma = \mathbb{Z}$, we define $\omega\colon \mathbb{Z} \to Aut(D)$ and $\mathbf{1}\colon \mathbb{Z}\times\mathbb{Z} \to D^\times$ by $\omega_\gamma = \tau^\gamma$ and $\mathbf{1}(\gamma,\delta) = 1$ for all γ, $\delta \in \mathbb{Z}$. With this choice,

ω and $\mathbf{1}$ satisfy the conditions (1.6)–(1.8), and the Mal'cev–Neumann ring $D((\mathbb{Z};\omega,\mathbf{1}))$ can be identified with a division ring of Laurent series:

$$D((\mathbb{Z};\omega,\mathbf{1})) \;=\; D((z^1;\tau)).$$

Similarly, rings of n-fold iterated twisted Laurent series can be identified with Mal'cev–Neumann rings with $\Gamma = \mathbb{Z}^n$.

1.2 Valuations on finite-dimensional division algebras

After proving in §1.2.1 a fundamental inequality relating the dimension of an extension of valued division rings to the product of the ramification index and residue degree, we focus in this section on division rings D that are finite-dimensional algebras over a field F, viewed as a subfield of the center $Z(D)$. Any valuation w on D restricts to a valuation v on F, and we may consider w as an extension of v. Thus, we may try to define valuations on D by extending a given valuation on F. A necessary and sufficient condition for the existence of such an extension is given in §1.2.2: see Th. 1.4. This condition features one of the most striking differences between commutative and noncommutative valuation theory: recall that for any extension field L of F, the valuation v extends to a valuation on L; often there are many such extensions. By contrast, Th. 1.4 below shows that a valuation on the center of a division algebra D has at most one extension to D, but may have none at all. Likewise, a composition of valuations cannot be defined without an added condition, which is given in §1.2.3. We conclude this section with various examples in §1.2.4–§1.2.9.

1.2.1 The fundamental inequality

Let D be a division ring with valuation v, and let E be any sub-division ring of D. Then the restriction $v|_E$ of v to E is clearly a valuation on E, and we have

$$\Gamma_E \subseteq \Gamma_D, \qquad \mathcal{O}_E = \mathcal{O}_D \cap E, \quad \text{and} \quad \mathfrak{m}_E = \mathfrak{m}_D \cap E;$$

hence, there is a canonical injection $\overline{E} \hookrightarrow \overline{D}$, which we will treat as an inclusion. We write $[D{:}E]_\ell$ (resp. $[D{:}E]_r$) for the dimension of D as a left (resp. right) E-vector space.

Proposition 1.3 (Fundamental Inequality).

$$[\overline{D}{:}\overline{E}]_\ell\,|\Gamma_D{:}\Gamma_E| \;\le\; [D{:}E]_\ell \quad \textit{and} \quad [\overline{D}{:}\overline{E}]_r\,|\Gamma_D{:}\Gamma_E| \;\le\; [D{:}E]_r.$$

Proof. We prove only the left inequality. The proof of the right one is analogous. Pick $\{d_i\}_{i\in I} \subseteq \mathcal{O}_D^\times$ such that the images $\{\overline{d_i}\}_{i\in I} \subseteq \overline{D}^\times$ form a base of

$\overline{D}$ as a left $\overline{E}$-vector space. Also pick a set of coset representatives $\{\delta_j\}_{j\in J}$ of Γ_E in Γ_D, and for each j choose some $c_j \in D$ with $v(c_j) = \delta_j$. To verify the inequality of the proposition, we show that $\{d_i c_j\}_{i\in I\, j\in J}$ is left E-linearly independent in D.

For this, suppose $\{i_1, \ldots, i_r\} \subseteq I$ and $\{j_1, \ldots, j_s\} \subseteq J$ are finite subsets and $\{a_{k\ell} \mid 1 \le k \le r,\ 1 \le \ell \le s\}$ is a set of elements in E. We want to show

$$\sum_{k=1}^{r}\sum_{\ell=1}^{s} a_{k\ell} d_{i_k} c_{j_\ell} \;=\; 0 \quad \text{implies} \quad a_{k\ell} \;=\; 0 \text{ for all } k,\, \ell.$$

For $\ell = 1, \ldots, s$, let

$$b_\ell = \sum_{k=1}^{r} a_{k\ell} d_{i_k} \quad \text{and} \quad \gamma_\ell = min\{v(a_{k\ell}) \mid 1 \le k \le r\} \in \Gamma_E \cup \{\infty\}.$$

We claim that $v(b_\ell) = \gamma_\ell$ for all ℓ. Assuming this equation holds, we have

$$v(b_\ell c_{j_\ell}) \;=\; \gamma_\ell + \delta_{j_\ell} \;\equiv\; \delta_{j_\ell} \ (mod\ \Gamma_E) \quad \text{if } b_\ell \ne 0.$$

Because the $v(b_\ell c_{j_\ell})$ are thus distinct for those ℓ with $b_\ell \ne 0$, we have

$$v\big(\sum_{k,\ell} a_{k\ell} d_{i_k} c_{j_\ell}\big) \;=\; v\big(\sum_{\ell} b_\ell c_{j_\ell}\big) \;=\; \min_{1\le \ell\le s} v(b_\ell c_{j_\ell}) \;=\; \min_{1\le \ell\le s} (\gamma_\ell + \delta_\ell).$$

Therefore, if $\sum a_{k\ell} d_{i_k} c_{j_\ell} = 0$ we must have $\gamma_\ell = \infty$ for all ℓ, i.e., $a_{k\ell} = 0$ for all k, ℓ, and the proof is complete.

To prove the claim, suppose $\gamma_\ell \ne \infty$. By reordering $i_1, \ldots, i_r$, we may assume $v(a_{1\ell}) = \gamma_\ell$. The $\overline{E}$-independence of the $\overline{d_i}$ shows $\sum_{k=1}^{r} \overline{a_{1\ell}^{-1} a_{k\ell}}\, \overline{d_{i_k}} \ne 0$; hence, $v\big(a_{1\ell}^{-1} \sum_{k=1}^{r} a_{k\ell} d_{i_k}\big) = 0$ and therefore

$$v(b_\ell) \;=\; v\big(\sum_{k=1}^{r} a_{k\ell} d_{i_k}\big) \;=\; v(a_{1\ell}) \;=\; \gamma_\ell,$$

proving the claim. □

1.2.2 Extension of a valuation from the center

Our focus henceforward will be on division rings that are finite-dimensional algebras over a field. Let $v \colon F \to \Gamma \cup \{\infty\}$ be a valuation on a field F. By replacing Γ by its divisible hull $\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$ (to which the ordering on Γ extends uniquely compatibly with the group structure), we may assume at the outset that Γ is divisible. Let D be a finite-dimensional division F-algebra. If v extends to a valuation w on D, then the fundamental inequality (Prop. 1.3) shows that $|w(D^\times){:}v(F^\times)| < \infty$. Since Γ is divisible and torsion-free, the inclusion $\Gamma_F = v(F^\times) \hookrightarrow \Gamma$ therefore extends uniquely to a monomorphism $\iota \colon w(D^\times) \hookrightarrow \Gamma$. Moreover, ι is order-preserving, since the ordering on $w(D^\times)$ is determined by the ordering on Γ_F. Therefore, we may assume without loss of generality that all valuations extending v to a finite-dimensional division algebra over F take their values in the divisible group Γ.

Throughout this section, we fix a valuation $v\colon F \to \Gamma \cup \{\infty\}$ where Γ is a divisible totally ordered abelian group, and we refer to the pair (F, v) as a *valued field*. The division algebras with center F are called *central* division algebras over F; they are always assumed to be finite-dimensional over F.

Theorem 1.4. *Let (F, v) be a valued field and let D be a (finite-dimensional) central division algebra over F. The valuation v extends to a valuation w on D if and only if v has a unique extension to each field L with $F \subseteq L \subseteq D$. When this condition holds, the valuation w is unique and is given in terms of the reduced norm Nrd and the index $\mathrm{ind}(D) = \sqrt{[D{:}F]}$ by the formula*

$$w(x) \;=\; \tfrac{1}{ind(D)}\, v\big(Nrd(x)\big) \qquad \text{for } x \in D^\times. \tag{1.9}$$

Moreover, $\mathcal{O}_D = \{x \in D \mid \ x \text{ is integral over } \mathcal{O}_F\}$.

The main technical tool for the proof is Wedderburn's Factorization Theorem on the minimal polynomial of elements in D. Before proving Th. 1.4 we gather its main consequences in the following proposition:

Proposition 1.5. *Let D be a central division F-algebra. Suppose w is a valuation on D extending v and let $P \subseteq Z(\overline{D})$ be the purely inseparable closure of $\overline{F}$ in $Z(\overline{D})$.*

(i) *For all $a \in D^\times$ we have $v\big(Nrd(a)\big) = ind(D)\, w(a)$.*
(ii) *Every element in $\mathcal{O}_D$ is integral over $\mathcal{O}_F$.*
(iii) *The field extension $Z(\overline{D})/\overline{F}$ is normal.*
(iv) *$P = Z(\overline{D})^{\theta_D(\Gamma_D)}$, the subfield fixed under $\theta_D(\Gamma_D)$ (where θ_D is defined in (1.1)).*
(v) *$Z(\overline{D})/P$ is Galois with abelian Galois group $\theta_D(\Gamma_D)$.*

Proof. Let $a \in D^\times$ and let $f(X) = X^n + \alpha_{n-1}X^{n-1} + \ldots + \alpha_0 \in F[X]$ be the minimal polynomial of a over F. Thus, $n = [F(a){:}F]$ and $Nrd(a) = (-1)^{ind(D)}\alpha_0^{ind(D)/n}$, so $v\big(Nrd(a)\big) = \frac{ind(D)}{n} v(\alpha_0)$. By Wedderburn's Factorization Theorem (see Lam [121, Th. 16.9, p. 265]), we may find conjugates of a,

$$a_1 \;=\; d_1 a d_1^{-1}, \quad \ldots, \quad a_n \;=\; d_n a d_n^{-1}$$

such that $f(X) = (X - a_1)\ldots(X - a_n)$. In particular, $\alpha_0 = (-1)^n a_1 \ldots a_n$. Since $w(a_i) = w(a)$ for all i, it follows that $v(\alpha_0) = n\, w(a)$; hence, $v\big(Nrd(a)\big) = ind(D) w(a)$, proving (i).

If $a \in \mathcal{O}_D$, then $a_i \in \mathcal{O}_D$ for all i; hence, $f(X) \in \mathcal{O}_D[X] \cap F[X] = \mathcal{O}_F[X]$, and (ii) follows.

Now, suppose $a \in \mathcal{O}_D$ and $\overline{a} \in Z(\overline{D})$. By definition of θ_D we have

$$\overline{a_i} \;=\; \theta_D\big(w(d_i)\big)(\overline{a}) \;\in\; Z(\overline{D});$$

hence, $\overline{f}(X)$ splits in $Z(\overline{D})[X]$. The minimal polynomial of $\overline{a}$ over $\overline{F}$ also splits over $Z(\overline{D})$ since it divides $\overline{f}$, hence (iii) is proved.

Assuming $\overline{a} \in Z(\overline{D})^{\theta_D(\Gamma_D)}$, we have $\overline{a_i} = \overline{a}$ for all i, hence $\overline{f}(X) = (X - \overline{a})^n$ and $\overline{a}$ is purely inseparable over $\overline{F}$. Thus, $Z(\overline{D})^{\theta_D(\Gamma_D)} \subseteq P$. Statements (iv) and (v) then readily follow by Galois theory. □

For the proof of Th. 1.4, we also need the following lemma:

Lemma 1.6. *Let L be a finite-degree field extension of F. If v has a unique extension to a valuation w on L, then for each $x \in L$ we have*

$$v(N_{L/F}(x)) \;=\; [L{:}F]\, w(x),$$

where $N_{L/F}$ denotes the norm from L to F.

Proof. Let K be a normal closure of L over F, and let v' be any extension of v to K. We have $N_{L/F}(x) = x_1 \dots x_n$ where $n = [L{:}F]$ and each x_i is a conjugate of x in K. For each i, there is an F-automorphism σ_i of K with $\sigma_i(x) = x_i$. Since $v' \circ \sigma_i|_L$ is a valuation of L extending v, we have $v' \circ \sigma_i|_L = w$. Hence, $v'(x_i) = v'(\sigma_i(x)) = w(x)$. Thus,

$$v(N_{L/F}(x)) \;=\; v'(N_{L/F}(x)) \;=\; v'(x_1) \dots v'(x_n) \;=\; [L{:}F]\, w(x). \qquad \square$$

Proof of Th. 1.4. Suppose v has a unique extension to each field L with $F \subseteq L \subseteq D$. We show that the formula (1.9) defines a valuation on D. Clearly, $w(x) = \infty$ if and only if $x = 0$, and $w(xy) = w(x) + w(y)$ for x, $y \in D$ since $Nrd(xy) = Nrd(x)\, Nrd(y)$. To prove $w(x+y) \geq min\big(w(x), w(y)\big)$ for $x, y \in D$, we may of course assume $y \neq 0$. Since $w(x+y) = w(xy^{-1} + 1) + w(y)$ and $w(x) = w(xy^{-1}) + w(y)$, it suffices to show

$$w(xy^{-1} + 1) \;\geq\; min(w(xy^{-1}), 0). \tag{1.10}$$

Let $L \subseteq D$ be any maximal subfield containing xy^{-1}. Since $N_{L/F} = Nrd|_L$, Lemma 1.6 shows that $w|_L$ is the unique valuation on L extending v. Therefore, (1.10) holds and w is a valuation on D.

Now, suppose v extends to a valuation w on D. Proposition 1.5(i) yields the formula (1.9), which shows that w is unique. If $a \in D$ is integral over $\mathcal{O}_F$, then we have

$$a^n \;=\; \alpha_{n-1} a^{n-1} + \dots + \alpha_0 \;\text{ for some } n \geq 1 \text{ and some } \alpha_{n-1}, \dots, \alpha_0 \in \mathcal{O}_F.$$

If $w(a) < 0$, then $w(a^n) < w(\alpha_i a^i)$ for $i = 0, \dots, n-1$, and the equality above is impossible. Therefore, $a \in \mathcal{O}_D$. By Prop. 1.5(ii), it follows that $\mathcal{O}_D$ consists of the elements in D that are integral over $\mathcal{O}_F$. Likewise, for any field L with $F \subseteq L \subseteq D$ the valuation ring $\mathcal{O}_L$ of $w|_L$ is integral over $\mathcal{O}_F$. But in any finite degree field extension of F the integral closure of $\mathcal{O}_F$ is the irredundant intersection of the valuation rings of all the extensions of v to L, see Engler–Prestel [73, Cor. 3.1.4, p. 60; Lemma 3.2.8, p. 64]. The integrality of $\mathcal{O}_L$ over $\mathcal{O}_F$ therefore implies that $w|_L$ is the only extension of v to L. □

Theorem 1.4 indicates the importance of Henselian valuations in the valuation theory of division algebras. Recall that the valuation v on F is Henselian[1] if and only if v has a unique extension to each field L algebraic over F; the extension of v to L is also clearly Henselian. Thus we have the following corollary to Th. 1.4:

Corollary 1.7. *Let D be a division algebra finite-dimensional over F. If v is a Henselian valuation on F, then v has a unique extension to a valuation on D.*

Proof. Let L be the center of D. Since $[L{:}F] < \infty$, v has a unique extension to a valuation v_L of L, and v_L is Henselian. Therefore, by Th. 1.4, v_L has a unique extension to a valuation v_D on D. Then, v_D is the unique extension of v to D, since any valuation on D restricts to a valuation on L. □

1.2.3 Composite valuations

When F is a field, we can "compose" a valuation v on F with any valuation u on $\overline{F}^{v}$ to obtain a valuation $u * v$ on F that is a refinement of v. This is the valuation whose ring is $\pi^{-1}(\mathcal{O}_{\overline{F}^{v},u})$, where $\pi\colon \mathcal{O}_{F,v} \to \overline{F}^{v}$ is the canonical surjection; see Engler–Prestel [73, p. 45]. For valuations on a division algebra, such a composition is possible only with an added condition:

Proposition 1.8. *Let D be a central division algebra over F and let w be a valuation on D. Let u be a valuation on $\overline{D}^{w}$. Then, the composite valuation $u|_{\overline{F}^{w}} * w|_F$ on F extends to a valuation on D if and only if u has a unique extension from $\overline{F}^{w}$ to $Z(\overline{D}^{w})$. When this extension exists, its valuation ring is $\pi^{-1}(\mathcal{O}_{\overline{D}^{w},u})$, where $\pi\colon \mathcal{O}_{D,w} \to \overline{D}^{w}$ is the canonical surjection.*

Here is why the added condition is needed: While the ring $V = \pi^{-1}(\mathcal{O}_{\overline{D}^{w},u})$ is always a total valuation ring of D, i.e., d or d^{-1} lies in V for all $d \in D^{\times}$, this V need not have the further property of being stable under inner automorphisms, which is required for V to be the valuation ring of a valuation on D. The added condition in Prop. 1.8 is equivalent to: $dVd^{-1} = V$ for all $d \in D^{\times}$.

Proof of Prop. 1.8. We write $\overline{D}$ for $\overline{D}^{w}$ and $\overline{F}$ for $\overline{F}^{w}$ throughout the proof. Let z be the valuation $u|_{\overline{F}^{w}} * w|_F$ on F. Suppose z extends to a valuation v on D. Take any element of $\overline{D}$, and write it as $\overline{d}$ for some $d \in \mathcal{O}_{D,w}$. Let L be the field $F(d)$, and set $\overline{L} = \overline{L}^{w}$. If u' is any valuation on $\overline{L}$ with $u'|_{\overline{F}} = u|_{\overline{F}}$, then $u' * w|_L$ and $u|_{\overline{L}} * w|_L$ are each valuations on L extending $v|_F = z$. Since

[1] Properties of Henselian valuations and Henselizations are reviewed in Appendix A.

z on F extends to v on D, by Th. 1.4 z has a unique extension to L. Hence, $u' * w|_L = v|_L = u|_{\overline{L}} * w|_L$; therefore, u' must coincide with $u|_{\overline{L}}$, since they each have the valuation ring $\pi(\mathcal{O}_{L,v})$. Thus, $u|_{\overline{L}}$ is the unique extension of the valuation $u|_{\overline{F}}$ on $\overline{F}$ to $\overline{L}$. Therefore, $u|_{\overline{F}}$ extends uniquely to $\overline{F}(\overline{d}) \subseteq \overline{L}$. It follows that $u|_{\overline{F}}$ must extend uniquely to $Z(\overline{D})$, since different extensions to $Z(\overline{D})$ would restrict to different extensions to $\overline{F}(\overline{d})$ for some $\overline{d} \in Z(\overline{D})$. The valuation ring $\mathcal{O}_{D,v}$ is the union of its restrictions, $\mathcal{O}_{D,v} = \bigcup_{L \in \mathcal{L}} \mathcal{O}_{L,v}$, where $\mathcal{L} = \{\text{fields } F(d) \mid d \in D\}$. But we saw above that $v|_L = u|_{\overline{L}} * w|_L$ for all $L \in \mathcal{L}$; hence, $\mathcal{O}_{L,v} = \pi^{-1}(\mathcal{O}_{\overline{L},u}) \cap L$. Thus,

$$\mathcal{O}_{D,v} \;=\; \bigcup_{L \in \mathcal{L}} \big(\pi^{-1}(\mathcal{O}_{\overline{L},u}) \cap L\big) \;=\; \pi^{-1}(\mathcal{O}_{\overline{D},u}).$$

Conversely, suppose $u|_{Z(\overline{D})}$ is the unique extension of $u|_{\overline{F}}$ to $Z(\overline{D})$. Let K be any subfield of $\overline{D}$. By Th. 1.4, $u|_{KZ(\overline{D})}$ is the unique extension of the valuation $u|_{Z(\overline{D})}$ from $Z(\overline{D})$ to the subfield $KZ(\overline{D})$ of $\overline{D}$. Hence, $u|_K$ must be the unique extension of $u|_{\overline{F}}$ from $\overline{F}$ to K. Now, let M be any field with $F \subseteq M \subseteq D$, and let y be a valuation on M extending z $(= u|_{\overline{F}} * w|_F)$ on F. Since $w|_F$ is a coarsening of z, there is a prime ideal $\mathfrak{p}$ of $\mathcal{O}_{M,y}$ with $\mathfrak{p} \cap \mathcal{O}_{F,z} = \mathfrak{m}_{F,w}$, which is a prime ideal of $\mathcal{O}_{F,z}$. Then, $y = \widetilde{u} * \widetilde{w}$, where $\widetilde{w}$ is the valuation on M with valuation ring the localization $(\mathcal{O}_{M,y})_{\mathfrak{p}}$ of $\mathcal{O}_{M,y}$ at $\mathfrak{p}$ and $\widetilde{u}$ is the valuation on $\overline{M}^{\widetilde{w}}$ with ring $\mathcal{O}_{M,y}/\mathfrak{p}$. Since $\mathcal{O}_{M,\widetilde{w}} \cap F = \mathcal{O}_{F,w}$, this $\widetilde{w}$ is an extension of $w|_F$ to M. Hence, as $w|_F$ extends to D, Th. 1.4 shows that $\widetilde{w} = w|_M$, so $\overline{M}^{\widetilde{w}} = \overline{M}^{w}$. Thus, $\widetilde{u}$ is an extension of $u|_{\overline{F}}$ from $\overline{F}$ to $\overline{M}^{w}$. We saw above that the only such extension is $u|_{\overline{M}^w}$. Hence, $y = u|_{\overline{M}^w} * w|_M$, which shows that z has a unique extension from F to M. Since this is true for each subfield M of D, Th. 1.4 shows that z extends to a valuation on D. □

Remark 1.9. The composite valuation v of Prop. 1.8 is denoted by $u * w$. Note that $\overline{D}^{u*w} = \overline{\overline{D}^{w}}^{u}$ and we have a short exact sequence of ordered abelian groups

$$0 \longrightarrow \Gamma_{\overline{D}^{w},u} \longrightarrow \Gamma_{D,u*w} \longrightarrow \Gamma_{D,w} \longrightarrow 0. \tag{1.11}$$

If there is an ordered group homomorphism $\Gamma_{D,w} \to \Gamma_{D,u*w}$ that is a splitting map for (1.11), then we have an ordered group isomorphism

$$\Gamma_{D,u*w} \;\cong\; \Gamma_{\overline{D}^{w},u} \times \Gamma_{D,w},$$

where the direct product is given the right-to-left lexicographic ordering (defined by $(\gamma,\delta) \le (\gamma',\delta')$ if and only if either $\delta < \delta'$, or $\delta = \delta'$ and $\gamma \le \gamma'$). We saw a case of this when we discussed iterated Laurent series in §1.1.3. This construction will also be used in Ex. 7.77.

If v is a valuation on D and w is any coarser valuation, i.e., $\mathcal{O}_{D,w} \supseteq \mathcal{O}_{D,v}$, then there is a "residue valuation" on $\overline{D}^{w}$ with associated valuation ring $\mathcal{O}_{D,v}/\mathfrak{m}_{D,w}$. This ring is invariant under inner automorphisms of $\overline{D}^{w}$ since