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Anca Capatina

Variational Inequalities and Frictional Contact Problems

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Variational Inequalities and Frictional Contact Problems

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In the memory of my father, Dumitru Stoleru

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Chapter 1
Introduction

Nowadays, the expression *Variational Inequalities and Contact Problems* can be considered as a syntagm since the variational methods have provided one of the most powerful techniques in the study of contact problems and, on the other hand, the variational formulations of the contact problems are, in most cases, variational inequalities.

We therefore considered a book on this subject as necessary, a book where the reader will find many results on variational inequalities and, at the same time, a detailed study of certain contact problems with non local Coulomb friction.

In the last 50 years, variational inequalities became a strong tool in the mathematical study of many nonlinear problems of physics and mechanics, as the complexity of the boundary conditions and the diversity of the constitutive equations lead to variational formulations of inequality type.

The theory of variational inequalities find its roots in the works of Signorini [38] and Fichera [14] concerning unilateral problems and, also, in the work of Ting [44] for the elasto-plastic torsion problem. The mathematical foundation of the theory was widened by the invaluable contributions of Stampacchia [41] and Lions and Stampacchia [26] and then developed by the French and the Italian school: Brézis [3, 4], Stampacchia [42], Lions [25], Mosco [28], Kinderlehrer and Stampacchia [22]. Concerning the approximation of the variational inequalities, we refer to the important contributions brought by Mosco [27], Glowinski et al. [17], or Glowinski [16].

We do not claim that this book covers all the aspects in the study of the variational inequalities. However, we intent to give the reader an overview on this huge subject in a unified form, containing a detailed and justified description of the results on existence, uniqueness, regularity or approximation of solutions of variational and quasi-variational inequalities, in the linear and nonlinear cases, for the static and quasistatic cases.

We also deal in this book with the study of certain static and quasistatic problems with friction whose weak formulations are variational or quasi-variational

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inequalities. More precisely, we address here frictional contact problems for a linearly elastic body which, under the influence of volume and surface forces, is in contact with a rigid foundation. The contact is modeled by Signorini's law, except for the last section where bilateral contact is considered. We also use a nonlocal version of Coulomb's friction law. Most of the results presented here are obtained by applying abstract results on variational inequalities.

The first results concerning the mathematical study of this kind of problems, in the case of Tresca's friction (i.e., with given friction), are due to Duvaut and Lions [12]. In the static case, important results concerning the study of contact problems of Signorini type with local or nonlocal friction have been obtained by Duvaut [11], Nečas et al. [29], Oden and Pires [31,32], Demkowicz and Oden [10], and Cocu [7]. In the quasistatic case, the first existence results were given by Andersson [1], Han and Sofonea [18], and Klarbring et al. [23] for problems with normal compliance. Their approach is based on incremental formulations obtained from the quasi-variational inequality by an implicit time discretization scheme. The same technique was used by Cocu et al. [9], Rocca [36], Andersson [2], and Cocu and Rocca [8] in their existence proofs for quasistatic problems of Signorini type with local or nonlocal friction or with friction and adhesion. The works of Panagiotopoulos [33,34], Glowinski et al. [17], Glowinski [16], Campos et al. [5], Kikuchi and Oden [21], Haslinger et al. [19], Hlaváček et al. [20], Shillor et al. [37], Eck et al. [13], and Sofonea and Matei [39] enriched, theoretically and numerically, the study of contact problems. Among those who developed algorithms of resolution of the unilateral contact problems with friction, let us quote Raous et al. [35], Sofonea et al. [40], and Lebon and Raous [24].

The book is divided into III parts and 9 chapters.

Part I reviews, in a general way, the fundamental definitions, notation and theorems of the functional analysis which will be essential to understand the following parts. So, Chap. 2 is a potpourri of standard topics on functional spaces, while Chap. 3 refers to spaces of vector-valued functions. The material we present in these two chapters is a classical one and can be found in many monographs. Also, throughout this book, when necessary, further basic results on functional analysis will be recalled.

Part II is concerned with the study of variational inequalities.

Chapter 4 presents some generally known existence and uniqueness results. More precisely, in Sect. 4.1 one considers elliptic variational inequalities of the first and second kind involving linear and continuous operators in Hilbert spaces (Sect. 4.1.1) or monotone and hemicontinuous operators in Banach spaces (Sect. 4.1.2). The results are established using projection or proximity operators, Weierstrass or Lax–Milgram theorems, Schauder or Banach fixed point theorems.

Section 4.2 deals with elliptic quasi-variational inequalities. In Sect. 4.2.1, we refer to the case of monotonous and hemicontinuous operators: the existence is obtained by using Kakutani fixed point theorem, while the uniqueness, only for strongly monotone operators, is obtained using Banach fixed point theorem. In Sect. 4.2.2 we consider the case of potential operators and we introduce and justify the concept of generalized solution of a quasi-variational inequality. We then apply, in Sect. 4.2.3, these results to prove the existence and the uniqueness of the generalized solution of a contact problem with friction for the operator of Hencky–Nadai theory.

Section 4.3 presents a strategy, rather new, for the study of a class of abstract implicit evolutionary quasi-variational inequalities which covers the variational formulation of many quasistatic contact problems. The method used rests, as in the typical cases, on incremental formulations.

In Chap. 5 we give two remarkable properties satisfied by the solutions of certain variational inequalities. In Sect. 5.1 one highlights a maximum principle which is then applied to a problem which models the flow of fluids through a porous medium and also to an obstacle problem. In Sect. 5.2, using the method of the translations due to Nirenberg [30] (as Brezis [4] did in his thesis for a scalar second order elliptic operator), local and global regularity results of the solutions of a class of variational inequalities of the second kind are established.

In Chap. 6 we present first a brief background on convex analysis, and we then recall some classical results of the Mosco et al. [6] (M-CD-M) duality theory in its form adapted by Telega [43] for the so-called implicit variational inequalities.

In Chap. 7 one can find details results on the discrete approximation of two general classes of variational inequalities. For the quasi-variational inequalities considered in Sect. 4.2.1, the convergence of an internal approximation is obtained in Sect. 7.1 and an abstract error estimate is given in Sect. 7.2. A convergence result for an internal approximation in space and a back difference scheme in time of implicit evolutionary quasi-variational inequalities introduced in Sect. 4.3 is proved in Sect. 7.3.

In Part III we study, in an almost exhaustive way, the problem of Signorini with nonlocal Coulomb friction in elasticity.

Chapter 8 deals with the static problem. The mechanical problem is described in Sect. 8.1 and its variational formulation is obtained in Sect. 8.2. The existence and, under certain assumptions on the data, the uniqueness of the solution are obtained in Sect. 8.3 by applying the theorems established in Sect. 4.2.1. Using the regularity results given in Sect. 5.2.2 and an argument due to Fichera [15], we get, in Sect. 8.4, a local regularity result for the solutions of the static problem. In Sect. 8.5 we derive two dual formulations, dual and dual condensed, which involve as unknown the stress field instead of the displacement field like in the case of the primal problem, i.e. the variational formulation considered in Sect. 8.2. The first dual formulation is obtained, by using Green's formula, from the mechanical problem in the same way as for the primal formulation. The second dual formulation, i.e. the dual condensed one, is a problem posed on the surface of possible contact only, obtained by applying the M-CD-M duality theory developed in Sect. 6.2. This condensed dual formulation could be useful in numerical calculations since one computes directly the stresses on the contact boundary and usually these are the quantities of interest. In Sect. 8.6 we consider a finite element approximation of the primal problem. We first obtain an error estimate, either directly or by applying the estimate given in Sect. 7.3. We then prove that a higher order of the approximation can be obtained for a suitable choice of the regularization which describes the nonlocal character of Coulomb law.

In Sect. 8.7 we consider the discretization by the equilibrium finite element method of the two stress formulations, i.e. the dual formulation and the dual condensed one. We prove the convergence of our approximations and we derive error estimates of these discretized problems in different cases of the data. Section 8.8 is devoted to the study of an optimal control problem related to the Signorini problem with nonlocal Coulomb friction. More precisely, one characterizes the coefficient of friction which leads to a given profile of displacements on the contact surface.

Chapter 9 deals with the quasistatic problem. In Sect. 9.1, using an implicit time discretization scheme and applying the results of Sect. 4.3, an existence result is obtained. We then consider, in Sect. 9.2, a space finite element approximation and an implicit time discretization scheme of this problem and, by using the results of Sect. 8.3, we prove the convergence of the approximation. In the last section we consider a mathematical model describing the quasistatic process of bilateral contact with friction between an elastic body and a rigid foundation. Our goal is to study a related optimal control problem which allows us to obtain a given profile of displacements on the contact boundary, by acting with a control on another part of the boundary of the body. Using penalization and regularization techniques, we derive the necessary conditions of optimality.

This book was written in the framework of the author's research activity within the Institute of Mathematics of the Romanian Academy, and the results presented here are partially based on the author's own research.

The book is intended to be self-contained and it addresses mathematicians, applied mathematicians, graduate students in mathematical and physical sciences as well researchers in mechanics and engineering.

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Part I
Preliminaries

Chapter 2
Spaces of Real-Valued Functions

This chapter is a brief background on spaces of continuous functions and some Sobolev spaces including basic properties, embedding theorems and trace theorems. Hence, we recall some classical definitions and theorems of functional analysis which will be used throughout this book. These results are standard and so they are stated without proofs; for more details and proofs, we refer the readers to the monographs [1, 3–7, 10, 11, 14].

In this book we only deal with real-valued functions. We assume that the reader is familiar with the basic concepts of general topology and functional analysis.

For a point $\boldsymbol{x} = (x_1, \cdots, x_d) \in \mathbb{R}^d$, we denote by D_i the differential operator $\dfrac{\partial}{\partial x_i}$ $(1 \leq i \leq d)$.

If $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_d)$ is a multi-index, then $D^{\boldsymbol{\alpha}}$ denotes the differential operator of order $\boldsymbol{\alpha}$, with $|\boldsymbol{\alpha}| = \sum_{i=1}^{d} \alpha_i$, defined by

$$D^{\boldsymbol{\alpha}} = D_1^{\alpha_1} \cdots D_d^{\alpha_d} = \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}\,.$$

Obviously, D_i^0 denotes the identity operator.

If $A \subset \mathbb{R}^d$, we denote by $C(A)$ the space of real continuous functions on A.

Let Ω be an open set in $\mathbb{R}^d$ with its boundary Γ. We denote by $\overline{\Omega} = \Omega \cup \Gamma$ the closure of Ω.

For any nonnegative integer m, let $C^m(\Omega)$, respectively $C^m(\overline{\Omega})$, be the space of real functions which, together with all their partial derivatives of orders $\boldsymbol{\alpha}$, with $|\boldsymbol{\alpha}| \leq m$, are continuous on Ω, respectively, on the closure $\overline{\Omega}$ of Ω in $\mathbb{R}^d$, i.e.

$$C^m(\Omega) = \{v \in C(\Omega)\,;\; D^{\boldsymbol{\alpha}} v \in C(\Omega) \text{ for } |\boldsymbol{\alpha}| \leq m\}\,. \tag{2.1}$$

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When $m = 0$, we abbreviate $C(\Omega) \equiv C^0(\Omega)$ and $C(\overline{\Omega}) \equiv C^0(\overline{\Omega})$. Any function in $C(\overline{\Omega})$ is bounded and uniformly continuous on Ω, thus it possesses a unique, bounded, and continuous extension to $\overline{\Omega}$.

Let

$$C^\infty(\Omega) = \bigcap_{m=0}^{\infty} C^m(\Omega)$$

be the space of infinitely differentiable functions on Ω.

If K is a subset of Ω, we shall write $K \subset\subset \Omega$ if $\overline{K} \subset \Omega$ and $\overline{K}$ is a compact (i.e., bounded and closed) subset of $\mathbb{R}^d$.

The support of a function $v : \Omega \to \mathbb{R}$ is defined as the closed subset

$$\text{supp } v = \overline{\{\boldsymbol{x} \in \Omega\, ;\ v(\boldsymbol{x}) \neq 0\}}\,. \tag{2.2}$$

We shall say that a function v has compact support in Ω if there exists a compact subset K of Ω such that $v(\boldsymbol{x}) = 0\ \forall \boldsymbol{x} \in \Omega \backslash K$ or, equivalently, supp $v \subset\subset \Omega$.

We shall denote by $C_0^m(\Omega)$ the subspace of $C^m(\Omega)$ consisting of all those functions which have compact support in Ω.

If $m < +\infty$ and Ω is bounded, then $C^m(\overline{\Omega})$ is a Banach space with the norm given by

$$\|v\|_{C^m(\overline{\Omega})} = \sum_{|\alpha| \le m} \max_{x \in \overline{\Omega}} |D^{\alpha} v(\boldsymbol{x})|\,. \tag{2.3}$$

In the sequel, for $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ two normed spaces with $X \subset Y$, we shall write $X \hookrightarrow Y$ to designate the continuously embedding of X in Y provided the identity operator $I : X \to Y$ is continuous. This is equivalent, since I is linear, to the existence of a constant C such that

$$\|u\|_Y \le C\,\|u\|_X \qquad \forall u \in X\,.$$

We also say that the normed space X is compactly embedded in the normed space Y and write $X \hookrightarrow_c Y$ if the identity operator I is compact, i.e. every bounded sequence in X has a subsequence converging in Y, or, equivalently, if $\{u_k\}_k$ is a sequence which converges weakly to u in X, and we write $u_k \rightharpoonup u$, then $\{u_k\}_k$ converges strongly to u in Y, and we write $u_k \to u$.

We denote by $L^p(\Omega)$, for $1 \le p < +\infty$, the space of (equivalence classes of) real functions v defined on Ω with the p-power absolutely integrable, i.e.

$$\int_{\Omega} |v(\boldsymbol{x})|^p \,\mathrm{d}x < \infty\,,$$

where $\mathrm{d}x = \mathrm{d}x_1 \mathrm{d}x_2 \dots \mathrm{d}x_d$ is the Lebesque measure. The elements of $L^p(\Omega)$, being equivalence classes of measurable functions, are identical if they are equal almost everywhere (a.e.) on Ω. Thus, we write $v = 0$ in $L^p(\Omega)$ if $v(x) = 0$ a.e. $x \in \Omega$.

We also denote by $L^\infty(\Omega)$ the space consisting of all (equivalence classes of) measurable real functions v that are essentially bounded on Ω, i.e. there exists a constant C such that $|v(\boldsymbol{x})| \le C$ a.e. on Ω.

The space $L^p(\Omega)$ endowed with the norm

$$\|v\|_{L^p(\Omega)} = \begin{cases} \left(\int\limits_\Omega |v(\boldsymbol{x})|^p \, \mathrm{d}x \right)^{1/p} & \text{if } 1 \le p < +\infty \\ \operatorname*{ess\,sup}_{\boldsymbol{x}\in\Omega} |v(\boldsymbol{x})| = \inf\{C;\, |v(\boldsymbol{x})| \le C \text{ a.e. } \boldsymbol{x} \in \Omega\} & \text{if } p = +\infty \end{cases} \tag{2.4}$$

is a Banach space. In addition, the space $L^p(\Omega)$ is separable if $1 \le p < +\infty$ and reflexive if $1 < p < +\infty$.

If $p \in [1, \infty]$, then the exponent conjugate to p is the number denoted by p' defined by the relation

$$\frac{1}{p} + \frac{1}{p'} = 1$$

where we used the convention

$$p' = \begin{cases} \infty & \text{if } p = 1\,, \\ 1 & \text{if } p = \infty\,. \end{cases}$$

From Riesz representation Theorem 4.1 for Hilbert spaces it follows that, for $p \in [1, +\infty)$, the dual space of $L^p(\Omega)$ is the space $(L^p(\Omega))' = L^{p'}(\Omega)$ where p' is the exponent conjugate to p. The dual space of $L^\infty(\Omega)$ is a space larger than $L^1(\Omega)$ (for more details, see [14, p. 118]).

In the case $p = 2$, the space $L^2(\Omega)$ is a Hilbert space with respect to the inner product

$$(u, v)_{L^2(\Omega)} = \int\limits_\Omega u(\boldsymbol{x})\, v(\boldsymbol{x})\, \mathrm{d}x\,. \tag{2.5}$$

Definition 2.1. We say that a measurable function v defined a.e. on Ω is locally p-integrable on Ω if $v \in L^p(A)$ for every measurable set $A \subset\subset \Omega$.

We shall denote by $L^p_{\mathrm{loc}}(\Omega)$ the space of all locally p-integrable functions on Ω.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^d$ be an open set. The following assertions hold.*

1) Let $1 < p, q < \infty$.

If $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, then $uv \in L^{\frac{pq}{p+q}}(\Omega)$.

If $u_n \to u$ in $L^p(\Omega)$ and $v_n \to v$ in $L^q(\Omega)$, then $u_n v_n \to uv$ in $L^{\frac{pq}{p+q}}(\Omega)$.

If $u \in L^p(\Omega)$ and $v \in L^{p'}(\Omega)$ where p' is the exponent conjugate to p, then $uv \in L^1(\Omega)$ and the Hölder's inequality holds:

$$\int_\Omega u(\boldsymbol{x})v(\boldsymbol{x})\,\mathrm{d}x \le \|u\|_{L^p(\Omega)}\|v\|_{L^{p'}(\Omega)}\,. \tag{2.6}$$

When $p = p' = 2$, we get the Cauchy–Schwartz inequality.

2) For $1 \le p \le \infty$, every Cauchy sequence in $L^p(\Omega)$ has a subsequence converging pointwise a.e. on Ω.

3) $L^p(\Omega) \subset L^1_{\mathrm{loc}}(\Omega) \quad \forall p$ with $1 \le p \le \infty$.

4) Let $v \in L^1_{\mathrm{loc}}(\Omega)$ be such that $\int_\Omega v(\boldsymbol{x})\varphi(\boldsymbol{x})\,\mathrm{d}x = 0 \quad \forall\, \varphi \in \mathscr{D}(\Omega)$. Then $v(\boldsymbol{x}) = 0$ a.e. on Ω.

5) $C_0^\infty(\Omega)$ is dense in $L^p(\Omega) \quad \forall p$ with $1 \le p < \infty$.

The following theorem gives an embedding result for the spaces $L^p(\Omega)$ and some of its consequences.

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^d$ be an open set with vol $(\Omega) = \int_\Omega \mathrm{d}x < \infty$. Then the following statements are valid.*

1) For all p, q such that $1 \le p \le q \le \infty$, we have $L^q(\Omega) \hookrightarrow L^p(\Omega)$ and

$$\|v\|_{L^p(\Omega)} \le (vol\,\Omega)^{\frac{1}{p}-\frac{1}{q}}\|v\|_{L^q(\Omega)} \quad \forall\, v \in L^q(\Omega)\,.$$

2) $\lim_{p\to\infty} \|v\|_{L^p(\Omega)} = \|v\|_{L^\infty(\Omega)} \quad \forall v \in L^\infty(\Omega)$.

3) Suppose that $v \in L^p(\Omega)$ for any $1 \le p < \infty$ and that there exists a constant C such that $\|v\|_{L^p(\Omega)} \le C$. Then $v \in L^\infty(\Omega)$.

To better understand what is the meaning of the differential operator $D^{\boldsymbol{\alpha}}v$ for functions v whose derivatives do not exist in the classical sense, we briefly remind the definition of distributions on Ω.

We denote by $\mathscr{D}(\Omega)$, called the space of test functions, the space $C_0^\infty(\Omega)$ equipped with the inductive limit topology as in the Schwartz theory of distributions [11].

Definition 2.2. A sequence $\{\varphi_k\}_k \subset C_0^\infty(\Omega)$ is said to converge to a function $\varphi \in C_0^\infty(\Omega)$ in (the sense of the space) $\mathscr{D}(\Omega)$, provided the following conditions are satisfied:

i) There exists a compact subset K of Ω such that supp $(\varphi_k - \varphi) \subset K\,,\ \forall k$

ii) $D^{\boldsymbol{\alpha}}\varphi_k \to D^{\boldsymbol{\alpha}}\varphi$ uniformly on $K\,,\ \forall\boldsymbol{\alpha}$ multi-index .

The dual space $\mathscr{D}'(\Omega)$ of $\mathscr{D}(\Omega)$ is called the space of (Schwartz) distributions (or, generalized functions). Hence, any distribution T is a linear and continuous functional on $\mathscr{D}(\Omega)$, i.e. $T(\varphi_k) \to T(\varphi)$ in $\mathbb{R}$ whenever $\varphi_k \to \varphi$ in $\mathscr{D}(\Omega)$. As dual of $\mathscr{D}(\Omega)$, the space $\mathscr{D}'(\Omega)$ is equipped with the weak-star topology: $T_k \to T$ in $\mathscr{D}'(\Omega)$ if and only if $T_k(\varphi) \to T(\varphi)$ in $\mathbb{R}$, for every $\varphi \in \mathscr{D}(\Omega)$.

Every distribution is infinitely differentiable in the following sense: if $T \in \mathscr{D}'(\Omega)$ then, for all multi-index $\boldsymbol{\alpha}$, the function $D^{\boldsymbol{\alpha}}T$ defined on $\mathscr{D}(\Omega)$ by

$$D^{\boldsymbol{\alpha}}T(\varphi) = (-1)^{|\boldsymbol{\alpha}|}T(D^{\boldsymbol{\alpha}}\varphi) \quad \forall\varphi \in \mathscr{D}(\Omega)\,. \tag{2.7}$$

is a distribution. In addition, the operator $D^{\boldsymbol{\alpha}}$ from $\mathscr{D}'(\Omega)$ into $\mathscr{D}'(\Omega)$ is continuous.

Any function $u \in L^1_{\mathrm{loc}}(\Omega)$ generates a distribution $T_u \in \mathscr{D}'(\Omega)$ defined by

$$T_u(\varphi) = \int_\Omega u(\boldsymbol{x})\varphi(\boldsymbol{x})\,\mathrm{d}x \quad \forall\varphi \in \mathscr{D}(\Omega)\,. \tag{2.8}$$

Therefore, for any multi-index $\boldsymbol{\alpha}$, there exists the $\boldsymbol{\alpha}$-th derivative of T_u, namely the distribution $D^{\alpha}T_u \in \mathscr{D}'(\Omega)$ defined by (2.7), i.e.

$$D^{\boldsymbol{\alpha}}T_u(\varphi) = (-1)^{|\boldsymbol{\alpha}|}T_u(D^{\boldsymbol{\alpha}}\varphi) \quad \forall\varphi \in \mathscr{D}(\Omega)\,.$$

But not any distribution is generated by a locally integrable function.

Definition 2.3. We shall say that the function $u \in L^1_{\mathrm{loc}}(\Omega)$ possesses the distributional (or generalized or weak) partial derivative of order $\boldsymbol{\alpha}$ on Ω, denoted by $D^{\boldsymbol{\alpha}}u$, if there exists a function $v_\alpha \in L^1_{\mathrm{loc}}(\Omega)$ which generates the distribution $D^{\alpha}T_u \in \mathscr{D}'(\Omega)$, i.e.

$$D^{\boldsymbol{\alpha}}T_u = T_{v_\alpha}\,.$$

Thus, from the last three relations, it follows that $D^{\boldsymbol{\alpha}}u = v_\alpha$ is the distributional partial derivative of u if $v_\alpha \in L^1_{\mathrm{loc}}(\Omega)$ satisfies

$$\int_\Omega u(\boldsymbol{x})\,D^{\boldsymbol{\alpha}}\varphi(\boldsymbol{x})\,\mathrm{d}x = (-1)^{|\boldsymbol{\alpha}|}\int_\Omega v_\alpha\,\varphi(\boldsymbol{x})\,\mathrm{d}x \quad \forall\varphi \in \mathscr{D}(\Omega)\,. \tag{2.9}$$

Obviously, the distributional derivative is uniquely defined up to a set of measure zero.

In fact, this definition generalizes the classical partial derivative, obtained, for a function $u \in C^{|\boldsymbol{\alpha}|}(\Omega)$, by integrating by parts $|\boldsymbol{\alpha}|$ times

$$\int_\Omega D^{\boldsymbol{\alpha}}u(\boldsymbol{x})\,\varphi(\boldsymbol{x})\,\mathrm{d}x = (-1)^{|\boldsymbol{\alpha}|}\int_\Omega u(\boldsymbol{x})\,D^{\boldsymbol{\alpha}}\varphi(\boldsymbol{x})\,\mathrm{d}x \quad \forall\varphi \in \mathscr{D}(\Omega)\,. \tag{2.10}$$

Of course, in this case, $D^{\boldsymbol{\alpha}} u$ is also a distributional partial derivative of u. However, it should be noted that the derivative in the sense of distributions of a function, even sufficiently smooth, may exist, even if it does not exist in the classical sense.

In particular, the relation (2.8) brings out a linear and continuous mapping $u \mapsto T_u$ from $L^p(\Omega)$ into $\mathscr{D}'(\Omega)$ and so, we may identify the distribution T_u with the integrable function u. The same identification may be made for $\mathscr{D}(\Omega)$. Thus, we have

$$\mathscr{D}(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow \mathscr{D}'(\Omega)\,.$$

Using this result and the definition (2.9), Sobolev [12] expanded in a natural way the space $L^p(\Omega)$ by considering those functions which, for some nonnegative integer m, possess distributional partial derivatives of all orders $|\boldsymbol{\alpha}| \leq m$ in $L^p(\Omega)$. This is the definition of the Sobolev space

$$W^{m,p}(\Omega) = \{v\,;\ D^{\boldsymbol{\alpha}} v \in L^p(\Omega)\,,\ \text{ for } |\boldsymbol{\alpha}| \leq m\}\,.$$

The space $W^{m,p}(\Omega)$ is a Banach space with the norm

$$\|v\|_{W^{m,p}(\Omega)} = \begin{cases} \left(\displaystyle\sum_{|\boldsymbol{\alpha}| \leq m} \|D^{\boldsymbol{\alpha}} v\|^p_{L^p(\Omega)} \right)^{1/p} & \text{if } p \in [1, \infty)\,, \\ \displaystyle\max_{|\boldsymbol{\alpha}| \leq m} \|D^{\boldsymbol{\alpha}} v\|_{L^\infty(\Omega)} & \text{if } p = \infty\,. \end{cases} \tag{2.11}$$

Obviously, $W^{0,p}(\Omega) = L^p(\Omega)$ for $p \in [1, \infty)$. The seminorm over $W^{m,p}(\Omega)$ is defined by

$$|v|_{W^{m,p}(\Omega)} = \begin{cases} \left(\displaystyle\sum_{|\boldsymbol{\alpha}| = m} \|D^{\boldsymbol{\alpha}} v\|^p_{L^p(\Omega)} \right)^{1/p} & \text{if } p \in [1, \infty)\,, \\ \displaystyle\max_{|\boldsymbol{\alpha}| = m} \|D^{\boldsymbol{\alpha}} v\|_{L^\infty(\Omega)} & \text{if } p = \infty\,. \end{cases} \tag{2.12}$$

We denote by $W_0^{m,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in the space $W^{m,p}(\Omega)$ for the norm $\|\cdot\|_{W^{m,p}(\Omega)}$. For $p \in [1, \infty)$, we have the following chain of embeddings

$$W_0^{m,p}(\Omega) \hookrightarrow W^{m,p}(\Omega) \hookrightarrow L^p(\Omega)$$

and, since $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$, it is clear that $W_0^{0,p}(\Omega) = L^p(\Omega)$.

It is easy to see that, if the open set Ω is bounded, the seminorm $|\cdot|_{W^{m,p}(\Omega)}$ is a norm over $W_0^{m,p}(\Omega)$ equivalent to the norm $\|\cdot\|_{W^{m,p}(\Omega)}$.

In the case $p = 2$, we use the notation

$$H^m(\Omega) = W^{m,2}(\Omega)\,.$$

Endowed with the scalar product

$$(u,v)_{H^m(\Omega)} = \sum_{\alpha \le m} (D^{\alpha} u, D^{\alpha} v)_{L^2(\Omega)}\,, \tag{2.13}$$

the Sobolev space $H^m(\Omega)$ is a Hilbert space. Also we denote $H_0^m(\Omega) = W_0^{m,2}(\Omega)$.

If Ω is bounded, then, without any hypothesis on the regularity of Ω, we have

$$H_0^1(\Omega) \hookrightarrow_c L^2(\Omega)\,.$$

Many different symbols are being used to denote these norms, when no confusion may occur: $\|\cdot\|_{m,p,\Omega}$ or $\|\cdot\|_{m,p}$ instead of $\|\cdot\|_{W^{m,p}(\Omega)}$, $\|\cdot\|_{m,\Omega}$ or $\|\cdot\|_m$ instead of $\|\cdot\|_{H^m(\Omega)}$ and $\|\cdot\|_{0,\Omega}$ or $\|\cdot\|_0$ instead of $\|\cdot\|_{L^2(\Omega)}$.

If $m \ge 1$ and $1 \le p < \infty$, we denote by $W^{-m,p'}(\Omega)$ the dual space of $W_0^{m,p}(\Omega)$, p' being the exponent conjugate to p (in fact, $W^{-m,p'}(\Omega)$ is the notation for a space of some distributions on Ω which is isometrically isomorphic to the dual space $(W_0^{m,p}(\Omega))'$; for details, see [1]). Endowed with the norm

$$\|f\|_{W^{-m,p'}(\Omega)} = \sup_{\substack{u \in W_0^{m,p}(\Omega) \\ u \ne 0}} \frac{\langle f, u\rangle}{\|u\|_{W^{m,p}(\Omega)}}\,,$$

the space $W^{-m,p'}(\Omega)$ is a Banach space which is separable and reflexive if $1 < p < \infty$. Here $\langle \cdot,\cdot \rangle$ is the duality pairing between $W^{-m,p'}(\Omega)$ and $W_0^{m,p}(\Omega)$.

We note that if X, Y are two Hilbert spaces such that $X \hookrightarrow Y$ dense, then (see, for instance, [2, p. 51]) $Y^* \hookrightarrow X^*$ dense, where Y^* and X^* denote their dual spaces.

If Ω is bounded, then $\mathscr{D}(\Omega)$ is dense in $H_0^m(\Omega)$, and so, we can identify the dual space $H^{-m}(\Omega)$ of $H_0^m(\Omega)$ with a subspace of $\mathscr{D}'(\Omega)$:

$$\mathscr{D}(\Omega) \subset H_0^m(\Omega) \subset L^2(\Omega) \subset H^{-m}(\Omega) \subset \mathscr{D}'(\Omega)\,.$$

Now, we notice that most of the important results involving Sobolev spaces are first obtained for regular functions and then extended to Sobolev spaces. The density theorems and the embedding theorems show how and whether an element of a Sobolev space can be approximated by smooth functions. Since these theorems require additional regularity properties for the open set Ω, we recall some definitions of them. Later, in Chaps. 5 and 8, we will use some of these assumptions on Ω for getting regularity properties of the solutions of some concrete variational inequalities.

Definition 2.4. We say that the open subset Ω of $\mathbb{R}^d$ has the cone property if there exists a finite open bounded cover $\{O_j\}_{j\in J}$ of the boundary Γ of Ω and, for any j, there exists a cone C_j with the vertex at 0, such that, for all $\boldsymbol{x} \in O_j \cap \Omega$, $\boldsymbol{x} + C_j$ do not intersect $O_j \cap \Gamma$.