

Chiara Esposito

Formality Theory

From Poisson
Structures
to Deformation
Quantization



Springer

SpringerBriefs in Mathematical Physics

Volume 2

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ISSN 2197-1757
ISBN 978-3-319-09289-8
DOI 10.1007/978-3-319-09290-4

ISSN 2197-1765 (electronic)
ISBN 978-3-319-09290-4 (eBook)

Library of Congress Control Number: 2014945958

Springer Cham Heidelberg New York Dordrecht London

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Preface

This book aims to be a survey of the theory of formal deformation quantization of Poisson manifolds, in the formalism developed by Kontsevich. It is intended to be a pedagogical introduction to mathematicians and mathematical physicists who are first approaching the subject. The main topics are the theory of Poisson manifolds, star products and their classification, deformations of associative algebras, and the formality theorem. Furthermore, the aim is to introduce the reader to the relevant physical motivations behind the purely mathematical constructions.

Despite the fact that deformation quantization is a broad research area, the pedagogical literature dealing with this topic, and in particular formal deformation quantization, is limited.

The construction of star products on symplectic manifolds can be found in [3], a monograph focused on Fedosov's approach. Recently, S. Waldmann [6] wrote a more extensive book on the argument, which is only available in German. Further interesting reviews on the subject can be found, for instance, in [1, 2, 4, 5]. In particular, [1] is a sketchy introduction to formal deformation quantization. A detailed, although brief, presentation of the Kontsevich theory from a purely mathematical point of view is included in [2, 4, 5]. It has to be noted that these reviews are oriented to researchers with a basic knowledge of Poisson geometry. There, the authors describe Kontsevich's formality theorem and its relation to the existence and classification of star products on a Poisson manifold. In [4] the author introduces the theory of star products and their classification on symplectic manifolds. The notes in [2, 5] are also focused on further developments of Kontsevich's theory. In particular, the author in [5] also introduces the Tamarkin approach.

The above list, although not exhaustive, represents the state of the art of the introductory literature on formal deformation quantization. At the present time, a book that covers both the basic mathematical tools and Kontsevich's theory, and one that is also designed for first-time readers, is missing. These notes aim to fill this gap writing a didactical exposition to the subject for nonexperts without assuming from the reader too many prerequisites. In particular, this book is addressed to mathematical physicists, who have basic knowledge of differential geometry, classical, and quantum mechanics.

We describe the Poisson structures and their role in classical mechanics, present the results of Kontsevich on the classification of star products on Poisson manifolds, and discuss the physical motivations underlying this theory.

Würzburg, May 2014

Chiara Esposito

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Acknowledgments

This work is based on the course given during the Summer School of Geometry, which took place in Ouargla (Algeria), 2012. I would like to thank the hospitality and support at the University Kasdi Merbah of Ouargla and in particular Mohamed Amine Bahayou for the nice organization during our stay in Algeria.

I started writing the proposal for this book during my stay in the Oberwolfach Mathematical Institute; I sincerely thank the organization that gave me the possibility of working on this project in such a beautiful and inspiring place.

Special thanks go to Peter Bongaart for his interest in these lectures and for his help during the writing process: without his enthusiasm this book would have never been written. Many thanks also to the Editor of these notes, Aldo Rampioni, for his kind support during all the stages of this project.

I want to thank my advisor, Ryszard Nest, who introduced me to this beautiful field of mathematics and helped me to understand it. Many thanks to Stefan Waldmann for his precious help; his comments have been fundamental to clarify many parts of this book. Thanks also to Thorsten Reichert, for his inspiring seminars on Dolgushev's proof.

I want to thank my friend, Romero Barbieri Solha, for his help in writing the proposal of this book. The Mathematics Department at Würzburg University has been a very pleasant and stimulating working place, and there are several people who contributed to this in particular I would like to thank my dear friends Alejandro Bolaños Rosales, Stephan Hachinger, Filippo Bracci, and Alessandra Zappoli for the help during the last months of writing and editing of this book. Finally, to all the persons who love me and support me, my parents, my brother and my sister, George, Fedele and Susana, Giovannino, my friends from physics, all my family: thank you!

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