Developments in Mathematics

Geoffrey Mason
Ivan Penkov
Joseph A. Wolf *Editors*

Developments and Retrospectives in Lie Theory

Algebraic Methods

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Developments in Mathematics

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Lie Groups, Lie Algebras and their Representations

Third Announcement -- October 1991

This is to announce a new program ***Lie Groups, Lie Algebras and Their Representations*** starting in the academic year 1991-92. We plan to meet at various University of California campuses for one weekend every other month during the year. The purpose of the program is to communicate results and ideas rather than to deliver polished presentations. The tentative format of the meetings is

Saturday: one late morning talk and two afternoon talks; these scheduled in advance.
Sunday: two morning talks and one early afternoon talk; some may be scheduled in advance and the rest will be scheduled Saturday.

The meetings are scheduled as follows:

Location	Dates	Local Organizer	e-mail address
Berkeley	October 19, 20	Joseph Wolf	jawolf@cartan.berkeley.edu
Riverside	December 7, 8	Ivan Penkov	penkov@ucrmath.ucr.edu
Davis	February 8, 9 ??	Alice Fialowski	fialowski@bolyai.ucdavis.edu
UCLA	April 4, 5	Robert Blattner	blattner@math.ucla.edu
UC Santa Cruz	May 30, 31	Daniel Goldstein	danny@cats.ucsc.edu

The program for the first meeting, Saturday October 19 and Sunday October 20, is

time	speaker	title
Sat 11:00	Joseph Wolf	Lie structures on direct limit groups and their completions (joint work with L. Natarajan and E. Rodríguez-Carrington)
Sat 02:00	Dmitri Fuchs	On the cohomology of Lie algebras of vector fields in the line and of Hamiltonian vector fields in the plane
Sat 04:00	Nicolai Reshetikhin	q-holonomic systems and quantum affine Lie algebras, I
Sun 09:00	Ivan Penkov	Infinite dimensional irreducible modules over finite dimensional Lie superalgebras
Sun 10:30	Yan Soibelman	Algebras of functions on compact quantum groups
Sun 01:00	Nicolai Reshetikhin	q-holonomic systems and quantum affine Lie algebras, II

These talks will take place in 9 Evans Hall on the Berkeley campus. The one hour scheduled time of any talk will be extended by the informal discussion that takes place during that talk.

There are no registration fees. No support is currently available. We hope to minimize participants' expenses by car-pooling and various informal arrangements for staying over Saturday night.

People interested in participating, or speaking and participating, at a meeting, should contact the local organizer for that meeting. People interested in helping with the local organization should contact both the local organizer and Joseph Wolf. If you wish to be on the mailing list for further announcements, and for meeting programs/schedules, write to Joseph Wolf.

Announcement of the First Conference, October 1991

Preface

The West Coast of the United States has a longstanding tradition in Lie theory, although before 1991 there had been no systematic cooperation between the various strongholds. In 1991, partially inspired by the arrival of some well–known Lie theorists from eastern Europe, the situation changed. A new structure emerged: a seminar that would meet at various University of California campuses three or four times a year. The purpose of the seminar was to foster contacts between researchers and graduate students at the various campuses by facilitating the sharing of ideas prior to formal publication. This idea quickly gained momentum, and became a great success. It was enthusiastically supported by graduate students. A crucial feature of the entire endeavor was the feeling of genuine interest for the work of colleagues and the strong desire to collaborate.

The first meeting of the new seminar "Lie Groups, Lie Algebras and their Representations" was held in Berkeley on October 19 and 20, 1991. On the second day of the seminar, excitement was made even more memorable by the historic Berkeley–Oakland fire, which we observed from Evans Hall. The original announcement for that meeting is reprinted here on page v. The phrase "The purpose of the program is to communicate results and ideas rather than to deliver polished presentations" quickly became, and still is, the guiding principle of the seminar. We never restricted ourselves to Lie Theory per se, and speakers from geometry, algebra, complex analysis, and other adjacent areas were often invited.

NSF travel grants were crucial to the success of the seminar series. These grants funded travel for speakers, graduate students and postdoctoral researchers, and we thank the National Science Foundation for its continued support.

Over the years our idea became widely popular. On occasion the seminar took place in Salt Lake City, in Stillwater, Oklahoma, and in Eugene, Oregon. In addition, colleagues from other regional centers of Lie theory and related areas picked up on our idea and created their own meeting series. This is how the "Midwest Lie Theory

Seminars", the "Midwest Group Theory Seminar", the "Southeastern Lie Theory Workshops", and other regional series emerged.

The California Lie Theory Seminar has now been alive and well for 23 years. Joe Wolf has always played a central role in the seminar, along with Geoff Mason and Ivan Penkov. When Ivan left for Germany in 2004, Susan Montgomery and Milen Yakimov joined the team of organizers.

Over the course of these 23 years, at about 20 talks per year, some 450 talks have been hosted by the seminar. It seemed unrealistic to give an overview of all of the topics covered over all these years, and similarly unrealistic to try to publish a comprehensive set of volumes. Rather, we settled on two retrospective volumes containing work representative of the seminar as a whole. We started with a list of participants who spoke more than once in the seminar, and invited them to submit work relevant to their seminar talks. For obvious reasons we did not hear from everyone. Nevertheless, there was a strong response, and the reader of these Volumes will find 26 research papers, all of which received strong referee reports, in the greater area of Lie Theory. We decided to split the papers into two volumes: "Algebraic Methods" and "Geometric/Analytic Methods". We thank Springer, the publisher of these volumes, and especially Ann Kostant and Elizabeth Loew, for their cooperation and assistance in this project.

This is the Algebraic Methods volume.

Santa Cruz, USA — Geoffrey Mason
Bremen, Germany — Ivan Penkov
Berkeley, USA — Joseph A. Wolf

Contents

Group Gradings on Lie Algebras, with Applications to Geometry. I

Yuri Bahturin, Michel Goze, and Elisabeth Remm

Abstract In this article, which is the first part of a sequence of two, we discuss modern approaches to the classification of group gradings on simple and nilpotent Lie algebras. In the second article we discuss applications and related topics in differential geometry.

Key words Lie and associative algebras of linear transformations • Graded algebras • Hopf algebras • Automorphisms • Involution gradings

Mathematics Subject Classification (2010): 16K20, 16T05, 16W10, 16W20, 16W50, 17B65, 17B70, 17B40.

1 Introduction

This article is the first is a sequence of two articles in which we discuss several topics related to the first author's presentations given at the workshops "Lie Groups, Lie Algebras and their Representations" at the University of Southern California and several campuses of the University of California. The main topics covered were the theory and representations of simple locally finite Lie algebras, Lie superalgebras, and group gradings on simple Lie algebras, with applications to differential geometry. The most recent talk, in 2010 at the University of Southern California, was devoted to a more abstract topic: the study of the distortion arising when one Lie algebra is embedded in another as a subalgebra.

At least in one of the areas, namely, in the theory of group gradings on simple Lie algebras, the development that followed was truly spectacular. Thanks to the efforts

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of several researchers in several countries, now we have a complete classification of abelian group gradings on all classical simple Lie algebras over an algebraically closed fields of characteristic different from 2. The detailed exposition of this theory can now be found in a monograph [19], which has incorporated the results of [6], where the above classification, up to the isomorphism, has been completed, with the exception of D_4 and $\mathfrak{psl}_3$. In the same monograph [19], the authors present the classification of group gradings on almost all exceptional algebras, which heavily depends on the study of various classes of nonassociative algebras, such as octonions, Albert algebra, and so on. In the case char $F = 0$, a description of fine gradings on simple Lie algebras, with the same exception, as above, has appeared in [22] and a description of all group gradings was obtained in [9] and [11].

Lately, the study of gradings on Lie algebras went beyond the boundaries of the area of classical or exceptional simple finite-dimensional algebras. In the paper [3] the authors classify up to isomorphism all abelian group gradings on finitary simple Lie algebras of linear transformations. In the paper [7] the authors classify all abelian group gradings on two types of restricted simple Lie algebras of Cartan type over algebraically closed field of characteristic $p > 3$.

These results became possible thanks to the extensive use of the methods involving the techniques of the theory of Hopf algebras. Probably the first papers where this approach was successfully implemented were [5] and [8]. This is a well-known fact that an algebra A over a field F graded by an abelian group G bears a canonical structure of the (right) comodule algebra over the group algebra $H = FG$ (see [24]). Conversely, the H-comodule algebra canonically becomes a G-graded algebra. If A is an associative envelope of a Lie algebra L then in some cases of interest to us, the H-comodule structure of L can be extended to the H-comodule structure of A. Then A becomes G-graded and the grading of L becomes the restriction of a G-grading of an associative algebra A. It is a usual pattern that the gradings of simple associative algebras are easier to determine.

The problem of extending the H-comodule structure from Lie to associative algebras in the case of finite-dimensional algebras is approached using the technique of affine group schemes [32]. These techniques, however, do not apply in the case of infinite-dimensional algebras. In the case where the algebras are infinite-dimensional (or just of sufficiently great dimension) quite another approach works. As it turns out, the question about the possibility of extending the H-comodule structure from a Lie to an associative algebra is analogous to the problem of extending automorphisms of Lie algebras to the automorphisms or negatives of antiautomorphisms of their associative envelopes. This latter problem, named after I. Herstein, was successfully solved in the 1990s using so-called *functional identities* (see [18]). An appropriate adaptation of the methods of this book made it possible to classify in [3] all abelian group gradings on simple finitary Lie algebras over algebraically closed fields of characteristic other than 2 and 3. Note that these results are also related to the theory of locally finite simple Lie algebras, which was one of the topics of research of several people related to the West Coast Lie Theory workshop “Lie Groups, Lie Algebras and their Representations”.

Since Cartan decompositions of semisimple Lie algebras are a special case of fine gradings, it is our strong belief that the classification of all gradings on simple Lie algebras should be important for the study of their structure, representations and other properties.

As examples, the study of graded identical relations of algebras lately became an important branch of the theory of algebras with polynomial identities (PI-algebras). Graded polynomial identities of algebras are often much easier to study, yet they define the ordinary ones.

2 Group Gradings of Lie Algebras

In what follows $\mathbb{K}$ is an algebraically closed field of characteristic other than 2. For more details on group gradings of algebras, see a recent monograph [19]. These definitions apply to other classes of algebras, with an appropriate change of notation.

2.1 Definition

Let L be a Lie algebra over $\mathbb{K}$ and S a set.

Definition 2.1. An S-grading Γ of L with support S is a decomposition

$$\Gamma : L = \bigoplus_{s \in S} L_s$$

of L as the sum on nonzero vector subspaces L_s satisfying the following condition. For any $s_1, s_2 \in S$ such that $[L_{s_1}, L_{s_2}] \neq 0$ there is $\mu(s_1, s_2) \in S$ such that $[L_{s_1}, L_{s_2}] \subset L_{\mu(s_1,s_2)}$.

If S is a subset of a group G such that $\mu(s_1, s_2) = s_1 s_2$ in G, we say that Γ is a *group grading by the group G with support S*. Such a group G is not defined uniquely, but for any group grading there is a universal grading group $U(\Gamma)$ such that any other grading group G of Γ is a factor group of $U(\Gamma)$. The universal group is given in terms of generators and defining relations if one chooses S as the set of generators and all the above equations $s_1 s_2 = \mu(s_1, s_2)$ as the set of defining relations.

2.2 Equivalent and Isomorphic Gradings

If $\Gamma' : L = \bigoplus_{s' \in S'} L'_{s'}$ is another grading of L, we say that Γ is *equivalent* to Γ' if there is an automorphism $\varphi \in \operatorname{Aut} L$ and a bijection $\sigma : S \to S'$ such that $\varphi(L_s) = L'_{\sigma(s)}$. It follows that $\sigma(\mu(s_1, s_2)) = \mu(\sigma(s_1), \sigma(s_2))$.

Two group gradings Γ and Γ' of a Lie algebra L by groups G and G' with supports S and S' are called *weakly isomorphic* if they are equivalent, as above, and the map $\sigma : S \to S'$ extends to the isomorphism of groups G to G'. The strongest relation is the *isomorphism* of G-gradings. In this case both Γ and Γ' are gradings by the same group; they have the same support and the isomorphism of groups σ is identity. As a result, we have $\varphi(L_g) = L'_g$, for any $g \in G$. In the case of group G-gradings with support S we will write $\Gamma : L = \bigoplus_{g\in G} L_g$, assuming that $L_g = \{0\}$ if $g \in G \setminus S$.

2.3 Refinements, Coarsenings, Fine Gradings

Definition 2.2. Let Γ, as above, and $\Gamma' : L = \bigoplus_{t\in T} L'_t$ be two gradings on L with supports S and T, respectively. We say that Γ is a *refinement* of Γ' or Γ' is a *coarsening* of Γ if for any $s \in S$ there exists $t \in T$ such that $L_s \subset L'_t$.

Coarsening and refinements often arise as follow. Let $\Gamma : L = \bigoplus_{g\in G} L_g$ and let $\varepsilon : G \to K$ be a homomorphism of groups. We set $L'_k = \bigoplus_{\varepsilon(g)=k} L_g$, for any $k \in K$. Then we obtain the image-grading $\varepsilon(\Gamma) = \bigoplus_{k\in K} L'_k$. If ε is an onto homomorphism, we say that $\varepsilon(\Gamma)$ is a *factor-grading* of the grading Γ. Clearly, $\varepsilon(\Gamma)$ a coarsening of Γ and Γ is a refinement of $\varepsilon(\Gamma)$.

A group grading Γ of L is called a *fine grading* if doesn't admit proper group refinements.

Remark 2.1. If Γ' is a refinement of Γ, then Γ viewed as a $U(\Gamma)$-grading is a factor grading of Γ' viewed as a $U(\Gamma')$-grading.

2.4 Abelian Groups Gradings

In the case of Lie algebras it is natural to assume that all groups involved in the group gradings are abelian. In fact, for many Lie algebras, like finite-dimensional simple ones, this is satisfied (see [11]) in the sense that the partial function $\mu : S \times S \to S$ appearing in the definition of the grading is symmetric or that the elements of the support in the group grading commute. So in what follows we always deal with gradings by abelian groups. In addition, when we study finite-dimensional algebras, the supports of the gradings are finite sets, so our groups are finitely generated.

Now given a finitely generated abelian group G, we denote by $\hat{G}$ the group of (1-dimensional) characters of G, that is, the group of all homomorphisms $\chi : G \to F^\times$ where $F^\times$ is the multiplicative group of the field F. If $\Gamma : L = \bigoplus_{g\in G} L_g$ is a grading of a Lie algebra L with a grading group G, there is an action of $\hat{G}$ by semisimple automorphisms of L given on the homogeneous elements of L by $\chi * x = \chi(g)x$ where $\chi \in \hat{G}$ and $x \in L_g$. If G is generated by the support

S of Γ, different characters act differently. Indeed, assume $\chi_1, \chi_2 \in \hat{G}$ are such that $\chi_1 * x = \chi_2 * x$, for any $x \in L$. Choose any $s \in S$ and $0 \neq x \in L_s$. In this case $\chi_1(s)x = \chi_1 * x = \chi_2 * x = \chi_2(s)x$. As a result, $\chi_1(s) = \chi_2(s)$, for any $s \in S$. Since χ_1 and χ_2 are homomorphisms and G is generated by S, we have $\chi_1 = \chi_2$, as claimed. This allows us, in this important case, to view $\hat{G}$ as a subgroup of Aut L. We will view Aut L as an algebraic group. When we study finite-dimensional algebras, then G is finitely generated abelian and so $\hat{G}$ is the group of characters of a finitely generated abelian group. If $G \cong \mathbb{Z}^m \times A$, where m is an integer, $m \geq 0$, and A a finite abelian group, then $\hat{G} \cong (F^\times)^m \times \hat{A}$. Such subgroups of algebraic groups, consisting of semisimple elements, are called *quasitori*.

A quasitorus is a generalization of the notion of a torus, which is an algebraic subgroup of Aut L isomorphic to $\hat{G} \cong (F^\times)^m$, for some m. A torus, which is not contained in a larger torus is called *maximal*. The following result is classical.

Theorem 2.1. *In any algebraic group any two maximal tori are conjugate by an inner automorphism.*

Another well-known result is often attributed to Platonov [27] (but see also [28]).

Theorem 2.2. *Any quasitorus is isomorphic to a subgroup in the normalizer of a maximal torus.*

Thus, if we find a maximal torus D in Aut L equal to its normalizer in Aut L, then for any grading of L by a finitely generated abelian group G, there is $\varphi \in$ Aut L such that $\varphi\hat{G}\varphi^{-1} \subset D$.

Every time we have a quasitorus Q in Aut L there is root space decomposition of L with roots from the group of (algebraic) characters $\mathfrak{X}(Q)$ for the group Q, the root subspace for $\lambda \in \mathfrak{X}(Q)$ given by

$$L_\lambda = \{x \in L \mid \alpha(x) = \lambda(\alpha)x \text{ for any } \alpha \in Q\}.$$

If $Q \subset D$ then, by duality, $\mathfrak{X}(Q)$ is a factor group of $\mathfrak{X}(D) \cong \mathbb{Z}^m$ where $m = \dim D$. The root space decomposition by D is the refinement of the root space decomposition by Q and so the grading by $\mathfrak{X}(Q)$ is a coarsening of a grading by $\mathfrak{X}(D) \cong \mathbb{Z}^m$.

Now assume that we deal with a grading of L by a finitely generated abelian group G, $\Gamma : L = \bigoplus_{g\in G} L_g$. Assume that $p = \operatorname{char} F$ and write $G = G_p \times G_{p'}$, where G_p is the Sylow p-subgroup and $G_{p'}$ its complement in G that has no elements of order p. If char $F = 0$, then $G = G_{p'}$. Let us consider the quasitorus $\hat{G} \subset$ Aut L. Then there is $\varphi \in$ Aut L such that $\varphi\hat{G}\varphi^{-1} \subset D$. Let us switch to another G-grading $\varphi(\Gamma) : L = \bigoplus_{g\in G} \varphi(L_g)$. The action of $\hat{G}$ on L induced by $\varphi(\Gamma)$ gives rise to another copy of $\hat{G}$ in Aut L, namely, $\varphi\hat{G}\varphi^{-1}$ and now this subgroup is a subgroup in D. Replacing D by another maximal torus $\varphi^{-1}D\varphi$ we may assume from the very beginning that $\hat{G} \subset D$. Then the root decomposition by D is a refinement of the root decomposition under the action of $\hat{G}$. Thus the grading

$\Gamma' : L = \bigoplus_{\lambda\in\mathfrak{X}(\hat{G})} L_\lambda$ induced by this root decomposition resulting from the action of $\hat{G}$ is a coarsening of the $\mathfrak{X}(D) \cong \mathbb{Z}^m$ grading of L induced by the action of the torus D.

In the case where G has no elements of order $p = \operatorname{char} F$, we have the original grading being a coarsening of the grading induced from the action of a maximal torus. Usually, this is some "standard" torus, and the grading induced by its action is also called "standard."

We summarize the above discussion as follows.

Theorem 2.3. *Let $\Gamma : L = \bigoplus_{g\in G} L_g$ be a grading of a finite-dimensional algebra L over an algebraically closed field $\mathbb{K}$ by a finitely generated abelian group G. If* $\operatorname{char}\mathbb{K} = p > 0$, *let G_p denote the Sylow p-subgroup of G. Consider the automorphism group $A =$* Aut *L of L and assume D is a maximal torus of A, of dimension m, equal to its normalizer in A. Then the factor-grading Γ/G_p is isomorphic to a factor-grading of the standard $\mathbb{Z}^m$-grading of L induced by the action of D on L.*

An important particular case is the following.

Theorem 2.4. *Let $\Gamma : L = \bigoplus_{g\in G} L_g$ be a grading of a finite-dimensional Lie algebra L over an algebraically closed field $\mathbb{K}$ by a finitely generated abelian group G. If* $\operatorname{char}\mathbb{K} = p > 0$, *assume G has no elements of order p. Consider the automorphism group $A =$* Aut *L of L and assume D is a maximal torus of A, of dimension m, equal to its normalizer in A. Then Γ is isomorphic to a factor-grading of the standard $\mathbb{Z}^m$-grading of L induced by the action of D on L.*

Even more special is the following.

Theorem 2.5. *Let $\Gamma : L = \bigoplus_{g\in G} L_g$ be a grading of a finite-dimensional algebra L over an algebraically closed field of characteristic zero $\mathbb{K}$ by a finitely generated abelian group G. Consider the automorphism group $A =$* Aut *L of L and assume D is a maximal torus of A, of dimension m, equal to its normalizer in A. Then Γ is isomorphic to a factor-grading of the standard $\mathbb{Z}^m$-grading of L induced by the action of D on L.*

2.5 Automorphism Group of a Grading

Since we completely classify gradings up to equivalence for certain classes of algebras, we quote some more results from [19]. Given a grading $\Gamma : L = \bigoplus_{s\in S} L_s$ of an algebra L, the subgroup of the group Aut L permuting the components of Γ is called the *automorphism group* of the grading Γ and denoted by Aut Γ. Each $\varphi \in$ Aut Γ defines a bijection on the support S of the grading: if $\varphi(L_s) = L_{s'}$, then $s \mapsto s'$ is the desired permutation $\sigma(\varphi)$, an element of the symmetric group Sym S. The kernel of the homomorphism $\varphi \mapsto \sigma(\varphi)$ is called the *stabilizer* of the grading

Γ, denoted by Stab Γ. Finally a subgroup of Stab Γ, whose elements are scalar maps on each graded component of Γ is called the *diagonal group* of Γ and denoted by Diag Γ.

Definition 2.3. Let $Q \subset$ Aut Γ be a quasitorus. Let Γ be the eigenspace decomposition of L with respect to Q. Then the quasitorus Diag Γ in Aut Γ is called the saturation of Q. We always have $Q \subset$ Diag Γ. If $Q =$ Diag Γ, then we say that Q is a *saturated* quasitorus.

A quasitorus Q is saturated if and only if the group $\mathscr{X}(Q)$ of algebraic characters of Q is $U(\Gamma)$, the universal group of Γ.

Proposition 2.1. *The equivalence classes of gradings on L are in one-to-one correspondence with the conjugacy classes of saturated quasitori in* Aut L.

Notice that if we already know that every quasitorus is conjugate to a subgroup of a fixed maximal torus and that two subgroups of the maximal torus are conjugate if and only if they are equal, we can say that the *equivalence classes of gradings are in one-to-one correspondence with the saturated subquasitori of a fixed maximal torus.*

2.6 Group Gradings and Actions of Hopf Algebras

In the case where the action of $\hat{\Gamma}$ by automorphisms does not completely reflect the G-grading, one can still find the transformations which are responsible for the gradings. For this one need to consider the group algebra $H = FG$ of the group G. The Hopf algebra structure on H, that is, the coproduct Δ, the counit ε and the antipode S are given as follows: $\Delta(g) = g \otimes g$, $\varepsilon(g) = 1$ and $S(g) = g^{-1}$, for any $g \in G$. Now let us consider the *finite dual* $K = H^\circ$ of H, which is just the ordinary dual H^* in the case where $|G| < \infty$, and the set of linear functions $f : H \to \mathbb{K}$ such that Ker f contains a two-sided ideal of finite codimension in H. All operations on K are defined by duality. The action of $f \in K$ on $A = \bigoplus_{g\in G} A_g$ is defined by setting $f * a = f(g)a$ if $a \in A_g$. If G is a finitely generated abelian group (or, more generally, a residually linear group), then K separates points of G and the G-grading can be recovered from the K-action. One has to set $A_g = \{x \in A \mid f * x = f(g)x \text{ for all } f \in K\}$.

If G is a finite group of order coprime to the characteristic of the base field, the basis of K is formed by the characters of G; they act as automorphisms and so we are bounced back to the situation described earlier. In the case where char $F = p > 0$ and G is an elementary abelian group of order p^n, it is known that K is a restricted enveloping algebra for an abelian p-Lie algebra spanned by n semisimple commuting derivations. In this case the study of gradings on an algebra is reduced to the study of action by derivations. In all other cases, although K has a very simple structure as an algebra, just the direct sum of the copies of the ground field, the action of the elements of K on the products of element of A can be extremely complex. However, this approach was successfully applied to the study of gradings on algebras of type A in characteristic $p > 0$ [5].

3 Transfer

If A is an associative algebra over a field $\mathbb{K}$, then we denote by $A^{(-)}$ the Lie algebra on the same vector space A under the bracket operation $[a, b] = ab - ba$, for all $a, b \in A$. By the Poincaré–Birkhoff–Witt Theorem, for any Lie algebra L there is an associative algebra A such that L is (isomorphic to) a subalgebra of $A^{(-)}$. If A is generated by (the image of) L, we say that A is an *associative envelope* of L. For any L one can choose the *universal enveloping algebra* $U(L)$ as an associative envelope, the drawback being that $U(L)$ is an infinite-dimensional algebra, for any nonzero L. In many cases, however, there are more manageable associative envelopes. For example, by the Ado–Iwasawa Theorem, every finite-dimensional Lie algebra has a finite-dimensional associative envelope. A Lie algebra consisting of linear transformations of a vector space V is a subalgebra of the associative algebra End (V), etc.

It was noted a while ago [26] that the study of group gradings on simple Lie algebras is closely related to the study of gradings on their associative envelopes. This is especially true in the case of classical simple Lie algebras over algebraically closed fields of characteristic zero for two reasons. First, because, as we saw, the study of gradings is equivalent to the study of quasitori in the automorphism groups of these algebras. Second, because all these algebras have matrix realizations and, with the exception of D_4, their automorphisms extend to the automorphisms or negatives of the automorphisms of the respective matrix algebras. The situation is especially benign in the case of algebras of the types B, C, D, except D_4. In this case, every automorphism of a Lie algebra of one of these types is given by a matrix conjugation. As a result, any G-grading of a Lie algebra, of the type $so(n)$ or $sp(n)$, n even, can be obtained by restriction from a G-grading of the matrix algebra M_n.

3.1 Affine Group Schemes

An object that most fully reflects the structure of gradings on a finite-dimensional algebra A by abelian groups is called the *automorphism group scheme* of A over a field $\mathbb{K}$, denoted by **Aut** A. This is a *representable* functor $\mathscr{F}$ from the category *Comm* of commutative associative unital algebras over $\mathbb{K}$ to the category *Ab* of abelian groups, which associates with each commutative associative unital algebra R the group $\mathrm{Aut}_R(A \otimes R)$. The value $\mathscr{F}(\mathbb{K})$ is just the ordinary Aut A. Being representable for a functor $\mathscr{G}$ means that there is there is a finitely generated Hopf algebra H such that $\mathscr{G}(R) = \mathrm{Alg}_{\mathbb{K}}(H, R)$, the group of algebra homomorphisms from H to R under the convolution product $(f * g)(h) = \sum_h f(h_1)g(h_2)$, where $\Delta(h) = \sum_h h_1 \otimes h_2$, h being an arbitrary element of H. Given a finitely generated abelian group G, the affine group scheme represented by the group algebra $H = \mathbb{K}G$ is denoted by G^D.

We quote the following results from [19, Section 1.4].

Proposition 3.1. *The G-gradings on an algebra A are in one-to-one correspondence with the morphisms of affine group schemes $G^D \to$ **Aut** A. Two G-gradings are isomorphic if and only if the corresponding morphisms are conjugate by an element of* Aut A.

Theorem 3.1. *Let A and B be finite-dimensional algebras. Assume that we have a morphism θ :* **Aut** *$A \to$* **Aut** *B. Then, for any abelian group G, we have a mapping $\Gamma \to \theta(\Gamma)$ from G-gradings of A to G-gradings of B. If $\Gamma \cong \Gamma'$, then $\theta(\Gamma) \cong \theta(\Gamma')$. If θ is an isomorphism and G is the universal grading group of a fine grading Γ, then $\theta(\Gamma)$ is a fine grading with universal group G.*

Using this theorem and the results about the gradings of nonassociative algebras, such as octonions or the Albert algebra, made it possible to describe abelian group gradings on simple Lie algebras of the types G_2 and F_4 in an amazing generality of arbitrary algebraically closed fields of characteristic not 2 or 3 in the case of G_2 and not 2, in the case of F_4. The group scheme **Aut** $L \cong$ **Aut O**, where **O** is the octonion algebra in the case where L is of the type G_2 and **Aut** $L \cong$ **Aut A** where **A** is the Albert algebra in the case where L is of the type F_4. For details see [19, Chapters 4, 5]. In both cases, the Lie algebra L is the derivation algebra of the respective nonassociative algebra $\mathscr{C}$ and the isomorphism of the group schemes is given by the adjoint map (see below).

The approach via group schemes paved the way for the classification of abelian group gradings on simple Cartan Lie algebras. These algebras arise as subalgebras of the derivation algebras of so-called *divided power algebras* (Witt algebras, type W; Special algebras, type S; Hamiltonian algebras, type H; Contact algebras, type K). For the classification of Cartan type Lie algebras see [29, 30]. A fairly recent paper [7] is devoted to the classification of all abelian group gradings on the restricted Lie algebras of the types W and S, with important information in the case S. In the case of restricted algebras, we only need to consider the derivation algebras of so the *truncated polynomial algebras*, that is, the algebras of the form $\mathscr{O}_p(n) = \mathbb{K}[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p)$, where $n \geq 1$, $p = \operatorname{char} \mathbb{K}$. Using the isomorphism of the group schemes Ad : **Aut** $\mathscr{O}_p(n) \to$ **Aut** $W_p(n)$, where Ad : $\varphi \to (D \to (\varphi^{-1} \circ D \circ \varphi))$ where φ is an automorphism of $\mathscr{O}_p(n)$ and D is a derivation of $\mathscr{O}_p(n)$ one easily transfers the gradings from $\mathscr{O}_p(n)$ to $W_p(n)$. For more details see [19, Chapter 7].

3.2 Group Gradings, Comodules and Functional Identities

Suppose a Lie algebra L over a field F is graded by an abelian group G. This is well known [24] to be equivalent to L being a (right) *H-comodule Lie algebra* over the group algebra $H = FG$, that is, to the existence of a Lie homomorphism $\rho : L \to L \otimes H$ such that

(1) $$(\rho \otimes \mathrm{id}_H)\rho = (\mathrm{id}_L \otimes \Delta)\rho$$

and

(2) $$(\mathrm{id}_L \otimes \varepsilon)\rho = \mathrm{id}_L.$$

In the case of a graded algebra, ρ is determined by $\rho(a_g) = a_g \otimes g$ where a_g is a homogeneous element of degree g. If L is a Lie subalgebra generating an associative algebra A and ρ extends to an associative homomorphism $\rho : A \to A \otimes H$, then A also becomes G-graded. One easily checks that (1) and (2) will still be satisfied because L generates A as an associative algebra. Since both L_g and A_g are defined as the sets of elements x in L and A satisfying $\rho(x) = x \otimes g$, we have $L_g = L \cap A_g$. In what follows, we will see that this extension of ρ from L to A indeed happens in the case where L is the Lie algebra of skew-symmetric elements in a central simple associative algebra over any field of characteristic not 2, provided that $\dim L \geq 21$. The case of L being infinite-dimensional is not excluded but is rather welcome! On the other hand, it is clear that some gradings of simple Lie algebras cannot be induced from the associative gradings. This is true for the grading of $\mathfrak{sl}(n)$ by $\mathbb{Z}_2$, which represents every matrix as the sum its of skew-symmetric and symmetric components. To correctly handle the situation arising, let us extend the Lie homomorphism $\rho : L \to L \otimes H$ to $\bar{\rho} : L \otimes H \to L \otimes H$ by setting $\bar{\rho}(x \otimes h) = \rho(x)(1 \otimes h)$. Clearly, this is a surjective Lie homomorphism. We want to extend $\bar{\rho}$ to $A \otimes H$.

Recall that a map σ from an associative algebra $\mathscr{A}'$ to a unital associative algebra $\mathscr{A}''$ is a *direct sum of a homomorphism and the negative of an antihomomorphism* if there exist central idempotent e_1 and e_2 in A'' with $e_1+e_2 = 1$ such that $x \mapsto e_1\sigma(x)$ is a homomorphism and $x \mapsto e_2\sigma(x)$ is the negative of an antihomomorphism. Maps with this property are clearly Lie homomorphisms; the nontrivial question is whether all Lie homomorphisms can be described in terms of such maps.

The answer to these questions is partially given in the following theorems from [2]. The technique used, so-called *Functional Identities*, was developed in course of solution of the famous Herstein problems about Lie homomorphisms of associative algebras (see [15–17]). At this time, the best source is the monograph by M. Brešar, M. Chebotar' and W. Martindale III [18].

We call a unital associative algebra over a field $\mathbb{K}$ *central* if its center equals $\mathbb{K} \cdot 1$.

Theorem 3.2. *Let A be a central simple algebra such that* $\dim_F A \geq 64$. *Let H be a unital commutative algebra. If A is not unital, then assume that H is finite dimensional. Then every surjective Lie homomorphism $\rho : [A, A] \otimes H \to [A, A] \otimes H$ can be extended to a direct sum of a homomorphism and the negative of an antihomomorphism $\sigma : A \otimes H \to A \otimes H$.*

Theorem 3.3. *Let* $\operatorname{char} F \neq 2$ *and let A be a central simple algebra such that* $\dim_F A \geq 441$. *Suppose that A has an involution and set $K = K_A$. Let H be a unital commutative algebra. If A is not unital, assume that H is finite dimensional. Then every surjective Lie homomorphism $\rho : [K, K] \otimes H \to [K, K] \otimes H$ can be extended to a homomorphism $\sigma : A \otimes H \to A \otimes H$.*

These transfer theorems and the techniques of affine group schemes have been successfully applied and the final classification of abelian group gradings on classical simple Lie algebras of types except D_4 over algebraically closed fields of characteristic other than 2 (in one case, different from 3) was obtained in [4]. The case of D_4 was accomplished by A. Elduque (see the details in [19]).

3.3 Case of Finitary Simple Algebras

Theorems 3.2 and 3.3 work well for infinite-dimensional algebras. For example, one can apply them to the case of Lie algebras L, which are direct limits of algebras $\{L_i = \mathfrak{sl}(n_i, \mathbb{K}) \mid k \in \mathbb{N}\}$, where each L_i is embedded in L_{i+1} by a diagonal embedding $X \to \operatorname{diag}(X, \ldots, X)$. In this case the unital algebra A is the direct limit of $\{A_i = M_{n_i}(\mathbb{K}) \mid k \in \mathbb{N}\}$, with the same embeddings. One could also consider the orthogonal or symplectic Lie algebras of skew-symmetric elements in A with respect to appropriate involutions. One drawback is that these theorems give no answer in the case of gradings of non-unital simple algebras by infinite groups, which is exactly the case when one deals with simple Lie algebras of finitary linear transformations. These algebras are direct limit of algebras of the same kind as before but the embedding is given by setting $X \to \operatorname{diag}(X, 0)$. Let us recall a few definitions.

An infinite-dimensional simple Lie algebra L of linear operators on an infinite-dimensional space over a field $\mathbb{K}$ is called *finitary* if L consists of linear operators of finite rank. These algebras were classified in [13] over any $\mathbb{K}$ with $\operatorname{char} \mathbb{K} = 0$ and in [14] over an algebraically closed $\mathbb{K}$ with $\operatorname{char} \mathbb{K} \neq 2, 3$. Under the latter assumption, finitary simple Lie algebras over $\mathbb{K}$ can be described in the following way.

Let U be an infinite-dimensional vector space over $\mathbb{K}$. Let $\Pi \subset U^*$ be a *total* subspace, i.e., for any $v \neq 0$ in U there is $f \in \Pi$ such that $f(v) \neq 0$. Let $\mathfrak{F}_\Pi(U)$ be the space spanned by the linear operators of the form $v \otimes f$, $v \in U$ and $f \in \Pi$, defined by $(v \otimes f)(u) = f(u)v$ for all $u \in U$. It is known from [23] that $R = \mathfrak{F}_\Pi(U)$ is a (non-unital) simple associative algebra.

The commutator $[R, R]$ is a simple Lie algebra, which is denoted by $\mathfrak{fsl}(U, \Pi)$. The algebra R admits an ($\mathbb{K}$-linear) involution if and only if there is a nondegenerate bilinear form $\Phi \colon U \times U \to \mathbb{K}$ that identifies U with Π and that is either symmetric or skew-symmetric. The set of skew-symmetric elements with respect to this involution, i.e., the set of $r \in R$ satisfying $\Phi(ru, v) + \Phi(u, rv) = 0$ for all $u, v \in U$, is a simple Lie algebra, which is denoted by $\mathfrak{fso}(U, \Phi)$ if Φ is symmetric and $\mathfrak{fsp}(U, \Phi)$ if Φ is skew-symmetric.

In [14] it is shown that if $\mathbb{K}$ is algebraically closed and char $\mathbb{K} \neq 2, 3$, then any finitary simple Lie algebra over $\mathbb{K}$ is isomorphic to one of $\mathfrak{fsl}(U, \Pi)$, $\mathfrak{fso}(U, \Phi)$ or $\mathfrak{fsp}(U, \Phi)$. The most important special case is that of countable (infinite) dimension. Then U has countable dimension and the isomorphism class of the Lie algebra does not depend on Π or Φ. Hence, there are exactly three finitary simple Lie algebras of countable dimension: $\mathfrak{sl}(\infty)$, $\mathfrak{so}(\infty)$ and $\mathfrak{sp}(\infty)$.

In the paper [3] we have given a complete classification of the abelian group gradings on finitary simple Lie algebras. This classification was obtained using the transfer to the case of associative algebras. Since the authors did not want to restrict themselves to the case of finite grading groups, as suggested by Theorems 3.2 or 3.3, they proved the following result.

To state the results, we set $\mathscr{A}^\sharp = \mathscr{A} + \mathbb{K}1$; thus, $\mathscr{A}^\sharp = \mathscr{A}$ if $1 \in \mathscr{A}$. If X is a subset of an associative algebra, then by $\langle X \rangle$ we denote the subalgebra generated by X.

Theorem 3.4. *Let $\mathscr{A}$ be any algebra satisfying $\mathfrak{F}_\Pi(U) \subset \mathscr{A} \subset \mathrm{End}\,(U)$, where U is infinite-dimensional, and let $\mathscr{L}$ be a noncentral Lie ideal of $\mathscr{A}$. If $\mathscr{H}$ is a unital commutative associative algebra, $\mathscr{M}$ is a Lie ideal of some associative algebra, and $\rho \colon \mathscr{M} \to \mathscr{L} \otimes \mathscr{H}$ is a surjective Lie homomorphism, then there exist a direct sum of a homomorphism and the negative of an antihomomorphism $\sigma \colon \langle \mathscr{M} \rangle \to \mathscr{A}^\sharp \otimes \mathscr{H}$ and a linear map $\tau \colon \mathscr{M} \to 1 \otimes \mathscr{H}$ such that $\tau([\mathscr{M}, \mathscr{M}]) = 0$ and $\rho(x) = \sigma(x) + \tau(x)$ for all $x \in \mathscr{M}$.*

If $\mathscr{A}$ is any associative algebra with an involution φ, then by $\mathscr{K}(\mathscr{A}, \varphi)$ we denote the Lie algebra of skew-symmetric elements in $\mathscr{A}$:

$$\mathscr{K}(\mathscr{A}, \varphi) = \{a \in \mathscr{A} \mid \varphi(a) = -a\}.$$

Theorem 3.5. *Let $\mathscr{A}$ be any algebra satisfying $\mathfrak{F}_\Pi(U) \subset \mathscr{A} \subset \mathrm{End}\,(U)$, where U is infinite-dimensional and the characteristic of the ground field is other than 2. Assume that $\mathscr{A}$ has an involution φ and let $\mathscr{L}$ be a noncentral Lie ideal of $\mathscr{K}(\mathscr{A}, \varphi)$. If $\mathscr{H}$ is a unital commutative associative algebra, $\mathscr{U}$ is an associative algebra with involution $*$, $\mathscr{M}$ is a Lie ideal of $\mathscr{K}(\mathscr{U}, *)$, and $\rho \colon \mathscr{M} \to \mathscr{L} \otimes \mathscr{H}$ is a surjective Lie homomorphism, then there exist a homomorphism $\sigma \colon \langle \mathscr{M} \rangle \to \mathscr{A}^\sharp \otimes \mathscr{H}$ and a linear map $\tau \colon \mathscr{M} \to 1 \otimes \mathscr{H}$ such that $\tau([\mathscr{M}, \mathscr{M}]) = 0$ and $\rho(x) = \sigma(x) + \tau(x)$ for all $x \in \mathscr{M}$.*

Note that in the case of simple algebras the map τ is always zero.

4 Group Gradings on Algebras of Finitary Linear Transformations

For the transfer approach to work, one needs to classify the gradings by groups on associative simple finitary algebras. In this section the grading group G does not need to be abelian.

Now the basic idea about the gradings in the case of finite-dimensional associative algebras (even graded Artinian algebras) is as follows. Let us call an algebra *graded simple* if it has no nontrivial proper ideals which are graded subspaces. Let us call a G-graded algebra a *graded division algebra* if any nonzero homogeneous element is invertible. An algebra is called graded left Artinian if it satisfies the descending chain condition for its left ideals. A well-known adaptation of the classical Wedderburn–Artin Theorem (see [25]) is the following.

Theorem 4.1. *Every G-graded left Artinian graded simple algebra A is isomorphic to an algebra of endomorphisms* $\mathrm{End}_D(V)$ *of a G-graded right vector space V over a graded division algebra D.*

Basically the same result holds in the case of simple associative algebras or finitary linear transformations, which belong to a wider class of primitive algebras with minimal ideals.

As shown in [23, IV.9], R is a primitive algebra with minimal one-sided ideals if and only if it is isomorphic to a subalgebra of $\mathfrak{L}_\Pi(U)$ containing the ideal $\mathfrak{F}_\Pi(U)$ where U is a right vector space over a division algebra Δ, Π is a total subspace of the left vector space U^* over Δ, $\mathfrak{L}_\Pi(U)$ is the algebra of all continuous Δ-linear operators on U, and $\mathfrak{F}_\Pi(U)$ is the set of all operators in $\mathfrak{L}_\Pi(U)$ whose image has finite dimension over Δ. The term *continuous* refers here to the topology on U with a neighbourhood basis at 0 consisting of the sets of the form $\ker f_1 \cap \ldots \cap \ker f_k$ where $f_1, \ldots, f_k \in \Pi$ and $k \in \mathbb{N}$. A linear operator $\mathscr{A}\colon U \to U$ is continuous with respect to this topology if and only if the adjoint operator $\mathscr{A}^*\colon U^* \to U^*$ leaves the subspace Π invariant. (In particular, if Δ is $\mathbb{R}$ and $\mathbb{C}$, U is a Banach space and Π consists of all bounded linear functionals on U, then a linear operator on U is continuous in our sense if and only if it is bounded.)

In our paper [3], we show that if an algebra R as above is given a grading by a group G, then it becomes a graded primitive algebra with minimal one-sided graded ideals. To state the graded analogue of the quoted result from [23, IV.9], let us remind that a linear transformation $f\colon M \to N$ of G-graded vector spaces is said to be *homogeneous of degree h* if $f(M_g) \subset N_{hg}$, for all $g \in G$. Thus, the set $\mathrm{Hom}^{\mathrm{gr}}(M, N)$ of finite sums of homogeneous maps from M to N is a G-graded vector space. A homomorphism of graded vector spaces is a homogeneous map of degree e.

4.1 Graded Division Algebras

In the case where $\mathbb{K}$ is algebraically closed and we consider a G-grading on $R = \mathfrak{F}_\Pi(U)$, which is a locally finite simple algebra with minimal ideals, we have that in the presentation of $R \cong \mathfrak{F}_W^{\mathrm{gr}}(V)$ in Theorem 4.4, the graded division algebra is isomorphic to a matrix algebra $M_\ell(\mathbb{K})$, as an ungraded algebra. If we restrict ourselves to the case where G is abelian (this is all we need when we deal with simple Lie algebras!) such graded division algebras have been described in [10] and classified up to isomorphism [4]. The following statement can be found in [19].

Theorem 4.2. *Let T be a finite abelian group and let $\mathbb{K}$ be an algebraically closed field. There exists a grading on the matrix algebra $M_n(\mathbb{K})$ with support T making $M_n(\mathbb{K})$ a graded division algebra if and only if* char $\mathbb{K}$ *does not divide n and $T \cong \mathbb{Z}_{\ell_1}^2 \times \cdots \times \mathbb{Z}_{\ell_r}^2$ where $\ell_1 \cdots \ell_r = n$. The isomorphism classes of such gradings are in one-to-one correspondence with nondegenerate alternating bicharacters $\beta : T \times T \to \mathbb{K}^\times$. All such gradings belong to one equivalence class.* □

A *standard realization* can be obtained as follows.

If R is finite-dimensional or a finitary algebra, then D is finite-dimensional and $T = \operatorname{Supp} D$ is a finite subgroup of G. Every homogeneous component of D is one-dimensional and, one can easily see that D is isomorphic to a group algebra of T twisted by a 2-cocycle $\sigma : T \times T \to \mathbb{K}^\times$, usually denoted by $\mathbb{K}^\sigma T$. This means that we can choose a basis X_t in each D_t, $t \in T$ and then we will have $X_t X_u = \sigma(t,u) X_{tu}$. Let us introduce an alternating bicharacter $\beta = \beta_\sigma$ by setting $\beta(t,u) = \sigma(t,u)\sigma(u,t)^{-1}$. This depends only on the cohomology class of σ, which defines the isomorphism class of $\mathbb{K}^\sigma T$. The simplicity of $\mathbb{K}^\sigma T$ is then equivalent to the nondegeneracy of β.

Suppose there exists a nondegenerate alternating bicharacter β on T. One easily shows that T admits a "symplectic basis", i.e., there exists a decomposition of T as the direct product of cyclic subgroups:

$$T = H_1' \times H_1'' \times \cdots \times H_r' \times H_r'' \tag{3}$$

such that $H_i' \times H_i''$ and $H_j' \times H_j''$ are β-orthogonal for $i \neq j$, and H_i' and H_i'' are in duality by β.

4.2 Pauli Gradings on Matrix Algebras

Given a matrix algebra $D = M_n(\mathbb{K})$ over a field $\mathbb{K}$ possessing a primitive nth root of 1, one can define a division grading by the group $T = \langle a \rangle_n \times \langle b \rangle_n \cong \mathbb{Z}_n^2$ on D, as follows. Let

(4)
$$X = X(n,\varepsilon) = \begin{bmatrix} \varepsilon^{n-1} & 0 & 0 \dots 0\,0 \\ 0 & \varepsilon^{n-2} & 0 \dots 0\,0 \\ \dots & & \\ 0 & 0 & 0 \dots \varepsilon\,0 \\ 0 & 0 & 0 \dots 0\,1 \end{bmatrix} \text{ and } Y = Y(n) = \begin{bmatrix} 0\; 1\, 0 \dots 0\, 0 \\ 0\; 0\, 1 \dots 0\, 0 \\ \dots \\ 0\; 0\, 0 \dots 0\, 1 \\ 1\; 0\, 0 \dots 0\, 0 \end{bmatrix}$$

be called *generalized Pauli matrices*. Clearly, X and Y are periodic matrices of order n each. Moreover, $YX = \varepsilon^{-1}XY$. For any $t = a^k b^l \in T$, we set $X_t = X^k Y^l$. There are n^2 of such different matrices, they form a basis of R and setting $D_t = \mathbb{K}X_t$, for each $t \in T$ turns D into a graded algebra. We call this grading a *Pauli grading* of $D = M_n(\mathbb{K})$ and denote by $\Pi(n,\varepsilon)$. Since for any $t \in T$ every nonzero element of D_t is invertible, D is a graded division algebras.

Using Pauli gradings allows us to describe the classes $[\sigma]$ such that β_σ is nondegenerate and the isomorphisms from $\mathbb{K}^\sigma T$ onto a matrix algebra, as follows. Denote by ℓ_i the order of H_i' and H_i''. (We may assume without loss of generality that ℓ_i are prime powers.) If we pick generators a_i and b_i for H_i' and H_i'', respectively, then $\varepsilon_i = \beta(a_i, b_i)$ is a primitive ℓ_i-th root of unity, and all other values of β on the elements $a_1, b_1, \dots, a_r, b_r$ are 1. We can scale the elements X_{a_i} and X_{b_i} so that $X_{a_i}^{\ell_i} = X_{b_i}^{\ell_i} = 1$. Then we consider the Kronecker product $M_{\ell_1}(\mathbb{K}) \otimes \cdots \otimes M_{\ell_r}(\mathbb{K})$ of matrix algebras $M_{\ell_i}(\mathbb{K})$, each with a Pauli grading $\Pi(\ell_i, \varepsilon_i)$. The degree of the product $X_1 \otimes \cdots \otimes X_r$ is equal to the product of the degrees of the factors. Then we map the generators X_{t_i} of $\mathbb{K}^\sigma T$, $t_i \in T_i$, $i = 1, \dots, r$, as follows:

$$X_{t_i} \mapsto I \otimes \cdots I \otimes X_{t_i} \otimes I \otimes \cdots I, \tag{5}$$

where the only nonzero factor on the ith position is the generalized Pauli matrix from the definition of the Pauli grading $\Pi(\ell_i, \varepsilon_i)$. Comparing defining relations and the dimensions shows that this is an isomorphism of graded algebras.

If we scale X_{a_i} and X_{b_i}, as above, to have $X_{a_i}^{\ell_i} = X_{b_i}^{\ell_i} = 1$, and set $X_{(a_1^{i_1}, b_1^{j_1}, \dots, a_r^{i_r}, b_r^{j_r})} = X_{a_1}^{i_1} X_{b_1}^{j_1} \cdots X_{a_r}^{i_r} X_{b_r}^{j_r}$, then

$$X_{(a_1^{i_1}, b_1^{j_1}, \dots, a_r^{i_r}, b_r^{j_r})} X_{(a_1^{i_1'}, b_1^{j_1'}, \dots, a_r^{i_r'}, b_r^{j_r'})} = \varepsilon_1^{-j_1 i_1'} \cdots \varepsilon_r^{-j_r i_r'} X_{(a_1^{i_1+i_1'}, b_1^{j_1+j_1'}, \dots, a_r^{i_r+i_r'}, b_r^{j_r+j_r'})}.$$

Hence, with this choice of X_t, we obtain a representative of the cohomology class $[\sigma]$ that is multiplicative in each variable, i.e., it is a *bicharacter* (not alternating unless T is trivial).

Summarizing, we derive the following.

Theorem 4.3. *If a matrix algebra $R = M_n(\mathbb{K})$, $\mathbb{K}$ an algebraically closed field, is turned into a G-graded division algebra, then there are $l_1, \dots, l_r$, $n = l_1 \cdots l_r$, a subgroup $T \cong \mathbb{Z}_{l_1}^2 \oplus \cdots \oplus \mathbb{Z}_{l_r}^2$ and $\varepsilon_1, \dots, \varepsilon_r$, where ε_i is an l_i*th *root of 1, for* *each $i = 1, \dots, r$, such that R is isomorphic as a graded algebra to the graded Kronecker product $M_{\ell_1}(\mathbb{K}) \otimes \cdots \otimes M_{\ell_r}(\mathbb{K})$ of matrix algebras, on each of which the grading is Pauli.*

In many arguments, however, it is convenient to simply use that any finite-dimensional G-graded division algebra D is a twisted group algebra of a group T, with basis X_t, $t \in T$, $D_t = \mathbb{K}X_t$ and $X_t X_{t'} = \sigma(t, t')X_{tt'}$ for some 2-cocycle $\sigma\colon T \times T \to \mathbb{K}^\times$. The isomorphism classes of such gradings are in bijection with the pairs (T, β) where $T \subset G$ is a subgroup of order ℓ^2 and $\beta\colon T \times T \to \mathbb{K}^\times$ is a nondegenerate alternating bicharacter. Here T is the support of the grading and β is given by $\beta(t, t') = \sigma(t, t')/\sigma(t', t)$, so we get

$$X_t X_{t'} = \beta(t, t')X_{t'} X_t \quad \text{for all} \quad t, t' \in T.$$

Note that char $\mathbb{K}$ cannot divide the order of T.

4.3 Graded Primitive Algebras with Minimal Graded Left Ideals

As mentioned in the introduction, we want to study gradings on the associative algebras of the form $\mathfrak{F}_\Pi(U)$ where U is a vector space over $\mathbb{K}$ and $\Pi \subset U^*$ is a total subspace. A natural class to which these algebras belong is the primitive algebras with minimal left ideals. In fact, the simple algebras in this class are precisely the algebras $\mathfrak{F}_\Pi(U)$ where U is a right vector space over a division algebra and $\Pi \subset U^*$ is a total subspace. We are now going to develop a graded version of the theory in [23, Chapter IV], which will apply to the setting we are interested in—thanks to the following lemma. All algebras in this section will be associative, but not necessarily unital. Unless indicated otherwise, the ground field $\mathbb{K}$ is arbitrary. (In fact, some results will also apply to rings.)

Lemma 4.1. *Let R be a primitive algebra (or ring) with minimal left ideals. Suppose that R is given a grading by a group G. Then R is graded primitive with minimal graded left ideals.*

Proof. Any nonzero left ideal of R is a faithful module because R is prime. Now consider R as a graded left R-module. According to [25, Proposition 2.7.3], the graded socle $S^{\mathrm{gr}}(R)$ always contains the ordinary socle $S(R)$. Hence R must contain a minimal graded left ideal I. Since I is a faithful graded simple R-module, R is graded primitive. □

4.4 A Structure Theorem

Let R be a G-graded algebra (or ring) and let V be a graded simple faithful left module over R. Let $D = \mathrm{End}_R^{\mathrm{gr}}(V)$. We will follow the standard convention of writing the elements of D on the right. By a version of Schur's Lemma (see e.g., [25, Proposition 2.7.1]), D is a graded division algebra, and thus V is a graded right vector space over D. Hence R is isomorphic to a graded subalgebra of $\mathrm{End}_D^{\mathrm{gr}}(V)$.

The *graded dual* $V^{\mathrm{gr}\,*}$ is defined as $\mathrm{Hom}_D^{\mathrm{gr}}(V, D)$. Note that $V^{\mathrm{gr}\,*}$ is a graded left vector space over D, with the action of D defined by $(df)(v) = df(v)$ for all $d \in D$, $v \in V$ and $f \in V^{\mathrm{gr}\,*}$. We will often use the symmetric notation (f, v) for $f(v)$.

Let $W \subset V^{\mathrm{gr}\,*}$ be a total graded subspace, i.e., the restriction of the mapping $(\,,\,)$ is a nondegenerate D-bilinear form $W \times V \to D$. Recall the graded subalgebras $\mathfrak{L}_W^{\mathrm{gr}}(V)$ and $\mathfrak{F}_W^{\mathrm{gr}}(V)$ of $\mathrm{End}_D^{\mathrm{gr}}(V)$ defined in the introduction: $\mathfrak{L}_W^{\mathrm{gr}}(V)$ is the set of all operators in $\mathrm{End}_D^{\mathrm{gr}}(V)$ that are continuous with respect to the topology induced by W and the subalgebra $\mathfrak{F}_W^{\mathrm{gr}}(V) \subset \mathfrak{L}_W^{\mathrm{gr}}(V)$ consists of all finite sums of operators of the form $v \otimes w$ where $v \in V$ and $w \in W$ are homogeneous and $v \otimes w$ acts on V as follows:

$$(v \otimes w)(u) = v(w, u) \quad \text{for all} \quad u, v \in V \ \text{ and } \ w \in W.$$

It is easy to see that V is a graded simple module for $\mathfrak{F}_W^{\mathrm{gr}}(V)$ and hence for $\mathfrak{L}_W^{\mathrm{gr}}(V)$ or any graded subalgebra of $\mathrm{End}_D^{\mathrm{gr}}(V)$ containing $\mathfrak{F}_W^{\mathrm{gr}}(V)$.

Suppose that R has a minimal graded left ideal I. Then either $I^2 = 0$ or $I = R\varepsilon$ where ε is a homogeneous idempotent (hence of degree e). Indeed, if $I^2 \neq 0$, then there is a homogeneous $x \in I$ such that $Ix \neq 0$ and hence $Ix = I$. The left annihilator L of x in R is a graded left ideal of R such that $I \not\subset L$ and hence $L \cap I = 0$. Let ε be an element of I such that $\varepsilon x = x$. Since x is homogeneous and every homogeneous component of ε is in I, we may assume that ε is homogeneous of degree e. Now $\varepsilon^2 x = \varepsilon x = x$ and hence $\varepsilon^2 - \varepsilon \in L$. Since $I \cap L = 0$, we conclude that $\varepsilon^2 = \varepsilon$. Since $Ix \neq 0$, it follows that $I\varepsilon \neq 0$ and hence $I\varepsilon = I$. Since we are assuming R graded primitive and hence graded prime, the case $I^2 = 0$ is not possible. Hence $I = R\varepsilon$ is a graded simple R-module.

The following is a graded version of a result in [23, III.5].

Lemma 4.2. *Let R be a graded primitive algebra (or ring) with a minimal graded left ideal I. Let V be a faithful graded simple left R-module. Then there exists $g \in G$ such that V is isomorphic to $I^{[g]}$ which is I, with the grading right shifted by g, as a graded R-module. Hence, all faithful graded simple left R-modules are isomorphic up to a right shift of grading.*

Proof. Since IV is a graded submodule of V, we have either $IV = 0$ or $IV = V$. But V is faithful, so $IV = V$. Pick a homogeneous $v \in V$ such that $Iv \neq 0$ and let $g = \deg v$. Then the map $I \to V$ given by $r \mapsto rv$ is a homomorphism of R-modules and sends I_h to V_{hg}, $h \in G$. By graded simplicity of I and V, this map is an isomorphism of R-modules. Hence $I^{[g]}$ is isomorphic to V as a graded R-module. □

Now we will obtain a graded version of the structure theorem in [23, IV.9].

Theorem 4.4. *Let R be a G-graded algebra (or ring). Then R is graded primitive with minimal graded left ideals if and only if there exists a G-graded division algebra D, a graded right vector space V over D and a total graded subspace $W \subset V^{\mathrm{gr}\,*}$ such that R is isomorphic to a graded subalgebra (subring) of $\mathfrak{L}_W^{\mathrm{gr}}(V)$ containing $\mathfrak{F}_W^{\mathrm{gr}}(V)$. Moreover, $\mathfrak{F}_W^{\mathrm{gr}}(V)$ is the only such subalgebra (subring) that is graded simple.*

4.5 An Isomorphism Theorem

We are going to investigate under what conditions two graded simple algebras described by Theorem 4.4 are isomorphic. By Lemma 4.2, V is determined by R up to isomorphism and shift of grading. If $\varphi \in \mathrm{End}_R^{\mathrm{gr}} V$ is homogeneous of degree t, then φ regarded as a map $V^{[g]} \to V^{[g]}$ will be homogeneous of degree $g^{-1}tg$. Indeed, $(V_h)\varphi \subset V_{ht}$, for all $h \in G$, can be rewritten as $(V_{hg}^{[g]})\varphi \subset V_{htg}^{[g]}$. Hence if $D = \mathrm{End}_R^{\mathrm{gr}}(V)$, then $\mathrm{End}_R^{\mathrm{gr}}(V^{[g]}) = {}^{[g^{-1}]}D^{[g]}$. It remains to include the total graded subspace $W \subset V^{\mathrm{gr}\,*}$ in the picture. It is convenient to introduce the following terminology. Fix a group G.

Definition 4.1. Let D and D' be G-graded division algebras (or rings) and let V and V' be graded right vector spaces over D and D', respectively. Let $\psi_0 \colon D \to D'$ be an isomorphism of G-graded algebras (or rings). A homomorphism $\psi_1 \colon V \to V'$ of G-graded vector spaces over $\mathbb{K}$ (or G-graded abelian groups) is said to be ψ_0-*semilinear* if $\psi_1(vd) = \psi_1(v)\psi_0(d)$ for all $v \in V$ and $d \in D$. The *adjoint* to ψ_1 is the mapping $\psi_1^* : (V')^{\mathrm{gr}\,*} \to V^{\mathrm{gr}\,*}$ defined by $(\psi_1^*(f))(v) = \psi_0^{-1}(f(\psi_1(v)))$ for all $f \in (V')^{\mathrm{gr}\,*}$ and $v \in V$.

One easily checks that ψ_1^* is ψ_0^{-1}-semilinear.

Definition 4.2. Let D and D' be G-graded division algebras (or rings), let V and V' be graded right vector spaces over D and D', respectively, and let W and W' be total graded subspaces of $V^{\mathrm{gr}\,*}$ and $(V')^{\mathrm{gr}\,*}$, respectively. An *isomorphism of triples* from (D, V, W) to (D', V', W') is a triple (ψ_0, ψ_1, ψ_2) where $\psi_0 \colon D \to D'$ is an isomorphism of graded algebras (or rings) while $\psi_1 \colon V \to V'$ and $\psi_2 \colon W \to W'$ are isomorphisms of graded vector spaces over $\mathbb{K}$ (or graded abelian groups) such that $(\psi_2(w), \psi_1(v)) = \psi_0((w, v))$ for all $v \in V$ and $w \in W$.

It follows that ψ_1 and ψ_2 are ψ_0-semilinear. Also, for given isomorphisms ψ_0 and ψ_1 there can exist at most one ψ_2 such that (ψ_0, ψ_1, ψ_2) is an isomorphism of triples. Such ψ_2 exists if and only if ψ_1 is ψ_0-semilinear and $\psi_1^*(W') = W$. Indeed, we can take ψ_2 to be the restriction of $(\psi_1^*)^{-1}$ to W. The condition $\psi_1^*(W') = W$ means that $\psi_1 \colon V \to V'$ is a homeomorphism with respect to the topologies induced by W and W'.

The following is a graded version of the isomorphism theorem in [23, IV.11].

Theorem 4.5. *Let G be a group. Let D and D' be G-graded division algebras (or rings), let V and V' be graded right vector spaces over D and D', respectively, and let W and W' be total graded subspaces of $V^{\mathrm{gr}\,*}$ and $(V')^{\mathrm{gr}\,*}$, respectively. Let R and R' be G-graded algebras (or rings) such that*

$$\mathfrak{F}_W^{\mathrm{gr}}(V) \subset R \subset \mathfrak{L}_W^{\mathrm{gr}}(V) \quad \text{and} \quad \mathfrak{F}_{W'}^{\mathrm{gr}}(V') \subset R' \subset \mathfrak{L}_{W'}^{\mathrm{gr}}(V').$$

If $\psi : R \to R'$ is an isomorphism of graded algebras, then there exist $g \in G$ and an isomorphism (ψ_0, ψ_1, ψ_2) from $({}^{[g^{-1}]}D^{[g]}, V^{[g]}, {}^{[g^{-1}]}W)$ to (D', V', W') such that

$$\psi(r) = \psi_1 \circ r \circ \psi_1^{-1} \quad \text{for all} \quad r \in R. \tag{6}$$

If other $g' \in G$ and isomorphism $(\psi_0', \psi_1', \psi_2')$ from $({}^{[(g')^{-1}]}D^{[g']}, V^{[g']}, {}^{[(g')^{-1}]}W)$ to (D', V', W') define ψ as above, then there exists a nonzero homogeneous $d \in D$ such that $g' = g \deg d$, $\psi_0'(x) = d^{-1}\psi_0(x)d$ for all $x \in D$, $\psi_1'(v) = \psi_1(v)d$ for all $v \in V$, and $\psi_2'(w) = d^{-1}\psi_2(w)$ for all $w \in W$.

As a partial converse, if (ψ_0, ψ_1, ψ_2) is an isomorphism of triples as above, then setting $\psi(r) = \psi_1 \circ r \circ \psi_1^{-1}$ defines an isomorphism of G-graded algebras (or rings) $\psi : \mathfrak{L}_W^{\mathrm{gr}}(V) \to \mathfrak{L}_{W'}^{\mathrm{gr}}(V')$ such that $\psi(\mathfrak{F}_W^{\mathrm{gr}}(V)) = \mathfrak{F}_{W'}^{\mathrm{gr}}(V')$.

Note that it follows that any isomorphism of graded algebras $\psi : R \to R'$ extends to an isomorphism $\mathfrak{L}_W^{\mathrm{gr}}(V) \to \mathfrak{L}_{W'}^{\mathrm{gr}}(V')$ and restricts to an isomorphism $\mathfrak{F}_W^{\mathrm{gr}}(V) \to \mathfrak{F}_{W'}^{\mathrm{gr}}(V')$.

4.6 Graded Simple Algebras with Minimal Graded Left Ideals

Fix a grading group G. In view of Theorems 4.4 and 4.5, graded simple algebras (or rings) with minimal graded left ideals are classified by the triples (D, V, W) where D is a graded division algebra (or ring), V is a right graded vector space over D and W is a total graded subspace of $V^{\mathrm{gr}\,*}$. For a fixed D, the triples (D, V, W) can be classified up to isomorphism as follows.

Let T be the support of D and let $\Delta = D_e$. Clearly, T is a subgroup of G and Δ is a division algebra (or ring). Consider the set G/T of left cosets and the set $T\backslash G$ of right cosets of T in G. The map $A \to A^{-1}$ is a bijection between G/T and $T\backslash G$. Clearly, the graded right D-modules ${}^{[g]}D$ and ${}^{[h]}D$ are isomorphic if and only if $gT = hT$ (similarly for graded left D-modules). Any graded right D-module V is a direct sum of modules of this form, which can be grouped into isotypic components. Namely, $V_A = \bigoplus_{a\in A} V_a$ is the isotypic component of V corresponding to ${}^{[g]}D$ where $A = gT$. Note that V_g is a right Δ-module and $V_A \cong V_g \otimes_\Delta {}^{[g]}D$ as graded right D-modules. Select a left transversal S for T, i.e., a set of representatives for