Developments in Mathematics

Geoffrey Mason
Ivan Penkov
Joseph A. Wolf *Editors*

Developments and Retrospectives in Lie Theory

Geometric and Analytic Methods

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Lie Groups, Lie Algebras and their Representations

Third Announcement -- October 1991

This is to announce a new program ***Lie Groups, Lie Algebras and Their Representations*** starting in the academic year 1991-92. We plan to meet at various University of California campuses for one weekend every other month during the year. The purpose of the program is to communicate results and ideas rather than to deliver polished presentations. The tentative format of the meetings is

Saturday: one late morning talk and two afternoon talks; these scheduled in advance.
Sunday: two morning talks and one early afternoon talk; some may be scheduled in advance and the rest will be scheduled Saturday.

The meetings are scheduled as follows:

Location	Dates	Local Organizer	e-mail address
Berkeley	October 19, 20	Joseph Wolf	jawolf@cartan.berkeley.edu
Riverside	December 7, 8	Ivan Penkov	penkov@ucrmath.ucr.edu
Davis	February 8, 9 ??	Alice Fialowski	fialowski@bolyai.ucdavis.edu
UCLA	April 4, 5	Robert Blattner	blattner@math.ucla.edu
UC Santa Cruz	May 30, 31	Daniel Goldstein	danny@cats.ucsc.edu

The program for the first meeting, Saturday October 19 and Sunday October 20, is

time	speaker	title
Sat 11:00	Joseph Wolf	Lie structures on direct limit groups and their completions (joint work with L. Natarajan and E. Rodríguez-Carrington)
Sat 02:00	Dmitri Fuchs	On the cohomology of Lie algebras of vector fields in the line and of Hamiltonian vector fields in the plane
Sat 04:00	Nicolai Reshetikhin	q-holonomic systems and quantum affine Lie algebras, I
Sun 09:00	Ivan Penkov	Infinite dimensional irreducible modules over finite dimensional Lie superalgebras
Sun 10:30	Yan Soibelman	Algebras of functions on compact quantum groups
Sun 01:00	Nicolai Reshetikhin	q-holonomic systems and quantum affine Lie algebras, II

These talks will take place in 9 Evans Hall on the Berkeley campus. The one hour scheduled time of any talk will be extended by the informal discussion that takes place during that talk.

There are no registration fees. No support is currently available. We hope to minimize participants' expenses by car-pooling and various informal arrangements for staying over Saturday night.

People interested in participating, or speaking and participating, at a meeting, should contact the local organizer for that meeting. People interested in helping with the local organization should contact both the local organizer and Joseph Wolf. If you wish to be on the mailing list for further announcements, and for meeting programs/schedules, write to Joseph Wolf.

Announcement of the First Conference, October 1991

Preface

The West Coast of the United States has a longstanding tradition in Lie theory, although before 1991 there had been no systematic cooperation between the various strongholds. In 1991, partially inspired by the arrival of some well-known Lie theorists from eastern Europe, the situation changed. A new structure emerged: a seminar that would meet at various University of California campuses three or four times a year. The purpose of the seminar was to foster contacts between researchers and graduate students at the various campuses by facilitating the sharing of ideas prior to formal publication. This idea quickly gained momentum, and became a great success. It was enthusiastically supported by graduate students. A crucial feature of the entire endeavor was the feeling of genuine interest for the work of colleagues and the strong desire to collaborate.

The first meeting of the new seminar Lie Groups, Lie Algebras and their Representations was held in Berkeley on October 19 and 20, 1991. On the second day of the seminar, excitement was made even more memorable by the historic Berkeley–Oakland fire, which we observed from Evans Hall. The original announcement for that meeting is reprinted here on page v. The phrase "The purpose of the program is to communicate results and ideas rather than to deliver polished presentations" quickly became, and still is, the guiding principle of the seminar. We never restricted ourselves to Lie Theory *per se*, and speakers from geometry, algebra, complex analysis, and other adjacent areas were often invited.

NSF travel grants were crucial to the success of the seminar series. These grants funded travel for speakers, graduate students and postdoctoral researchers, and we thank the National Science Foundation for its continued support.

Over the years our idea became widely popular. On occasion the seminar took place in Salt Lake City, in Stillwater, Oklahoma, and in Eugene, Oregon. In addition, colleagues from other regional centers of Lie theory and related areas picked up on our idea and created their own meeting series. This is how the "Midwest Lie Theory Seminars", the "Midwest Group Theory Seminar", the "Southeastern Lie Theory Workshops", and other regional series emerged.

The California Lie Theory Seminar has now been alive and well for 23 years. Joe Wolf has always played a central role in the seminar, along with Geoff Mason and Ivan Penkov. When Ivan left for Germany in 2004, Susan Montgomery and Milen Yakimov joined the team of organizers.

Over the course of these 23 years, at about 20 talks per year, some 450 talks have been hosted by the seminar. It seemed unrealistic to give an overview of all of the topics covered over all these years, and similarly unrealistic to try to publish a comprehensive set of volumes. Rather, we settled on two retrospective volumes containing work representative of the seminar as a whole. We started with a list of participants who spoke more than once in the seminar, and invited them to submit work relevant to their seminar talks. For obvious reasons we did not hear from everyone. Nevertheless, there was a strong response, and the reader of these Volumes will find 26 research papers, all of which received strong referee reports, in the greater area of Lie Theory. We decided to split the papers into two volumes: "Algebraic Methods" and "Geometric/Analytic Methods". We thank Springer, the publisher of these volumes, and especially Ann Kostant and Elizabeth Loew, for their cooperation and assistance in this project.

This is the Geometric/Analytic Methods volume.

Santa Cruz, CA, USA — Geoffrey Mason
Bremen, Germany — Ivan Penkov
Berkeley, CA, USA — Joseph A. Wolf

Contents

Group Gradings on Lie Algebras and Applications to Geometry: II

Yuri Bahturin, Michel Goze, and Elisabeth Remm

Abstract This paper is devoted to some applications of the theory of group gradings on Lie algebras to two topics of differential geometry, such as generalized symmetric manifolds and affine structures on nilmanifolds.

Key words Symmetric manifolds • Nilmanifolds • Affine connections • Filiform Lie algebras • Graded algebras

Mathematics Subject Classification (2010): 17B20, 17B30, 17B70, 17B40, 53C35, 53C05, 57S25.

1 Introduction

This paper is a sequel to [4]. Here we discuss two topics in differential geometry, closely related to the theory of gradings on Lie algebras which we developed in that earlier paper. These topics are as follows: the generalization of the theory of symmetric manifolds to the case where the local symmetries form the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ and Milnor's problem about the existence of affine structure on nilmanifolds. This latter topic quickly leads to the necessity of describing abelian group gradings on filiform nilpotent Lie algebras. As we showed in [4], in the case of filiform Lie algebras of nonzero rank, all abelian group gradings of a filiform Lie algebra are isomorphic to the coarsenings of a standard grading produced by the action of the maximal torus. In this paper, we classify these gradings up to equivalence (see Sect. 2) and also produce some results on filiform Lie algebras of zero rank, also called characteristically nilpotent.

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The results on the first topic have been reported to the Lie Theory Workshop at UC Berkeley in 2006. Since there was some later development in this area, we describe these results in some detail in Sect. 3.

In this geometric part the field of coefficients $\mathbb{K}$ has characteristic zero, and starting from Sect. 3.3, $\mathbb{K}$ is $\mathbb{R}$, or $\mathbb{C}$, and for a real Lie algebra $\mathfrak{g}$ its complexification $\mathfrak{g} \otimes \mathbb{C}$ will be denoted $\mathfrak{g}_{\mathbb{C}}$.

2 Gradings on Filiform Algebras of Nonzero Rank, Up to Equivalence

Recall that two gradings of an algebra are equivalent if there is an automorphism of this algebra permuting the components of the grading. In what follows we use a theorem in [4] according to which any grading of a filiform Lie algebra of nonzero rank is isomorphic, hence equivalent, to a grading where the elements of an adapted (case L_n and A_n^p) or quasi adapted (case Q_n and B_n^p) basis are homogeneous. Let $\mathcal{G}$ be the grading group and, for each $i = 1, 2, \ldots, n$, d_i denote the degree of the ith element of this basis. So if $\{X_1, X_2, \ldots, X_n\}$ is the adapted basis of $\mathfrak{g}$, then $d_i = \deg X_i$, $i = 1, 2, \ldots, n$, in the case of L_n and A_n^p and $d_i = \deg Y_i$, $i = 1, 2, \ldots, n$, in the case of Q_n and B_n^p, where $Y_1 = X_1 + X_2$ and $Y_i = X_i$, for $i = 2, \ldots, n$. Since $\mathfrak{g}$ is generated by X_1, X_2 (respectively, Y_1, Y_2) it follows that knowing $a = d_1$ and $b = d_2$ automatically gives values for the remaining d_i, $i = 3, \ldots, n$.

2.1 Gradings on L_n and A_n^p

Let $\{X_1, X_2, \ldots, X_n\}$ be an adapted basis of a filiform Lie algebra $\mathfrak{g}$ of type L_n or A_n^p. Since $[X_1, X_i] = X_{i+1}$ for $i = 2, \ldots, n-1$, we know that $d_i = a^{i-2}b$ for $i = 3, \ldots, n$. However, if $\mathfrak{g}$ is of the type A_n^p, we already have $b = a^{p+1}$. So in this case, $d_i = a^{p+i-1}$. In the case of L_n, the universal group of any grading is the factor-group of the free abelian group with free basis a, b by the relations satisfied by a, b, while in the case of A_n^p this is a factor-group of the free abelian group of rank 1 or rank 2 but we have to consider, in the latter case, that $b = a^{p+1}$.

Let us first consider the case of L_n. Any grading is a coarsening of the standard grading, which we denote by Γ_{st}. If all d_i are pairwise different then there is no coarsening and we have the standard grading.

Case 1. If $d_1 = d_l$, for $2 \le l \le n$, then the grading is a coarsening of the grading

$$\Gamma_0^l : \mathfrak{g} = \langle X_2 \rangle \oplus \cdots \oplus \langle X_1, X_l \rangle \oplus \cdots \oplus \langle X_n \rangle.$$

Since this is indeed a grading of $\mathfrak{g}$, our claim follows. The universal group of this grading is the factor-group of the free abelian group generated by a, b by a single relation $a = a^{l-2}b$, that is, the group $\mathbb{Z}$.

Case 2. If $d_i = d_j$, for $2 \le i < j \le n$, then the grading is a coarsening of a grading

$$\Gamma_k^0 : \mathfrak{g} = \langle X_1 \rangle \oplus [X_2]_k \oplus \cdots \oplus [X_{k+1}]_k.$$

Here $[X_i]_k$ is the span of the set of all X_j, $2 \le j \le n$ such that k divides $i - j$. This easily follows if we choose k to be the least positive with $d_2 = d_{2+k}$.

The universal group of this grading is the factor-group of the free abelian group generated by a, b by a single relation $b = a^k b$, that is, the group $\mathbb{Z}_k \times \mathbb{Z}$.

Any further coarsening of Γ_0^l is clearly also the coarsening of a Γ_k^0, so we can restrict ourselves to consider only the coarsenings of the latter grading.

Case 3. Any proper coarsening of Γ_k^0, which does not decrease k, is equivalent to one of the following.

$$\Gamma_k^l : \mathfrak{g} = [X_2]_k \oplus \cdots \oplus (\langle X_1 \rangle \oplus [X_l]_k) \oplus \cdots \oplus [X_{k+1}]_k.$$

Indeed, any further coarsening of $[X_l]_k$ will decrease k so we have to assume that $d_1 = d_l$, for $2 \le l \le n$, proving our claim. The universal group of this grading is the factor group of the group $\mathbb{Z}_k \times \mathbb{Z}$ by additional relation $a = a^{l-2}b$, that is, the group $\mathbb{Z}_k$.

Clearly, any further coarsening will lead to the decreasing of k, and so any grading is equivalent to one of the previous gradings.

Notice that these gradings are pairwise inequivalent. First, we have to look at the number of homogeneous components. Then it becomes clear that we only need to distinguish between the gradings with different values of the superscript parameter $l = 2, \ldots, n$. In this case, if an automorphism φ maps Γ_0^l to Γ_0^m, or Γ_k^l to Γ_k^m, where $m > l$, then $\varphi(\langle X_1, X_l \rangle) = \langle X_1, X_m \rangle$. But then $\varphi(X_{l+1}) = \varphi([X_1, X_l]) = [\varphi(X_1), \varphi(X_l)] = \alpha_{m+1}X_{m+1}$, $\varphi(X_{l+2}) = \alpha_{m+2}X_{m+2}$, etc. Finally, $\varphi(X_{n-m+l+1}) = 0$, which is impossible because $2 \le n - m + l + 1 \le n$.

As a result, we have the following.

Theorem 2.1. *Let $\mathfrak{g}$ be a filiform Lie algebra of the type L_n. If $\mathfrak{g}$ is $\mathcal{G}$-graded, then there exists a graded homogeneous adapted basis $\{X_1, X_2, \ldots, X_n\}$. If d_i denotes the degree of X_i, then any $\mathcal{G}$-grading is equivalent to one of the following pairwise inequivalent gradings:*

(1) Γ_{st}, $U(\Gamma_{\mathrm{st}}) = \mathbb{Z}^2$, $d_1 = (1, 0)$, $d_i = (i - 2, 1)$, $i = 2, \ldots, n$.
(2) Γ_0^l, $U(\Gamma_0^l) = \mathbb{Z}$, $2 \le l \le n$, $d_1 = 1$, $d_i = i - l + 1$, $i = 2, \ldots, n$.
(3) Γ_k^0, $U(\Gamma_k^0) = \mathbb{Z}_k \times \mathbb{Z}$, $1 \le k \le n-2$, $d_1 = (\overline{1}, 0)$, $d_i = (\overline{i-2}, 1)$, $i = 2, \ldots, n$.
(4) Γ_k^l, $U(\Gamma_k^l) = \mathbb{Z}_k$, $1 \le k \le n - 2$, $2 \le l \le k + 1$, $d_1 = \overline{1}$, $d_i = \overline{i - l + 1}$, $i = 2, \ldots, n$.

The total number of pairwise inequivalent gradings of L_n is equal to $1 + (n-1) + (n-2) + (1 + 2 + \cdots + (n-2)) = \dfrac{(n-1)(n+2)}{2}$.

In the case of A_n^p, we need to consider the coarsenings of the standard grading, which we denote here by Γ_{st}, where $U(\Gamma_{\mathrm{st}}) \cong \mathbb{Z}$, with free generator a, $d_1 = a$, $d_i = a^{i+p-1}$, where $i = 2, \ldots, n$. Clearly, in this case, any proper coarsening leads to relations $a^i = a^j$, hence $a^{i-j} = e$, for different $1 \le i, j \le n$. If m is the greatest common divisor of all such $i - j$, then the universal group is $\mathbb{Z}_m$. Any value of m between 1 and $n + p - 2$ is possible. Indeed, if $p \le m \le n + p - 2$, then $d_1 = d_{m-p+2}$. If $1 \le m \le n - 2$, then $d_2 = d_{m+2}$. But $1 \le p \le n - 4 < n - 2$, and so any m between 1 and $n + p - 2$ is available. Let us denote the grading corresponding to m by $\Gamma(m)$. If $1 \le m \le p - 1$, $\Gamma(m)$ is similar to Γ_m^0 of L_n, then

$$\Gamma(m) : \mathfrak{g} = \langle X_1 \rangle \oplus [X_2]_m \oplus \cdots \oplus [X_{m+1}]_m.$$

If $n - 1 \le m \le n + p - 1$ and $l = m - p + 2$, then $\Gamma(m)$ is similar to Γ_0^l of L_n:

$$\Gamma(m) : \mathfrak{g} = \langle X_2 \rangle \oplus \cdots \oplus \langle X_1, X_l \rangle \oplus \cdots \oplus \langle X_n \rangle.$$

If $p \le d \le n - 2$, then we have the gradings similar to Γ_m^l of L_n:

$$\Gamma(m) : \mathfrak{g} = [X_2]_m \oplus \cdots \oplus (\langle X_1 \rangle \oplus [X_l]_m) \oplus \cdots \oplus [X_{m+1}]_m.$$

Here l is a number between 2 and n such that $l + p - 1 \equiv 1 \bmod m$. The pairwise inequivalence of all the above gradings follows, as in the case of L_n.

Theorem 2.2. *Let $\mathfrak{g}$ be a filiform Lie algebra of the type A_n^p. If $\mathfrak{g}$ is $\mathcal{G}$-graded, then there exists a graded homogeneous adapted basis $\{X_1, X_2, \ldots, X_n\}$. If d_i denotes the degree of X_i, then any $\mathcal{G}$-grading is equivalent to one of the following non equivalent gradings:*

(1) Γ_{st}, $U(\Gamma_{\mathrm{st}}) = \mathbb{Z}$, $d_1 = 1$, $d_i = p + i - 1$, $i = 2, \ldots, n$.
(2) $\Gamma(m)$, $U(\Gamma(m)) = \mathbb{Z}_m$, $1 \le m \le n + p - 2$, $d_1 = \overline{1}$, $d_i = \overline{p + i - 1}$, $i = 2, \ldots, n$.

The total number of pairwise inequivalent gradings of A_n^p is equal to $n + p - 2$.

2.2 Gradings on Q_n and B_n^p

Let $\{Y_1, Y_2, \ldots, Y_n\}$ be a quasi-adapted basis of a filiform Lie algebra $\mathfrak{g}$ of type Q_n or B_n^p. Since $[Y_1, Y_i] = Y_{i+1}$ for $i = 2, \ldots, n-2$, we know that $d_i = a^{i-2}b$ for $i = 3, \ldots, n-1$. Also, $[Y_i, Y_{n-i+1}] = (-1)^{i+1} Y_n$ which assigns to d_n the value of $d_n = a^{n-3}b^2$. If $\mathfrak{g}$ is of the type B_n^p, we already have $b = a^{p+1}$. So in this case, $d_i = a^{p+i-1}$, for $2 \le i \le n-1$, and $d_n = a^{n+2p-1}$. In the case of Q_n, the universal group of any grading is the factor group of the free abelian group with free basis a,b by the relations satisfied by a,b while in the case of B_n^p, this is a factor-group of the free abelian group of rank 1 or rank 2 but we have to consider, in the latter case, that $b = a^{p+1}$.

Let us first consider the case of Q_n. We remember that $n = 2m$, for some $m \geq 2$. Now any grading is a coarsening of the standard grading, which we denote by $\overline{\Gamma}_{\mathrm{st}}$. If all d_i are pairwise different, then there is no coarsening and we have the standard grading.

Case 1. If $d_1 = d_n$, then the grading is a coarsening of the grading

$$\overline{\Gamma}(1,n) : \mathfrak{g} = \langle Y_2\rangle \oplus \cdots \oplus \langle Y_{n-1}\rangle \oplus \langle Y_1, Y_n\rangle.$$

Since this is indeed a grading of $\mathfrak{g}$, our claim follows. The universal group of this grading is the factor group of the free abelian group generated by a,b by a single relation $a = a^{n-3}b^2$, that is, the group $\mathbb{Z}\times\mathbb{Z}_2$.

Case 2. If $d_1 = d_l$, for $2 \leq l \leq n-1$, then $a = a^{l-2}b$. In this case also $d_n = a^{n-3}b^2 = a^{n-l}b = d_{n-l+2}$. Hence, for all q satisfying $l+q = n+2$, except $l = 2$, or $q = 2$, the grading is a coarsening of one of the following gradings. If $l \neq q$, then we have

$$\overline{\Gamma}_0^l : \mathfrak{g} = \langle Y_2\rangle \oplus \cdots \oplus \langle Y_1, Y_l\rangle \oplus \cdots \oplus \langle Y_m, Y_n\rangle \oplus \langle Y_{n-1}\rangle.$$

If $l = q = m+1$, then we have

$$\overline{\Gamma}_0^{m+1} : \mathfrak{g} = \langle Y_2\rangle \oplus \cdots \oplus \langle Y_1, Y_{m+1}, Y_n\rangle \oplus \cdots \oplus \langle Y_{n-1}\rangle.$$

Notice that this grading is a coarsening of $\overline{\Gamma}(1,n)$.

In the exceptional cases, $l = 2$ or $q = 2$, we have the following.

If $l = 2$, then we have

$$\overline{\Gamma}_0^2 : \mathfrak{g} = \langle Y_1, Y_2\rangle \oplus \cdots \oplus \langle Y_1, Y_l\rangle \oplus \cdots \oplus \langle Y_n\rangle.$$

If $q = 2$, then we have

$$\overline{\Gamma}_0^n : \mathfrak{g} = \langle Y_2, Y_n\rangle \oplus \langle Y_3\rangle \oplus \cdots \oplus \langle Y_{n-1}\rangle \oplus \langle Y_1\rangle.$$

Since all these are indeed gradings of $\mathfrak{g}$, our claim follows. The universal group of this grading is the factor-group of the free abelian group generated by a,b by a single relation $a = a^{l-2}b$, that is, the group $\mathbb{Z}$.

Case 3. If $d_i = d_j$, for $2 \leq i < j \leq n-1$, then the grading is a coarsening of a grading

$$\overline{\Gamma}_k^0 : \mathfrak{g} = \langle Y_1\rangle \oplus [Y_2]_k \oplus \cdots \oplus [Y_{k+1}]_k \oplus \langle Y_n\rangle,$$

where k is such that $1 \le k \le n-3$. Here $[Y_i]_k$ is the span of the set of all Y_j, $2 \le j \le n$, such that k divides $i-j$. This easily follows if we choose k to be the least positive with $d_2 = d_{2+k}$.

The universal group of this grading is the factor-group of the free abelian group generated by a, b by a single relation $b = a^k b$, that is, the group $\mathbb{Z}_k \times \mathbb{Z}$.

Any coarsening of $\overline{\Gamma}(1,n)$ (Case 1) is either $\overline{\Gamma}_0^{m+1}$ or is a coarsening of some $\overline{\Gamma}_k^0$. Any coarsening of a grading $\overline{\Gamma}_0^l$ (Case 2) is also a coarsening of some $\overline{\Gamma}_k^0$, so we can restrict ourselves to consider only the coarsenings of the latter gradings. This shows that any grading is either standard, or equivalent to one of the gradings in Cases 1–3 or is a proper coarsening of a grading $\overline{\Gamma}_k^0$. Let us choose $\overline{\Gamma}_k^0$ with minimal possible k.

Case 4. Any proper coarsening of $\overline{\Gamma}_k^0$, which does not change k, is equivalent to one of the following:

$$\overline{\Gamma}(1,n)_k : \mathfrak{g} = [Y_2]_k \oplus \cdots \oplus [Y_{k+2}]_k \oplus \langle Y_1, Y_n \rangle,$$

where $1 \le k \le n-3$, or

$$\overline{\Gamma}_k^l : \mathfrak{g} = [Y_2]_k \oplus \cdots \oplus (\langle Y_1 \rangle \oplus [Y_l]_k) \oplus \cdots \oplus ([Y_q]_k \oplus \langle Y_n \rangle) \oplus \cdots \oplus [Y_{k+1}]_k,$$

where $1 \le k \le n-3$, $2 \le l, q \le k+1$, $l + q \equiv n + 2 \bmod k$.

The universal group of $\overline{\Gamma}(1,n)_k$ is $\mathbb{Z}_k \times \mathbb{Z}_2$, whereas, in the case of $\overline{\Gamma}_k^l$, the universal group is $\mathbb{Z}_k$.

Indeed, any further coarsening of $[Y_l]_k$ will decrease k, so we have to assume that $d_1 = d_l$, for $2 \le l \le n$, proving our claim. Clearly, any further coarsening will lead to further decreasing of k, and so any grading is equivalent to one of the previous gradings.

Notice that these gradings are pairwise inequivalent. First, the gradings with different universal groups are not equivalent. As a result, we only need to distinguish between the gradings in the sets $\overline{\Gamma}_0^l$, $2 \le l \le n$ (the universal group $\mathbb{Z}$) and $\overline{\Gamma}_k^l$, $2 \le l \le k+1$ (the universal group $\mathbb{Z}_k$). This is done exactly in the same way, except for the cases where Y_1 is in a component which is not two-dimensional. However, such a grading can only be mapped to a grading with the same property because Y_1 is the only element among Y_i with $(\mathrm{ad} Y_i)^{n-2} \neq 0$.

As a result, we have the following.

Theorem 2.3. *Let $\mathfrak{g}$ be a filiform Lie algebra of the type Q_n. If $\mathfrak{g}$ is $\mathcal{G}$-graded, then there exists a graded homogeneous quasi-adapted basis $\{Y_1, Y_2, \ldots, Y_n\}$. If d_i denotes the degree of Y_i, then any $\mathcal{G}$-grading is equivalent to one of the following pairwise equivalent gradings:*

(1) $\overline{\Gamma}_{\mathrm{st}}$, $U(\overline{\Gamma}_{\mathrm{st}}) = \mathbb{Z}^2$, $d_1 = (1,0)$, $d_i = (i-2,1)$, $2 \le i \le n-1$, $d_n = (n-3,2)$.
(2) $\overline{\Gamma}(1,n)$, $U(\overline{\Gamma}(1,n)) = \mathbb{Z} \times \mathbb{Z}_2$, $d_1 = (1,\bar{0}) = d_n$, $d_i = (i-2,\bar{1})$, $2 \le i \le n-1$.

(3) $\overline{\Gamma}^l_0$, $U(\overline{\Gamma}^l_0) = \mathbb{Z}$, $2 \le l \le n$, $d_1 = 1$, $d_i = i - l + 1$, $2 \le i \le n-1$, $d_n = n - 2l + 3$.
(4) $\overline{\Gamma}^0_k$, $U(\overline{\Gamma}^0_k) = \mathbb{Z}_k \times \mathbb{Z}$, $1 \le k \le n-3$, $d_1 = (\overline{1}, 0)$, $d_i = (\overline{i-2}, 1)$, $d_n = (\overline{n-3}, 2)$.
(5) $\overline{\Gamma}(1,n)_k$, $U(\overline{\Gamma}(1,n)_k) = \mathbb{Z}_k \times \mathbb{Z}_2$, $1 \le k \le n-3$, $d_1 = (\overline{1}, \overline{0}) = d_n$, $d_i = (\overline{i-2}, \overline{1})$, $2 \le i \le n-1$.
(6) $\overline{\Gamma}^l_k$, $U(\overline{\Gamma}^l_k) = \mathbb{Z}_k$, $1 \le k \le n-3$, $2 \le l \le k+1$, $d_1 = \overline{1}$, $d_i = \overline{i - l + 1}$, $d_n = \overline{n - 2l + 3}$.

The total number of pairwise inequivalent gradings of Q_n is equal to $1 + 1 + (n-1) + (n-3) + (n-3) + (1 + 2 + \cdots + (n-3)) = \dfrac{(n-1)(n+2)}{2} - 1$.

In the case of B^p_n, we need to consider the coarsenings of the standard grading, which we denote here by Γ_{st}, where $U(\Gamma_{\mathrm{st}}) \cong \mathbb{Z}$, with free generator a, $d_1 = a$, $d_i = a^{i+p-1}$, where $i = 2, \ldots, n-1$, $d_n = a^{n+2p-1}$. Clearly, in this case, any proper coarsening leads to relations $a^i = a^j$, hence $a^{i-j} = e$, for different $1 \le i, j \le n$. If m is the greatest common divisor of all such $i - j$, then the universal group is $\mathbb{Z}_m$. Any value of m between 1 and $n + p - 3$ is possible, similar to the case of A^p_n. One more possible isolated value for m appears if we choose $d_1 = d_n$. Then $m = n + 2p - 3$. Let us denote the grading corresponding to m by $\overline{\Gamma}(m)$.

If $1 \le m \le p-1$, $\overline{\Gamma}(m)$ is similar to $\overline{\Gamma}^0_m$ of Q_n, then

$$\overline{\Gamma}(m) : \mathfrak{g} = \langle Y_1 \rangle \oplus [Y_2]_m \oplus \cdots \oplus [Y_{m+1}]_m \oplus \langle Y_n \rangle.$$

If $p \le m \le n-3$, then we have gradings similar to $\overline{\Gamma}^l_m$ of Q_n:

$$\overline{\Gamma}(m) : \mathfrak{g} = [Y_2]_m \oplus \cdots \oplus (\langle Y_1 \rangle \oplus [Y_l]_m) \oplus \cdots \oplus ([Y_q]_m \oplus \langle Y_n \rangle) \oplus \cdots \oplus [Y_{m+1}]_m.$$

If $n - 2 \le m \le n + p - 3$, $\overline{\Gamma}(m)$ is similar to $\overline{\Gamma}^l_0$ of Q_n, then

$$\overline{\Gamma}(m) : \mathfrak{g} = \langle Y_2 \rangle \oplus \cdots \oplus \langle Y_1, Y_l \rangle \oplus \cdots \oplus \langle Y_m, Y_n \rangle \oplus \langle Y_{n-1} \rangle.$$

Here $l = m - p + 2$.

If $m = n + 2p - 3$, then $\overline{\Gamma}(m)$ is similar to $\overline{\Gamma}(1, n)$:

$$\overline{\Gamma}(n + 2p - 3) : \mathfrak{g} = \langle Y_2 \rangle \oplus \cdots \oplus \langle Y_{n-1} \rangle \oplus \langle Y_1, Y_n \rangle.$$

The pairwise inequivalence of all the above gradings follows, as in the case of Q_n.

Theorem 2.4. *Let $\mathfrak{g}$ be a filiform Lie algebra of the type B^p_n. If $\mathfrak{g}$ is $\mathcal{G}$-graded, then there exists a graded homogeneous adapted basis $\{Y_1, Y_2, \ldots, Y_n\}$. If d_i denotes the degree of Y_i, then any $\mathcal{G}$-grading is equivalent to one of the following non equivalent gradings:*

(1) $\overline{\Gamma}_{\mathrm{st}}$, $U(\overline{\Gamma}_{\mathrm{st}}) = \mathbb{Z}$, $d_1 = 1$, $d_i = p + i - 1$, $i = 2, \ldots, n-1$, $d_n = n + 2p - 2$.
(2) $\overline{\Gamma}(m)$, $U(\overline{\Gamma}(m)) = \mathbb{Z}_m$, $1 \leq m \leq n + p - 3$ or $m = n + 2p - 3$, $d_1 = \overline{1}$, $d_i = \overline{p + i - 1}$, $i = 2, \ldots, n$, $d_n = \overline{n + 2p - 2}$.

The total number of pairwise inequivalent gradings of B_n^p is equal to $n + p - 3$.

2.3 Characteristically Nilpotent Lie Algebras

Definition 2.1. A finite-dimensional $\mathbb{K}$-Lie algebra $\mathfrak{g}$ is called characteristically nilpotent if any derivation of $\mathfrak{g}$ is nilpotent. It is called characteristically unipotent if the group $\mathrm{Aut}(\mathfrak{g})$ of the automorphisms of $\mathfrak{g}$ is unipotent.

A characteristically nilpotent Lie algebra has rank 0 and the Lie algebra of derivations Der $(\mathfrak{g})$ is nilpotent (but not necessarily characteristically nilpotent). In this case also Aut $(\mathfrak{g})$ is nilpotent. However, this group does not have to be unipotent. It is unipotent if $\mathfrak{g}$ is characteristically unipotent. Of course, if $\mathfrak{g}$ is characteristically unipotent, it is characteristically nilpotent.

Examples. (1) The simplest example [1], denoted by $\mathfrak{n}_{7,4}$ in terminology of [17], is seven-dimensional and given by

$$\begin{cases} [X_1, X_i] = X_{i+1}, \ 2 \leq i \leq 6, \\ [X_2, X_3] = -X_6, \\ [X_2, X_4] = -[X_5, X_2] = -X_7, \\ [X_3, X_4] = X_7. \end{cases}$$

This Lie algebra is filiform. Any automorphism of $\mathfrak{n}_{7,4}$ is unipotent. It is defined on the generators X_1, X_2 by

$$\begin{cases} \sigma(X_1) = X_1 + a_2X_2 + a_3X_3 + a_4X_4 + a_5X_5 + a_6X_6 + a_7X_7, \\ \sigma(X_2) = X_2 + b_3X_3 + \frac{b_3^2 - a_2}{2}X_4 + b_5X_5 + b_6X_6 + b_7X_7. \end{cases}$$

It follows that Aut $(\mathfrak{n}_{7,4})$ is a ten-dimensional unipotent Lie group and $\mathfrak{n}_{7,4}$ is also characteristically unipotent. Let us note that any $\sigma \in$ Aut $(\mathfrak{n}_{7,4})$ of finite order is equal to the identity.

(2) The first example of characteristically nilpotent Lie algebra was given by Dixmier and Lister [9]. It can be written as

$$\begin{cases} [X_1, X_2] = X_5, \quad [X_1, X_3] = X_6, \quad [X_1, X_4] = X_7, \quad [X_1, X_5] = -X_8, \\ [X_2, X_3] = X_8, \quad [X_2, X_4] = X_6, \quad [X_2, X_6] = -X_7, \ [X_3, X_4] = -X_5, \\ [X_3, X_5] = -X_7, \ [X_4, X_6] = -X_8. \end{cases}$$

It is an eight-dimensional nilpotent Lie algebra of nilindex 3, and it is not filiform. Let us note that nil-index 3 is the lowest possible nil-index for a characteristically nilpotent Lie algebra. Its automorphisms group Aut $(\mathfrak{g})$ is not unipotent. For example, the linear map given by

$$\begin{cases} \sigma(X_1) = X_5,\ \sigma(X_5) = X_1, \\ \sigma(X_2) = X_7,\ \sigma(X_7) = X_2, \\ \sigma(X_4) = X_8,\ \sigma(X_8) = X_4, \\ \sigma(X_3) = -X_3,\ \sigma(X_6) = -X_6 \end{cases}$$

is a non-unipotent automorphism of $\mathfrak{g}$ of order 2. In the case where char $\mathbb{K} \neq 2$, σ defines a nontrivial $\mathbb{Z}_2$-grading Γ of $\mathfrak{g}$:

$$\Gamma: \quad \mathfrak{g} = \langle X_1 + X_5, X_2 + X_7, X_4 + X_8\rangle \oplus \langle X_1 - X_5, X_2 - X_7, X_4 - X_8, X_3, X_6\rangle.$$

2.4 Structure of Characteristically Nilpotent Lie Algebras

It is known from [13] that any filiform $(n+1)$-dimensional Lie algebra over an algebraic field of characteristic 0 is defined by its Lie bracket μ with $\mu = \mu_0 + \psi$ where μ_0 is the Lie multiplication of L_{n+1} and ψ a 2-cocycle of $Z^2(L_{n+1}, L_{n+1})$ satisfying $\psi \circ \psi = 0$, that is, ψ is also a $(n+1)$-dimensional Lie multiplication. Let us consider the natural $\mathbb{Z}$-grading of L_{n+1}:

$$L_{n+1} = \bigoplus_{i \in \mathbb{Z}} L_{n+1,i},$$

where $L_{n+1,1}$ is generated by e_0, e_1 and $L_{n+1,i}$ by e_i for $i = 2, \dots, n$, and other subspaces are zero. This grading induces a $\mathbb{Z}$-grading in the spaces of cochains of the Chevalley–Eilenberg complex of L_{n+1}:

$$\mathcal{C}_p^k(L_{n+1}, L_{n+1}) = \{\phi \in \mathcal{C}_k(L_{n+1}, L_{n+1}),\ \phi(L_{n+1,i_1}, \dots, L_{n+1,i_k}) \subset L_{n+1,i_1+\dots+i_k+p}\}.$$

Since $d(\mathcal{C}_p^k(L_{n+1}, L_{n+1})) \subset \mathcal{C}_p^{k+1}(L_{n+1}, L_{n+1})$, we deduce a grading in the spaces of cocycles and coboundaries. Let

$$H_p^k(L_{n+1}, L_{n+1}) = Z_p^k(L_{n+1}, L_{n+1}) / B_p^k(L_{n+1}, L_{n+1})$$

be the corresponding grading in the Chevalley–Eilenberg cohomological spaces of L_{n+1}. We put

$$F_0 H^k(L_{n+1}, L_{n+1}) = \bigoplus_{p \in Z} H_p^k(L_{n+1}, L_{n+1}),$$

$$F_1 H^k(L_{n+1}, L_{n+1}) = \bigoplus_{p \geq 1} H_p^k(L_{n+1}, L_{n+1}).$$

We have

Proposition 2.1. *Let $\psi_{k,s}$, $1 \leq k \leq n-1$, $2k \leq s \leq n$ be the 2-cocycle in $\mathcal{C}^2(L_{n+1}, L_{n+1})$ defined by*

- $\psi_{k,s}(e_k, e_{k+1}) = e_s$,
- $\psi_{k,s}(e_i, e_{i+1}) = 0$ *if* $i \neq k$,
- $\psi_{k,s}(e_i, e_j) = 0$ *if* $i > k$,
- $\psi_{k,s}(e_i, e_j) = (-1)^{k-i} C_{j-k-1}^{k-i} e_{i+j+s-2k-1}$, $1 \leq i \leq k < j-1 \leq n-1$, $0 \leq i+j-2k-1 \leq n-s$

Then the family of $\psi_{k,s}$ with $1 \leq [n/2]-1$, $4 \leq s \leq n$, forms a basis of $F_1 H^2(L_{n+1}, L_{n+1})$.

Note that the cocycles $\psi_{k,s}$ also satisfy $\psi_{k,s}(e_k, e_j) = e_{j+s-k-1}$ when $k < j$.

Proposition 2.2. *If $\mu = \mu_0 + \psi$ is the Lie multiplication of a filiform $(n+1)$-dimensional Lie algebra, then $[\psi] \in F_1 H^2(L_{n+1}, L_{n+1})$ if n is even or $\psi \in F_1 H^2(L_{n+1}, L_{n+1}) + [\psi_{(n-1)/2,n}]$ if n is odd, where $[\psi]$ denote the class in $H^2(L_{n+1}, L_{n+1})$ of the 2-cocycle ψ.*

Examples. (1) Any seven-dimensional filiform Lie algebra can be written as $\mu = \mu_0 + \psi$ with

$$\psi = a_{1,4}\psi_{1,4} + a_{1,5}\psi_{1,5} + a_{1,6}\psi_{1,6} + a_{2,6}\psi_{2,6}.$$

(2) Any eight-dimensional filiform Lie algebra can be written as $\mu = \mu_0 + \psi$ with

$$\psi = a_{1,4}\psi_{1,4} + a_{1,5}\psi_{1,5} + a_{1,6}\psi_{1,6} + a_{1,7}\psi_{1,7} + a_{2,6}\psi_{2,6} + a_{2,7}\psi_{2,7} + a_{3,7}\psi_{3,7}.$$

(3) Any nine-dimensional filiform Lie algebra can be written as $\mu = \mu_0 + \psi$ with

$$\psi = a_{1,4}\psi_{1,4} + a_{1,5}\psi_{1,5} + a_{1,6}\psi_{1,6} + a_{1,7}\psi_{1,7} + a_{1,8}\psi_{1,8} + a_{2,6}\psi_{2,6} + a_{2,7}\psi_{2,7} + a_{2,8}\psi_{2,8} + a_{3,8}\psi_{3,8}.$$

(4) Any ten-dimensional filiform Lie algebra can be written as $\mu = \mu_0 + \psi$ with

$$\psi = a_{1,4}\psi_{1,4} + a_{1,5}\psi_{1,5} + a_{1,6}\psi_{1,6} + a_{1,7}\psi_{1,7} + a_{1,8}\psi_{1,8} + a_{1,9}\psi_{1,9} + a_{2,6}\psi_{2,6} + a_{2,7}\psi_{2,7} + a_{2,8}\psi_{2,8} + a_{2,9}\psi_{2,9} + a_{3,8}\psi_{3,8} + a_{3,9}\psi_{3,9} + a_{4,9}\psi_{4,9}.$$

To recognize characteristically nilpotent Lie algebras among the filiform Lie algebras, we can use the notion of a *sill* algebra. Recall that a filiform Lie algebra $\mathfrak{g}$ such that gr $(\mathfrak{g})$ is isomorphic to L_{n+1} can be written in an adapted basis as

$$[e_0, e_i] = e_{i+1},\ i = 1 \cdots, n-1,\ [e_i, e_j] = \sum_{r=1}^{n-i-j} a_{ij}^r e_{i+j+r},\ 1 \leq i < j \leq n-2.$$

A filiform Lie algebra $\mathfrak{g}$ such that gr $(\mathfrak{g})$ is isomorphic to Q_{n+1} can be written in an quasi-adapted basis:

$$[Z_0, Z_i] = Z_{i+1},\ i = 1\cdots, n-2,\ \ [Z_i, Z_{n-i}] = (-1)^i Z_n,$$
$$\left[Z_i, Z_j\right] = \sum_{r=1}^{n-i-j} b_{ij}^r Z_{i+j+r}$$

for $1 \le i < j \le n-2$.

Definition 2.2 ([14]). Let $\mathfrak{g}$ be a $(n+1)$-dimensional filiform Lie algebra such that gr $(\mathfrak{g})$ is isomorphic to L_{n+1}. The sill algebra of $\mathfrak{g}$ is defined by

$$[e_0, e_i] = e_{i+1},\ i = 1, \ldots, n-1,\ \ [e_i, e_j] = a_{ij}^r e_{i+j+r}$$

where $r \neq 0$ is the smallest index such that $a_{ij}^r \neq 0$ for some (i, j).

If gr $(\mathfrak{g})$ is isomorphic to Q_{n+1}, then the sill algebra is defined by

(1) If $b_{ij}^{n-i-j} = 0$, then

$$[Z_0, Z_i]=Z_{i+1},\ i=1,\ldots,n-2,\ [Z_i, Z_{n-i}]=(-1)^i Z_n,\ [Z_i, Z_j]=b_{ij}^r Z_{i+j+r}$$

for $1 \le i < j \le n-2$ where $r \neq 0$ is the smallest index such that $b_{ij}^r \neq 0$ for some (i, j).

(2) If $b_{ij}^{n-i-j} \neq 0$ for some (i, j), then

$$[Z_0, Z_i]=Z_{i+1},\ i = 1,\ldots,n-2,\ [Z_i, Z_{n-i}]=(-1)^i Z_n,\ [Z_i, Z_j]=b_{ij}^{n-i-j} Z_n$$

for $1 \le i < j \le n-2$.

Proposition 2.3 ([26]). *A filiform Lie algebra is characteristically nilpotent if and only if it is not isomorphic to its sill algebra.*

Examples. (1) Any seven-dimensional filiform characteristically nilpotent Lie algebra can be written $\mu = \mu_0 + \psi$ with

(a) $\psi = \psi_{1,5} + \psi_{1,6}$,
(b) $\psi = \psi_{1,4} + \psi_{1,6}$,
(c) $\psi = \psi_{1,5} + \psi_{2,6}$.

(2) Any eight-dimensional filiform characteristically nilpotent Lie algebra can be written as $\mu = \mu_0 + \psi$ with

(a) $\psi = \psi_{1,4} + a_{1,5}\psi_{1,5} - a_{2,6}\psi_{2,6} + \psi_{3,7}$.
(b) $\psi = \psi_{1,5} + a_{1,6}\psi_{1,6} + \psi_{3,7}$.
(c) $\psi = a_{1,4}\psi_{1,4} + \psi_{2,6} + \psi_{2,7}$.
(d) $\psi = \psi_{1,5} + \psi_{2,6}$.
(e) $\psi = \psi_{1,4} + a_{1,6}\psi_{1,6} + \psi_{2,7}$.

(f) $\psi = a_{1,5}\psi_{1,5} + \psi_{1,6} + \psi_{2,7}$.
(g) $\psi = a_{1,4}\psi_{1,4} + \psi_{1,6} + \psi_{1,7}$.
(h) $\psi = \psi_{1,4} + \psi_{1,6}$.
(i) $\psi = \psi_{1,4} + \psi_{1,7}$.
(j) $\psi = \psi_{1,5} + \psi_{1,6}$.

2.5 $\mathbb{Z}_2$-Gradings of Filiform Characteristically Nilpotent Lie Algebras

Since characteristically nilpotent Lie algebras have zero rank, they do not admit $\mathbb{Z}$-gradings. The following shows that there exists a class of such Lie algebras admitting $\mathbb{Z}_2$-gradings.

Proposition 2.4. *Let $\mu_0 + \psi$ the multiplication of a $(n+1)$-dimensional characteristically nilpotent Lie algebra $\mathfrak{g}$ such that $\mathrm{gr}\,(\mathfrak{g})$ is isomorphic to L_{n+1}. If*

$$\psi = \sum a_{k,2s}\psi_{k,2s}$$

or

$$\psi = \sum a_{k,2s+1}\psi_{k,2s+1},$$

then this Lie algebra admits a $\mathbb{Z}_2$-grading.

Proof. Since $\psi_{k,2s}(e_i, e_j) = a e_{i+j+2s-2k-1}$, the Lie algebra $\mu_0 + \psi$ is characteristically nilpotent as soon as we have in the sum ψ two terms $\psi_{k,2s}$ and $\psi_{k',2s'}$ such that $s - k \neq s' - k'$. Let us consider two $\mathbb{Z}_2$-gradings of the vector space $\mathfrak{g}$ as follows.

- $\mathfrak{g} = \langle e_1, e_3, \ldots, e_{2p\pm1}\rangle \bigoplus \langle e_0, e_2, e_4, \ldots, e_{2p}\rangle$
- $\mathfrak{g} = \langle e_2, e_4, \ldots, e_{2p}\rangle \bigoplus \langle e_0, e_1, e_3, \ldots, e_{2p\pm1}\rangle$;

(the $\pm$ sign means that we consider the cases n odd and n even at the same time).

We consider the first vectorial decomposition. From our description of gradings on L_n we can see that this is a grading of μ_0. It is sufficient then to check that this is a grading of $\psi_{k,2s}$. A cocycle $\psi_{k,s}$ is homogeneous, that is it satisfies $\psi_{k,s}(\mathfrak{g}_i, \mathfrak{g}_j) \subset \mathfrak{g}_{i+j(\mathrm{mod}\,2)}$ if s is even. In fact

$$\psi_{k,s}(e_{2i+1,2j+1}) = \lambda e_{2i+2j-2k+1+s}$$

where λ is a nonzero constant and $2i + 2j - 2k + 1 + s$ is odd if and only if s is even. Likewise

$$\psi_{k,s}(e_{2i,2j}) = \lambda e_{2i+2j-2k-1+s}$$

with $\lambda \neq 0$ and $2i+2j-2k-1+s$ is odd if and only if s is even. Since moreover

$$\psi_{k,s}(e_{2i}, e_{2j+1}) = e_{2i+2j-2k+s},$$

$2i+2j-2k+s$ is even as soon as s is even. Thus the existence of this grading implies that s is even.

Similarly, the vectorial decomposition of the second type, is a $\mathbb{Z}_2$-grading of $\mu_0 + \psi_{k,s}$ if s is odd. □

Proposition 2.5. *Let $\mu_0 + \psi$ the multiplication of a $(n+1)$-dimensional characteristically nilpotent Lie algebra $\mathfrak{g}$ such that* gr $(\mathfrak{g})$ *is isomorphic to Q_{n+1}. If*

$$\psi = \sum a_{k,2s}\psi_{k,2s} + \psi_{\frac{n-1}{2},n}$$

or

$$\psi = \sum a_{k,2s+1}\psi_{k,2s+1} + \psi_{\frac{n-1}{2},n},$$

then this Lie algebra admits a $\mathbb{Z}_2$-grading.

Proof. Since any $\mathbb{Z}_2$-grading on $\mathfrak{g}$ induces the same grading on gr $(\mathfrak{g}) = Q_{n+1}$, we consider the vectorial decompositions of $\mathfrak{g}$:

- $\mathfrak{g} = \langle e_2, e_4, \dots, e_{n-1}\rangle \oplus \langle e_0, e_1, e_3, \dots, e_n\rangle$,
- $\mathfrak{g} = \langle e_0 + e_1, e_n\rangle \oplus \langle e_1, e_2, \dots, e_{n-1}\rangle$,
- $\mathfrak{g} = \langle e_1, e_3, \dots, e_{n-2}\rangle \oplus \langle e_0 + e_1, e_2, e_4, \dots, e_{n-1}, e_n\rangle$.

If the multiplication of $\mathfrak{g}$ is given by $\mu_0 + \psi$ with $\psi = \sum a_{k,2s+1}\psi_{k,2s+1} + \psi_{\frac{n-1}{2},n}$, then the first vectorial decomposition is also a $\mathbb{Z}_2$-grading. If the multiplication of $\mathfrak{g}$ is given by $\mu_0 + \Psi$ with $\psi = \sum a_{k,2s}\psi_{k,2s} + \psi_{\frac{n-1}{2},n}$, then the third vectorial decomposition is also a $\mathbb{Z}_2$-grading. If the second vectorial decomposition is a grading, then $\mathfrak{g}$ is not characteristically nilpotent. □

Corollary 2.1. *There exists an infinite family of graded characteristically nilpotent filiform Lie algebras.*

Proof. We consider the nine-dimensional filiform Lie algebras given by

$$\mu = \mu_0 + \psi_{1,4} + \alpha\psi_{2,6} + \psi_{2,8} + \frac{3\alpha^2}{\alpha+2}\psi_{3,8}$$

with $\alpha \neq 0$ and -2. These Lie algebras admits a $\mathbb{Z}_2$-grading and for two different values of α we have non isomorphic Lie algebras [12]. Moreover, since $\alpha \neq 0$, these Lie algebras are characteristically nilpotent. □

2.6 $\mathbb{Z}_k$-Gradings, $k > 2$, of Filiform Characteristically Nilpotent Lie Algebras

We assume that $k > 2$. A $\mathbb{Z}_k$-grading of L_n is equivalent to one of the following

$$\Gamma_k^l : L_n = \sum_{i=2}^{l-1} [X_i]_k \oplus (\langle X_1 \rangle \oplus [X_l]_k) \oplus \sum_{j=l+1}^{k+1} [X_j]_k,$$

where l is a parameter satisfying $2 \leq l \leq k+1$. The homogeneous component of this grading corresponding to the identity of $\mathbb{Z}_k$ is $[X_{l-1}]_k$. Two cocycles ψ_{h_1,s_1} and ψ_{h_2,s_2} send an homogeneous component (in particular $[X_{l-1}]_k$) in another homogeneous component if and only if

$$s_1 - 2h_1 = s_2 - 2h_2 (\mathrm{mod}\, k).$$

We deduce

Proposition 2.6. *Any filiform characteristically nilpotent Lie algebra* $\mathfrak{g}$ *such that* gr $(\mathfrak{g})$ *is isomorphic to* L_{n+1} *whose Lie multiplication is of the form*

$$\mu = \mu_0 + \sum_{i \in I} a_{h_i,s_i} \psi_{h_i,s_i}$$

with

$$s_i - 2h_i = s_j - 2h_j (\mathrm{mod}\, k), \; k < n-2$$

for any $i, j \in I$ *admits a* $\mathbb{Z}_k$*-grading.*

Proposition 2.7. *For any* k*, there exists a* $\mathbb{Z}_k$*-graded characteristically nilpotent filiform Lie algebra.*

Proof. If fact, we consider the filiform Lie algebra of dimension $n = k+5$ given by

$$\mu = \mu_0 + \psi_{1,4} + \psi_{1,4+k},$$

that is

$$\begin{cases} \mu(e_0, e_i) = e_{i+1}, \; i = 2, \ldots, k+3, \\ \mu(e_1, e_2) = e_4 + e_{k+4}, \\ \mu(e_1, e_i) = e_{i+2}, \; i = 3, \ldots, k+2. \end{cases}$$

The Jacobi conditions are satisfied. The sill algebra is given by $\mu_0 + \psi_{1,4}$ and is not isomorphic to μ as soon as $k \neq 0$. Then this Lie algebra is characteristically nilpotent and from the previous proposition, it is $\mathbb{Z}_k$-graded. ◻

3 $\mathcal{G}$-symmetric spaces

3.1 Definition

Let G be a Lie group and H a closed subgroup of G.

Definition 3.1. Let $\mathcal{G}$ be a finite abelian group. A homogeneous space $M = G/H$ is called *$\mathcal{G}$-symmetric* if

(1) The Lie group G is connected,
(2) The group G is effective on G/H (i.e., the Lie algebra $\mathfrak{h}$ of H does not contain a nonzero proper ideal of the Lie algebra $\mathfrak{g}$ of G),
(3) There is an injective homomorphism

$$\rho : \mathcal{G} \to \mathrm{Aut}\,(G)$$

such that if $G^{\mathcal{G}}$ is the closed subgroup of all elements of G fixed by $\rho(\mathcal{G})$ and $(G^{\mathcal{G}})_e$ the identity component of $G^{\mathcal{G}}$, then

$$(G^{\mathcal{G}})_e \subset H \subset G^{\mathcal{G}}.$$

Examples: Symmetric and *k*-symmetric Spaces. If $\mathcal{G} = \mathbb{Z}_2$, then the notion of $\mathbb{Z}_2$-symmetric spaces corresponds to the classical notion of symmetric spaces [19]. If $\mathcal{G} = \mathbb{Z}_p$ with p a prime number, we find again the p-manifolds in the sense of Ledger–Obata [21].

We denote by ρ_γ the automorphism $\rho(\gamma)$ for any $\gamma \in \mathcal{G}$. If H is connected, we have

$$\begin{cases} \rho_{\gamma_1} \circ \rho_{\gamma_2} = \rho_{\gamma_1\gamma_2}, \\ \\ \rho_\varepsilon = Id, \\ \\ \rho_\gamma(g) = g, \ \forall \gamma \in \Gamma \Longleftrightarrow g \in H. \end{cases},$$

where ε is the identity element of $\mathcal{G}$. Each automorphism ρ_γ of G, $\gamma \in \mathcal{G}$, induces an automorphism of $\mathfrak{g}$, denoted by τ_γ and given by $\tau_\gamma = (T\rho_\gamma)_e$ where $(Tf)_x$ is the tangent map of f at the point x.

Lemma 3.1. *The map* $\tau : \mathcal{G} \longrightarrow \mathrm{Aut}\,(\mathfrak{g})$ *given by*

$$\tau(\gamma) = (T\rho_\gamma)_e$$

is an injective homomorphism of groups.

Proof. Let γ_1, γ_2 be in $\mathcal{G}$. Then $\rho_{\gamma_1} \circ \rho_{\gamma_2} = \rho_{\gamma_1\gamma_2}$. It follows that $(T\rho_{\gamma_1})_e \circ (T\rho_{\gamma_2})_e = (T\rho_{\gamma_1}\rho_{\gamma_1})_e = (T\rho(\gamma_1\gamma_2))_e$, that is, $\tau(\gamma_1\gamma_2) = \tau(\gamma_1)\tau(\gamma_2)$. Now let us assume that $\tau(\gamma) = \mathrm{Id}_{\mathfrak{g}}$. Then $(T\rho_\gamma)_e = \mathrm{Id} = (T\rho_\varepsilon)_e$. But ρ_γ is uniquely determined by the corresponding tangent automorphism of $\mathfrak{g}$. Then $\rho_\gamma = \rho_\varepsilon$ and $\gamma = \varepsilon$. □

Then we have the infinitesimal version of a $\mathcal{G}$-symmetric space. A $\mathcal{G}$-symmetric Lie algebra is a triple $(\mathfrak{g}, \mathfrak{h}, \tau)$ consisting of a Lie algebra $\mathfrak{g}$, a Lie subalgebra $\mathfrak{h}$ and an injective homomorphism of group $\tau : \mathcal{G} \longrightarrow \mathrm{Aut}\,(\mathfrak{g})$ such that $\mathfrak{h}$ consists of all elements X of $\mathfrak{g}$ satisfying $\tau(\gamma)(X) = X$ for all $\gamma \in \mathcal{G}$. Thus, if G/H is a $\mathcal{G}$-symmetric space, the triple $(\mathfrak{g}, \mathfrak{h}, \tau)$ is a $\mathcal{G}$-symmetric Lie algebra. Conversely, if $(\mathfrak{g}, \mathfrak{h}, \tau)$ is a $\mathcal{G}$-symmetric Lie algebra and if G is a connected, simply connected Lie group whose Lie algebra is $\mathfrak{g}$ and H is a connected subgroup associated with $\mathfrak{h}$, then G/H is a $\mathcal{G}$-symmetric space.

3.2 $\mathcal{G}$-Grading of a $\mathcal{G}$-Symmetric Lie Algebra

Let $(\mathfrak{g}, \mathfrak{h}, \tau)$ be a $\mathcal{G}$-symmetric Lie algebra. Since $\tau : \mathcal{G} \to \mathrm{Aut}\,(\mathfrak{g})$ is an injective homomorphism, the image $\mathcal{G}_1$ of $\mathcal{G}$ is a finite abelian subgroup of $\mathrm{Aut}(\mathfrak{g})$ isomorphic to $\mathcal{G}$. For each $\gamma \in \mathcal{G}$, let $(\mathfrak{g}_{\mathbb{C}})_\gamma$ the subspace of $\mathfrak{g}_{\mathbb{C}}$ given by

$$(\mathfrak{g}_{\mathbb{C}})_\gamma = \{X \mid \gamma(X) = \tau(\gamma)(X)\},$$

The vector space decomposition $\mathfrak{g}_{\mathbb{C}} = \bigoplus_{\gamma \in \mathcal{G}} (\mathfrak{g}_{\mathbb{C}})_\gamma$ is just a standard weight decomposition under the action of an abelian semisimple group of linear transformations over an algebraically closed field.

Proposition 3.1. *If* $(\mathfrak{g}, \mathfrak{h}, \tau)$ *is a $\mathcal{G}$-symmetric Lie algebra, then the complex Lie algebra* $\mathfrak{g}_{\mathbb{C}}$ *admits a $\mathcal{G}$-grading. If* $\tau(\gamma)^2 = \mathrm{Id}$ *for any $\gamma \in \mathcal{G}$, then we have a $\mathcal{G}$-grading on* $\mathfrak{g}$ *itself.*

Explicitly, since $\mathcal{G}_1$ is a finite abelian subgroup of $\mathrm{Aut}\,(\mathfrak{g})$, one can write $\mathcal{G}_1$ as $\mathcal{G}_1 = K_1 \times \ldots \times K_p$ where K_i is a cyclic group of order r_i. Let κ_i be a generator of K_i. The automorphisms κ_i satisfy

$$\begin{cases} \kappa_i^{r_i} = \mathrm{Id}, \\ \kappa_i \circ \kappa_j = \kappa_j \circ \kappa_i, \end{cases}$$

for all $i, j = 1, \ldots, p$. These automorphisms are simultaneously diagonalizable. If ξ_i is a primitive r_i^{th} root of 1, then the eigenspaces

$$\mathfrak{g}_{s_1,\ldots,s_p} = \{X \in \mathfrak{g} \text{ such that } \kappa_i(X) = \xi_i^{s_i} X,\ i = 1, \ldots, p\}$$

give the following grading of $\mathfrak{g}$ by $\mathbb{Z}_{r_1} \times \ldots \times \mathbb{Z}_{r_p}$:

$$\mathfrak{g} = \bigoplus_{(s_1,\ldots,s_p) \in \mathbb{Z}_{r_1} \times \ldots \times \mathbb{Z}_{r_p}} \mathfrak{g}_{s_1,\ldots,s_p}.$$

Actually, the actions of a group $\mathcal{G}$ by automorphisms of an algebra and the gradings by $\mathcal{G}$ of the same algebra are closely connected in a much more general setting. This we explained in detail in [4]. Here we quickly recall the main idea.

Assume that the Lie algebra $\mathfrak{g}$ is $\mathcal{G}$-graded where $\mathcal{G}$ is an finite abelian group. Let $\hat{\mathcal{G}}$ be the dual group of $\mathcal{G}$, that is, the group of characters of $\mathcal{G}$. If we assume that a Lie algebra $\mathfrak{g}$ is $\mathcal{G}$-graded, then we obtain a natural action of $\hat{\mathcal{G}}$ by linear transformations on $\mathfrak{g}_{\mathbb{C}}$. If $\chi \in \hat{\mathcal{G}}$ and $X \in \mathfrak{g}_\gamma$ then

$$\chi(X) = \chi(\gamma)X.$$

Since for $X \in \mathfrak{g}_{\gamma_1}$ and $Y \in \mathfrak{g}_{\gamma_2}$ we have $[X, Y] \in \mathfrak{g}_{\gamma_1\gamma_2}$, it follows that

$$\chi([X, Y]) = \chi(\gamma_1\gamma_2)[X, Y] = [\chi(\gamma_1)X, \chi(\gamma_2)Y] = [\chi(X), \chi(Y)], \tag{1}$$

that is, $\hat{\mathcal{G}}$ acts by Lie automorphisms on $\mathfrak{g}$. In this case there is a canonical homomorphism

$$\alpha : \hat{\mathcal{G}} \to \mathrm{Aut}(\mathfrak{g}_{\mathbb{C}}) \text{ given by } \alpha(\chi)(X) = \chi(X). \tag{2}$$

If for any γ in the support of the grading, we have $\gamma^2 = 1$, then the action is defined even on $\mathfrak{g}$ itself and the above homomorphism maps $\hat{\mathcal{G}}$ onto a subgroup of Aut $(\mathfrak{g})$. Conversely, if $\hat{\mathcal{G}}$ acts on $\mathfrak{g}_{\mathbb{C}}$ by automorphisms, then setting $(\mathfrak{g}_{\mathbb{C}})_\gamma = \{X \mid \chi(X) = \chi(\gamma)X\}$ provides $\mathfrak{g}_{\mathbb{C}}$ with a $\mathcal{G}$-grading.

Proposition 3.2. *Let $\mathcal{G}$ be a finite abelian group and $\hat{\mathcal{G}}$, the group of complex characters of $\mathcal{G}$.*

(a) *A complex Lie algebra $\mathfrak{g}$ is $\mathcal{G}$-graded if and only if the dual group $\hat{\mathcal{G}}$ maps homomorphically onto a finite abelian subgroup of* Aut $(\mathfrak{g})$, *by the canonical homomorphism α.*
(b) *A real Lie algebra $\mathfrak{g}$ is $\mathcal{G}$-graded, with $\gamma^2 = 1$ for each γ in the support of this grading, if and only if there is a homomorphism $\alpha : \hat{\mathcal{G}} \to$* Aut $(\mathfrak{g})$ *such that $\alpha(\chi)^2 = \mathrm{id}_{\mathfrak{g}}$ for any $\chi \in \hat{\mathcal{G}}$.*
(c) *In both cases above, the support generates $\mathcal{G}$ if and only if the canonical mapping α has trivial kernel, that is, $\hat{\mathcal{G}}$ is isomorphic to a (finite abelian) subgroup of* Aut $(\mathfrak{g})$.

From Lemma 3.1 we derive the following. Let $M = G/H$ be a $\mathcal{G}$-symmetric space. Then $\mathfrak{g}$ is graded by the dual group of $\mathcal{G}$. Since $\mathcal{G}$ is abelian, the groups $\mathcal{G}$ and $\hat{\mathcal{G}}$ are isomorphic (a noncanonical isomorphism). To simplify notations, we will identify $\mathcal{G}$ and its dual, this permits one to speak of $\mathcal{G}$-symmetric spaces and $\mathcal{G}$-graded Lie algebras (in place of $\hat{\mathcal{G}}$-graded Lie algebras).

Proposition 3.3. *If $M = G/H$ is a $\mathcal{G}$-symmetric space, then the complex Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$, where $\mathfrak{g}$ is the Lie algebra of G, is $\mathcal{G}$-graded and if $\mathcal{G} = \mathbb{Z}_2^k$, then the real Lie algebra $\mathfrak{g}$ of G is $\mathcal{G}$-graded. The subgroup of $\mathcal{G}$ generated by the support of the grading is $\mathcal{G}$ itself.*

Proof. Indeed, by Lemma 3.1, $\alpha : \mathcal{G} \to \text{Aut }(\mathfrak{g})$ is an injective homomorphism, so all our claims follow by Proposition 3.2. □

To study $\mathcal{G}$-symmetric spaces, we need to start with the study of $\mathcal{G}$-graded Lie algebras. But in a general case, if G is a connected Lie group corresponding to $\mathfrak{g}$, the $\mathcal{G}$-grading of $\mathfrak{g}$ or $\mathfrak{g}_{\mathbb{C}}$ does not necessarily give a $\mathcal{G}$-symmetric space G/H. Some examples are given in [7], even in the symmetric case. Still, if G is simply connected, Aut (G) is a Lie group isomorphic to Aut $(\mathfrak{g})$ and the $\mathcal{G}$-grading of $\mathfrak{g}$ determines a structure of $\mathcal{G}$-symmetric space from G.

Proposition 3.4. *Let $\mathcal{G} = \mathbb{Z}_2^k$, with the identity element ε and $\mathfrak{g}$ a real $\mathcal{G}$-graded Lie algebra such that the subgroup generated by the support of the grading equals $\mathcal{G}$ and the identity component $\mathfrak{h} = \mathfrak{g}_\varepsilon$ of the grading does not contain a nonzero ideal of $\mathfrak{g}$. If G is a connected simply connected Lie group with Lie algebra $\mathfrak{g}$ and H a Lie subgroup associated with $\mathfrak{h}$, then the homogeneous space $M = G/H$ is a $\mathcal{G}$-symmetric space.*

3.3 $\mathcal{G}$-Symmetries on a $\mathcal{G}$-Symmetric Space

Given a $\mathcal{G}$-symmetric space $(G/H, \mathcal{G})$ it is easy to construct, for each point x of the homogeneous space $M = G/H$, a subgroup of the group $\text{Diff}(M)$ of diffeomorphisms of M, isomorphic to $\mathcal{G}$, which has x as an isolated fixed point. We denote by $\overline{g}$ the class of $g \in G$ in M. If e is the identity of G, $\gamma \in \mathcal{G}$, we set

$$s_{(\gamma,\overline{e})}(\overline{g}) = \overline{\rho_\gamma(g)}.$$

If $\overline{g}$ satisfies $s_{(\gamma,\overline{e})}(\overline{g}) = \bar{g}$, then $\overline{\rho_\gamma(g)} = \overline{g}$, that is $\rho_\gamma(g) = gh_\gamma$ for $h_\gamma \in H$. Thus $h_\gamma = g^{-1}\rho_\gamma(g)$. But $\mathcal{G} \cong \hat{\mathcal{G}}$ is a finite abelian group. If p_γ is the order of γ, then $\rho_{\gamma^{p_\gamma}} = Id$. Then

$$h_\gamma^2 = g^{-1}\rho_\gamma(g)\rho_\gamma(g^{-1})\rho_{\gamma^2}(g) = g^{-1}\rho_{\gamma^2}(g).$$

Applying induction, and considering $(h_\gamma)^m \in H$ for any m, we have

$$(h_\gamma)^m = g^{-1}\rho_{\gamma^m}(g).$$

For $m = p_\gamma$ we obtain

$$(h_\gamma)^{p_\gamma} = e.$$

If g is near the identity element of G, then h_γ is also close to the identity and $h_\gamma^{p_\gamma} = e$ implies $h_\gamma = e$. Then $\rho_\gamma(g) = g$. This is true for all $\gamma \in \mathcal{G}$ and thus $g \in H$. It follows that $\overline{g} = \overline{e}$ and that the only fixed point of $s_{(\gamma,\overline{e})}$ is $\overline{e}$. In conclusion, the family $\{s_{(\gamma,\overline{e})}\}_{\gamma \in \mathcal{G}}$ of diffeomorphisms of M satisfy