Lecture Notes in Applied and Computational Mechanics 73

Jean-François Ganghoffer
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Similarity and Symmetry Methods

Applications in Elasticity and Mechanics of Materials

Springer

Lecture Notes in Applied and Computational Mechanics

Volume 73

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Editors

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ISSN 1613-7736 ISSN 1860-0816 (electronic)
ISBN 978-3-319-08295-0 ISBN 978-3-319-08296-7 (eBook)
DOI 10.1007/978-3-319-08296-7
Springer Cham Heidelberg New York Dordrecht London

Library of Congress Control Number: 2014944340

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface

On June 06–09, 2013, the EUROMECH Workshop *Similarity, Symmetry and Group Theoretical Methods in Mechanics* took place in Varna, Bulgaria.

It brought together many scientists from European countries and USA, and focused on the current state of the art in the field of similarity methods in mechanics. The aim of this Workshop was to bring together researchers who apply similarity and symmetry analysis to theoretical and engineering problems in both solid and fluid mechanics, researchers who are developing significant extensions of these methods so that they can be applied more widely, and numerical analysts who develop and use such methods in numerical schemes.

The scientific program of the Workshop was built around main speakers who gave an overview of the field in the form of short lecture courses delivered by

Nail H. Ibragimov—Group Analysis as a Microscope of Mathematical Modeling,

George Bluman—Some Recent Developments in Finding Systematically Conservation Laws and Nonlocal Symmetries for Partial Differential Equations, and

Charles-Michel Marle—Symmetries of Hamiltonian Dynamical Systems, Momentum Maps and Reduction.

The two organizers are deeply grateful to EUROMECH for the provided support making possible the first in this new series of scientific meetings. This Springer volume contains lecture notes written by the principal speakers of the Workshop which are complemented by a few shorter contributions dealing with specific problems.

The Editors hope very much that this volume gives a modern overview of the similarity and symmetry methods and shows applications of this active field of research in mechanics and will serve as a reference in the years to come.

Nancy, April 2014 — Jean-François Ganghoffer
Sofia — Ivaïlo Mladenov

Contents

Some Recent Developments in Finding Systematically Conservation Laws and Nonlocal Symmetries for Partial Differential Equations

George Bluman and Zhengzheng Yang

Abstract This chapter presents recent developments in finding systematically conservation laws and nonlocal symmetries for partial differential equations. There is a review of local symmetries, including Lie's algorithm to find local symmetries in evolutionary form and their applications. The Direct Method for finding local conservation laws is reviewed and its relationship to and extension of Noether's theorem are discussed. Moreover, it is shown how symmetries, including discrete symmetries may yield additional conservation laws from known conservation laws. Systematic procedures are presented to seek nonlocally related PDE systems for a given PDE system with two independent variables. In particular, these procedures include the use of conservation laws, point symmetries, and subsystems (including subsystems arising after appropriate invertible transformations of variables) to obtain trees of equivalent nonlocally related PDE systems. In turn, it is shown how the calculation of point symmetries of such nonlocally related systems leads to the discovery of nonlocal symmetries for a given PDE system. The situation of systematically constructing useful nonlocally related systems in multidimensions is considered. Many illustrative examples are provided.

1 Introduction

This chapter is concerned with recent developments in finding conservation laws (CLs) and nonlocal symmetries for partial differential equations (PDEs). It focuses on recent research of the authors and some of the first author's collaborators, including Stephen Anco, Alexei Cheviakov, Temuer Chaolu, Jean-François Ganghoffer, Nataliya

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J.-F. Ganghoffer and I. Mladenov (eds.), *Similarity and Symmetry Methods*, Lecture Notes in Applied and Computational Mechanics 73, DOI: 10.1007/978-3-319-08296-7_1,

Ivanova, Sukeyuki Kumei, Ian Lisle, Alex Ma, Greg Reid, Vladimir Shtelen and Thomas Wolf. Much of the material in this chapter appears in more detail in [1, 2].

In the latter part of the 19th century, Sophus Lie initiated his studies on continuous groups of transformations (Lie groups of transformations) in order to put order to, and thereby extend systematically, the hodgepodge of heuristic techniques for solving ordinary differential equations (ODEs). In particular, Lie showed the following.

- The problem of finding a Lie group of point transformations leaving invariant a differential equation (*point symmetry* of a differential equation) is systematic and reduces to solving a related linear system of determining equations for the coefficients (infinitesimals) of its *infinitesimal generator*.
- A point symmetry of an ODE leads to reducing systematically the order of an ODE (irrespective of any imposed initial conditions).
- A point symmetry of a PDE leads to finding systematically special solutions called *invariant* (*similarity*) *solutions*.
- A point symmetry of a differential equation generates a one-parameter family of solutions from any known solution of the differential equation that is not an invariant solution.

However there were limitations to the applicability of Lie's work.

- There were a restricted number of applications for point symmetries, especially for PDE systems.
- Few differential equations have point symmetries.
- For PDE systems having point symmetries, the invariant solutions arising from point symmetries normally yield only a small submanifold of the solution manifold of the PDE system and hence few posed boundary value problems can be solved.
- There was the computational difficulty of finding point symmetries.

Since the end of the 19th century there have been significant extensions of Lie's work on symmetries of PDEs to extend its range of applicability.

- Further applications of point symmetries have been found to include linearizations, other mappings and solutions of boundary value problems. In particular, knowledge of the point symmetries of a nonlinear PDE system (contact symmetries in the case of a scalar PDE), allows one to determine whether the system can be mapped invertibly to a linear system and yields a procedure to find such a mapping when one exists [2–4]. Knowledge of the point symmetries of a linear PDE system with variable coefficients allows one to determine whether the system can be mapped invertibly to a linear system with constant coefficients and yields a procedure to find such a mapping when one exists [2, 3].
- Extensions of the spaces of symmetries of a given PDE system to include local symmetries (higher-order symmetries) as well as nonlocal symmetries [2, 5–8].
- Extension of the applications of symmetries to include variational symmetries that yield conservation laws for variational systems [2, 8].

- Extension of variational symmetries to more general multipliers and resulting conservation laws for essentially any given PDE system [2, 8–11].
- The discovery of further solutions that arise from the extension of Lie's method to the "nonclassical method" as well as other generalizations [2, 12, 13].
- The development of symbolic computation software to solve efficiently the (overdetermined) linear system of symmetry and/or multiplier determining equations as well as related calculations for solving the nonlinear systems of determining equations arising when one uses the nonclassical method [14–18].

1.1 What is a Symmetry of a PDE System and How to Find One?

A symmetry (discrete or continuous) of a PDE system is any transformation of its solution manifold into itself, i.e., a symmetry transforms (maps) any solution of a PDE system to a solution of the same system. In particular, continuous symmetries of a PDE system are continuous deformations of its solutions to solutions of the same PDE system. Hence continuous symmetries are defined topologically and not restricted to just point or local symmetries. Thus, in principle, any nontrivial PDE has symmetries. The problem is to find and use the symmetries of a given PDE system. Practically, to find symmetries of a given PDE system, one considers transformations, acting locally on the variables of some finite-dimensional space, which leave invariant the solution manifold of the PDE system and its differential consequences. However, these variables do not have to be restricted to just the independent and dependent variables of the given PDE system.

Higher-order symmetries (*local symmetries*) arise when the solutions of the linear determining equations for infinitesimals are allowed to depend on a finite number of derivatives of dependent variables of the PDE system.

- Infinitesimals for a point symmetry in evolutionary form allow at most linear dependence on first derivatives of dependent variables of a PDE system.
- Infinitesimals for a contact symmetry in evolutionary form (only exists for a scalar PDE) allow arbitrary dependence on at most first derivatives of the dependent variable of a scalar PDE.

In making the extension from point and contact symmetries to higher-order symmetries, it is essential to realize that the linear determining equations for local symmetries are the linearized system (*Fréchet derivative*) of the given PDE system that holds for all of its solutions. Globally, point and contact symmetries act on finite-dimensional spaces whereas higher-order symmetries act on infinite-dimensional spaces consisting of the dependent and independent variables of a given PDE system as well as all of their derivatives. Well-known integrable equations of mathematical physics such as the Korteweg-de-Vries equation have an infinite number of higher-order symmetries [19].

Another extension is to consider solutions of the determining equations where infinitesimals have an ad-hoc dependence on nonlocal variables such as integrals of the dependent variables [20–23]. For some PDEs, such *nonlocal symmetries* can be found formally through *recursion operators* that depend on inverse differentiation. Integrable equations such as the sine-Gordon and cubic Schrödinger equations have an infinite number of such nonlocal symmetries.

1.2 Conservation Laws

In her celebrated 1918 paper [5], Emmy Noether showed that if a DE system admits a variational principle, then any local transformation group leaving invariant the action integral for its Lagrangian density, i.e., a *variational symmetry*, yields a *local conservation law*. Conversely, any local CL of a variational DE system arises from a variational symmetry, and hence there is a direct correspondence between local CLs and variational symmetries (Noether's theorem).

However there are limitations in the use of Noether's theorem.

- Its application is restricted to variational systems. In particular, a given DE system, *as written*, is variational if and only if its linearized system is self-adjoint.
- One has the difficulty of finding local symmetries of the action integral. In general, not all local symmetries of a variational DE system are variational symmetries.
- The use of Noether's theorem to find local conservation laws is coordinate-dependent.

The *Direct Method* for finding CLs allows one to find local CLs more generally for a given DE system. A CL of a given DE system is a divergence expression that vanishes on all solutions of the DE system. Local CLs arise from scalar products formed by linear combinations of *local CL multipliers* (factors that are functions of independent and dependent variables and their derivatives) multiplying each DE in the system. This scalar product is annihilated by the Euler operators associated with each of its dependent variables without restricting these variables in the scalar product to solutions of the system of DEs, i.e., the dependent variables are replaced by arbitrary functions of the independent variables.

If a given DE system, *as written*, is variational, then local CL multipliers correspond to variational symmetries. In the variational situation, using the Direct Method, local CL multipliers satisfy a linear system of determining equations that includes the linearizing system of the given DE system augmented by additional determining equations that taken together correspond to the action integral being invariant under the associated variational symmetry.

More generally, in using the Direct Method for any given DE system, the local CL multipliers are the solutions of an easily found linear determining system that includes the adjoint system of the linearizing DE system [1, 2, 9–11].

For any set of local CL multipliers, usually one can directly find the fluxes and density of the corresponding local CL and, if this proves difficult, there is an integral formula that yields them without the need of a specific functional (Lagrangian) even in the case when the given DE system is variational [9–11].

One can compare the number of local symmetries and the number of local CLs of a given DE system. When a DE system is variational, i.e., its linearized system is self-adjoint, then local CLs arise from a subset of its local symmetries and the number of linearly independent local CLs cannot exceed the number of higher-order symmetries. In general, this will not be the case when a system is not variational. Here a given DE system can have more local conservation laws than local symmetries as well as vice versa.

For any given DE system, a transformation group (continuous or discrete) that leaves it invariant yields an explicit formula that maps a CL to a CL of the same system, whether or not the given system is variational. If the transformation group is a one-parameter Lie group of point (or contact) transformations, then in terms of a parameter expansion a given CL can map into more than one additional CL for the given DE system [2, 24].

1.3 Nonlocally Related Systems and Nonlocal Symmetries

Systematic procedures have been found to seek nonlocal symmetries of a given PDE system through applying Lie's algorithm to nonlocally related systems. In particular, to apply symmetry methods to PDE systems, one needs to work in some specific coordinate frame in order to perform calculations. A procedure to find symmetries that are nonlocal and yet are local in some related coordinate frame involves embedding a given PDE system in another PDE system obtained by adjoining nonlocal variables in such a way that the resulting nonlocally related PDE system is equivalent to the given system. Consequently, any local symmetry of the nonlocally related system yields a symmetry of the given system (The converse also holds). A local symmetry of the nonlocally related system, with the corresponding infinitesimals for the variables of the given PDE system having an essential dependence on nonlocal variables, yields a nonlocal symmetry of the given PDE system.

There are two known systematic ways to find such an embedding.

- Each local CL of a given PDE system yields a nonlocally related system. For each local CL, one can introduce a potential variable(s). Here the nonlocally related system is the given PDE system augmented by a corresponding potential system [2, 25–27].
- Each point symmetry of a given PDE system yields a nonlocally related system. Here, as a first step, the given PDE system naturally yields a locally related PDE system (intermediate system) arising from the canonical coordinates of the point symmetry. In turn, the intermediate system has a natural CL which yields a nonlocally related system (inverse potential system) for the given PDE system [28, 29]. The intermediate system plays the role of a potential system for the inverse potential system.

If a local symmetry of such a nonlocally related system has an essential dependence on nonlocal variables when projected to the given system, then it yields a nonlocal symmetry of the given PDE system. It turns out that many PDE systems have such systematically constructed nonlocal symmetries. Furthermore, one can find additional nonlocal symmetries of a given PDE system through seeking local symmetries of an equivalent subsystem of the given system or one of its constructed nonlocally related systems provided that such a subsystem is nonlocally related to the given PDE system.

There are many applications of nonlocally related systems.

- Invariant solutions of nonlocally related systems (arising from CLs or point symmetries) can yield further solutions of a given PDE system.
- Since a point symmetry-based or CL-based nonlocal symmetry is a local symmetry of a constructed nonlocally related system, it generates a one-parameter family of solutions from any known solution (that is not an invariant solution) of such a nonlocally related system. In turn, this yields a one-parameter family of solutions from any known solution of the given PDE system.
- Local CLs of such nonlocally related systems can yield nonlocal CLs of a given PDE system if their local CL multipliers have an essential dependence on nonlocal variables.

Still wider classes of nonlocally related systems can be constructed systematically for a given PDE system. One can further extend embeddings through the effective use of local CLs to systematically construct trees of nonlocally related but equivalent PDE systems. If a given PDE system has n local CLs, then each CL yields potentials and corresponding potential systems. From the n local CLs, one can directly construct up to $2^n - 1$ independent nonlocally related systems of PDEs by considering corresponding potential systems individually (n singlets), in pairs ($n(n-1)/2$ couplets), ..., taken all together (one n-plet). Any of these systems could lead to the discovery of new nonlocal symmetries and/or nonlocal CLs of the given PDE system or any of the other nonlocally related systems. Such nonlocal CLs could yield further nonlocally related systems, etc. Furthermore, subsystems of such nonlocally related systems could yield further nonlocally related systems. Correspondingly, a tree of nonlocally related, and equivalent, systems is constructed for a given PDE system [2, 30, 31].

The situation in the case of multidimensional PDE systems (i.e., with at least three independent variables) is especially interesting. Here one can show that nonlocal symmetries and nonlocal CLs arising from the CL-based approach cannot arise from potential systems unless they are augmented by gauge constraints [2, 32, 33].

There exist many applications of such systematically constructed nonlocally related systems that further extend the use of symmetry methods for PDE systems.

- Through such constructions, one can systematically relate Eulerian and Lagrangian coordinate descriptions of gas dynamics and nonlinear elasticity. In particular, for the Eulerian coordinate description, a subsystem of the potential system arising from conservation of mass, naturally yields the corresponding description in Lagrangian coordinates [2, 30, 31, 34, 35].
- For a given class of PDEs with constitutive functions, one finds trees of nonlocally related systems yielding symmetries and CLs with respect to various forms of its constitutive functions.

- One can systematically seek noninvertible mappings of nonlinear PDE systems to linear PDE systems. Consequently, further nonlinear PDE systems can be mapped into equivalent linear PDE systems beyond those obtained through invertible mappings [2, 27, 36].
- One can systematically extend the class of linear PDE systems with variable coefficients that can be mapped into equivalent linear PDE systems with constant coefficients through inclusion of noninvertible mappings [2, 37, 38].

The rest of this chapter is organized as follows. In Sect. 2, we review local symmetries, Lie's algorithm to find local symmetries in evolutionary form, applications of local symmetries and as examples consider the heat equation and the Kortweg-de Vries equation. In Sect. 3, we consider the construction of conservation laws, introduce the Direct Method and its relationship to Noether's theorem, and show how symmetries could yield additional CLs from known CLs. As examples, we consider nonlinear telegraph equations, the Korteweg-de Vries equation, the Klein-Gordon equation, and nonlinear wave equations. In Sect. 4, we present systematic procedures to seek nonlocally related systems and nonlocal symmetries of a given PDE system with two independent variables. We introduce conservation law and point symmetry based methods as well as the use of subsystems to obtain trees of equivalent nonlocally related PDE systems. As examples, we focus on nonlinear wave equations, nonlinear telegraph equations, planar gas dynamics equations, and nonlinear reaction diffusion equations. In Sect. 5, we consider the situation of nonlocality in multidimensions. We show that if one directly applies the CL-based method to a single CL, then it is necessary to append a gauge constraint relating potential variables of the resulting vector potential system when seeking nonlocal symmetries. Some open problems are discussed.

2 Local Symmetries

Lie's algorithm for seeking point symmetries can be extended to seek more general local symmetries admitted by PDE systems. In the extension of Lie's algorithm, one uses differential consequences of the given PDE system, i.e., invariance of a given PDE system is understood to include its differential consequences. Here it is important to consider the infinitesimal generators for point symmetries in their *evolutionary form* where the independent variables are themselves invariant and the action of a group of point transformations is strictly an action on the dependent variables of the PDE system, so that solutions are *directly mapped into other solutions* under the group action. This allows one to readily extend Lie's algorithm to seek *contact symmetries* (only existing for scalar PDEs) where now the components of infinitesimal generators for dependent variables can depend at most on the first derivatives of the dependent variable of a given scalar PDE (if this dependence is at most linear on the first derivatives, then a contact symmetry is a point symmetry).

A contact symmetry is equivalent to a point transformation acting on the space of the given independent variables, the dependent variable and its first derivatives and, through this, can be naturally extended to a point transformation acting on the space of the given independent variables, the dependent variable and its derivatives to any finite order greater than one.

Lie's algorithm can be still further extended by allowing the infinitesimal generators in evolutionary form to depend on derivatives of dependent variables to any finite order. This allows one to calculate symmetries that are called *higher-order symmetries*. In the scalar case, contact symmetries are first-order symmetries. Otherwise, higher-order symmetries are not equivalent to point transformations acting on a finite-dimensional space including the independent variables, the dependent variables and their derivatives to some finite order. Higher-order symmetries are local symmetries in the sense that the components of the dependent variables in their infinitesimal generators depend at most on a finite number of derivatives of a given PDE system's dependent variables so that their calculation only depends on the local behaviour of solutions of a given PDE system.

Local symmetries include point symmetries, contact symmetries and higher-order symmetries. Local symmetries are uniquely determined when infinitesimal generators are represented in evolutionary form.

Sophus Lie considered contact symmetries. Emmy Noether introduced the notion of higher-order symmetries in her celebrated paper on conservation laws [5]. The well-known infinite sequences of conservation laws of the Korteweg-de Vries (KdV) and sine-Gordon equations are directly related to admitted infinite sequences of local symmetries obtained through the use of recursion operators [19].

Consider a given scalar PDE of order k

$$R(x, t, u, \partial u, \ldots, \partial^k u) = 0 \tag{1}$$

with independent variables (x, t) and dependent variable $u(x, t)$; $\partial^j u$ denotes the jth order partial derivatives of $u(x, t)$ appearing in the PDE (1). In *evolutionary form*, the *local symmetries of order* p of a PDE (1), in terms of their infinitesimal generators

$$\eta(x, t, u, \partial u, \ldots, \partial^p u)\frac{\partial}{\partial u}$$

are the solutions $\eta(x, t, u, \partial u, \ldots, \partial^p u)$ of its linearized system (*Fréchet derivative*)

$$\left[\frac{\partial R}{\partial u}\eta + \frac{\partial R}{\partial u_x}\mathrm{D}_x\eta + \frac{\partial R}{\partial u_t}\mathrm{D}_t\eta + \frac{\partial^2 R}{\partial {u_x}^2}(\mathrm{D}_x)^2\eta + \cdots\right]_{\substack{R=0,\\ \mathrm{D}_x R = 0,\\ \mathrm{D}_t R = 0,\\ \vdots}} = 0$$

in terms of *total derivative operators*

$$\mathrm{D}_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + \cdots$$
$$\mathrm{D}_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{xt} \frac{\partial}{\partial u_x} + \cdots$$

and holding for all solutions $u = \theta(x, t)$ of the PDE (1) and its differential consequences.

A local symmetry of order p, $\eta(x, t, u, \partial u, \ldots, \partial^p u)\frac{\partial}{\partial u}$ (including its natural extension to action on derivatives), maps *any* solution $u = \theta(x, t)$ of PDE (1) (that is not an invariant solution of PDE (1)) into a one-parameter (ε) family of solutions of PDE (1) given by the expression

$$u = \left(\mathrm{e}^{\varepsilon\left(\eta \frac{\partial}{\partial u} + (\mathrm{D}_x \eta)\frac{\partial}{\partial u_x} + (\mathrm{D}_t \eta)\frac{\partial}{\partial u_t} + \cdots\right)} u \right)\Bigg|_{u=\theta(x,t)}$$

and is equivalent to the transformation

$$\begin{aligned} x^* &= x \\ t^* &= t \\ u^* &= \mathrm{e}^{\varepsilon\left(\eta \frac{\partial}{\partial u} + (\mathrm{D}_x \eta)\frac{\partial}{\partial u_x} + (\mathrm{D}_t \eta)\frac{\partial}{\partial u_t} + \cdots\right)} u \\ &= u + \varepsilon \eta(x, t, u, \partial u, \ldots, \partial^p u) + O(\varepsilon^2). \end{aligned}$$

If $p = 1$, then the first order symmetry is equivalent to the *contact symmetry*

$$\begin{aligned} x^* &= x + \varepsilon \frac{\partial \eta}{\partial u_x} + O(\varepsilon^2) \\ t^* &= t + \varepsilon \frac{\partial \eta}{\partial u_t} + O(\varepsilon^2) \\ u^* &= u + \varepsilon \left(u_x \frac{\partial \eta}{\partial u_x} + u_t \frac{\partial \eta}{\partial u_t} - \eta \right) + O(\varepsilon^2) \\ u_x^* &= u_x + \varepsilon \left(-u_x \frac{\partial \eta}{\partial u} - \frac{\partial \eta}{\partial x} \right) + O(\varepsilon^2) \\ u_t^* &= u_t + \varepsilon \left(-u_t \frac{\partial \eta}{\partial u} - \frac{\partial \eta}{\partial t} \right) + O(\varepsilon^2). \end{aligned}$$

If a first order symmetry has an infinitesimal of the form

$$\eta(x, t, u, \partial u) = \xi(x, t, u)u_x + \tau(x, t, u)u_t - \omega(x, t, u)$$

then it is equivalent to the *point symmetry*

$$x^* = x + \varepsilon\xi(x,t,u) + O(\varepsilon^2)$$
$$t^* = t + \varepsilon\tau(x,t,u) + O(\varepsilon^2)$$
$$u^* = u + \varepsilon\omega(x,t,u) + O(\varepsilon^2).$$

2.1 Example 1: The Heat Equation

The heat equation

$$R = u_t - u_{xx} = 0$$

has the point symmetries [12, 13]

$$\mathrm{X}_1 = u_x \frac{\partial}{\partial u}, \qquad \mathrm{X}_2 = u_t \frac{\partial}{\partial u}, \qquad \mathrm{X}_3 = (xu_x + 2tu_t)\frac{\partial}{\partial u}$$
$$\mathrm{X}_4 = (xtu_x + t^2u_t + [\tfrac{1}{4}x^2 + \tfrac{1}{2}t]u)\frac{\partial}{\partial u}$$
$$\mathrm{X}_5 = (tu_x + \tfrac{1}{2}xu)\frac{\partial}{\partial u}, \quad \mathrm{X}_6 = u\frac{\partial}{\partial u}.$$

2.2 Example 2: The Korteweg-de Vries Equation

The Korteweg-de Vries (KdV) equation

$$R = u_t + uu_x + u_{xxx} = 0$$

has an infinite sequence of higher-order symmetries given by

$$(\mathbf{R}^n)u_x, \qquad n = 0, 1, 2, \ldots$$

in terms of the recursion operator [19]

$$\mathbf{R} = (\mathrm{D}_x)^2 + \tfrac{2}{3}u + \tfrac{1}{3}u_x(\mathrm{D}_x)^{-1}.$$

Specifically, one obtains corresponding nonlocal symmetries

$$u_x\frac{\partial}{\partial u}, \quad (uu_x + u_{xxx})\frac{\partial}{\partial u}$$
$$(\tfrac{5}{6}u^2u_x + 4u_xu_{xx} + \tfrac{5}{3}uu_{xxx} + u_{xxxxx})\frac{\partial}{\partial u}, \ldots .$$

For a given PDE system, local symmetries can be used to determine

- specific invariant solutions.
- a one-parameter family of solutions from “any” known solution.
- whether it can be linearized by an invertible transformation and find the linearization when it exists [3, 4, 21].
- whether an inverse scattering transform exists.
- whether a given linear PDE with variable coefficients can be invertibly mapped into a linear PDE with constant coefficients and find such a mapping when it exists [39, 40].

3 Construction of Conservation Laws

In this section, we consider the problem of finding the *local conservation laws* for a given PDE system. In particular, we present the Direct Method for the construction of CLs. In the Direct Method one first derives the determining equations yielding the multipliers (*local CL multipliers*). Following this, one finds the fluxes and densities of corresponding local CLs. It is shown that a subset of the determining equations for local CL multipliers includes the adjoint equations of the determining equations yielding the local symmetries (in evolutionary form) of a given PDE system. The self-adjoint case is especially interesting since here the given PDE system is variational and thus the local CL multipliers are also local symmetries (the converse is false) of the given PDE system. A comparison is made with the classical Noether theorem. Further connections between symmetries and CLs are presented. In particular, it is shown how a symmetry of a PDE system maps a known CL to a CL of the same PDE system. In the case of a local symmetry it is shown that a parameter expansion could yield more than one new CL from a known CL.

3.1 Uses of Conservation Laws

Conservation laws can yield constants of motion for any posed boundary value problem for a given PDE system. For this reason, for global convergence of an approximation scheme, it is important to preserve CLs, at least those CLs considered to be of importance for a particular posed boundary value problem.

From knowledge of the local CL multipliers for a given nonlinear PDE system, one can determine whether it can be mapped invertibly to a linear PDE system and set up the equations to find such a mapping when one exists [2].

In Sect. 4, it will be shown how one can use local CLs to find nonlocally related systems for a given PDE system. In turn, invariant solutions arising from local symmetries of such a nonlocally related system could yield further solutions of the given PDE system beyond those obtained as invariant solutions arising from local symmetry reductions. Moreover, the computation of local CLs of a nonlocally related system could yield nonlocal CLs of a given PDE system and to noninvertible linearizations of nonlinear PDE systems.

3.2 Direct Method for Construction of Conservation Laws

Consider a given system $\mathbf{R}\{x; u\}$ of N PDEs of order k with n independent variables $x = (x^1, \ldots, x^n)$ and m dependent variables $u(x) = (u^1(x), \ldots, u^m(x))$

$$R^\sigma[u] = R^\sigma(x, u, \partial u, \ldots, \partial^k u) = 0, \quad \sigma = 1, \ldots, N. \tag{2}$$

A *local conservation law* of the PDE system (2) is an expression

$$\mathrm{D}_i \Phi^i[u] = \mathrm{D}_1 \Phi^1[u] + \cdots + \mathrm{D}_n \Phi^n[u] = 0 \tag{3}$$

holding for any solution of the PDE system (2). In (3), the operators D_i, $i = 1, \ldots, n$ are total derivative operators.

Definition 1 A PDE system $\mathbf{R}\{x; u\}$ (2) is *totally non-degenerate* if (2) and its differential consequences have maximal rank and are locally solvable.

The proof of the following theorem appears in [11].

Theorem 1 *Suppose* $\mathbf{R}\{x; u\}$ *(2) is a totally non-degenerate PDE system. Then for every nontrivial local conservation law*

$$\mathrm{D}_i \Phi^i[u] = \mathrm{D}_i \Phi^i(x, u, \partial u, \ldots, \partial^r u) = 0$$

of (2), there exists a set of multipliers, called local conservation law multipliers*,*

$$\Lambda_\sigma[U] = \Lambda_\sigma(x, U, \partial U, \ldots, \partial^l U), \qquad \sigma = 1, \ldots, N$$

such that

$$\mathrm{D}_i \Phi^i[U] \equiv \Lambda_\sigma[U] R^\sigma[U]$$

holds for arbitrary $U(x)$.

Definition 2 The *Euler operator* with respect to U^j is the operator

$$\mathrm{E}_{U^j} = \frac{\partial}{\partial U^j} - \mathrm{D}_i \frac{\partial}{\partial U_i^j} + \cdots + (-1)^s \mathrm{D}_{i_1} \cdots \mathrm{D}_{i_s} \frac{\partial}{\partial U_{i_1 \cdots i_s}^j} + \cdots .$$

The proofs of the following two theorems follow from direct computations.

Theorem 2 *For* any divergence expression $\mathrm{D}_i \Phi^i[U]$*, one has*

$$\mathrm{E}_{U^j}(\mathrm{D}_i\Phi^i[U]) \equiv 0, \qquad j = 1, \ldots, m.$$

Theorem 3 *Let* $F[U] = F(x, U, \partial U, \ldots, \partial^s U)$. *Then*

$$\mathrm{E}_{U^j} F[U] \equiv 0, \qquad j = 1, \ldots, m$$

holds for arbitrary $U(x)$ *if and only if*

$$F[U] \equiv \mathrm{D}_i \Psi^i(x, U, \partial U, \ldots, \partial^{s-1} U)$$

for some set of functions $\{\Psi^i(x, U, \partial U, \ldots, \partial^{s-1} U)\}$.

The next theorem follows directly from Theorems 2 and 3.

Theorem 4 *A set of local multipliers* $\{\Lambda_\sigma(x, U, \partial U, \ldots, \partial^l U)\}$ *yields a divergence expression for PDE system (2) if and only if*

$$\mathrm{E}_{U^j}(\Lambda_\sigma(x, U, \partial U, \ldots, \partial^l U) R^\sigma(x, U, \partial U, \ldots, \partial^k U)) \equiv 0, \qquad j = 1, \ldots, m \tag{4}$$

holds for arbitrary $U(x)$.

3.2.1 Summary of Direct Method to Find Local CLs

The Direct Method to find local CLs for a given PDE system (2) can be summarized as follows. Further details can be found in [2, 10, 11].

1. Seek multipliers of the form $\Lambda_\sigma[U] = \Lambda_\sigma(x, U, \partial U, \ldots, \partial^l U)$ with derivatives $\partial^l U$ to some specified order l.
2. Obtain and solve the determining Eq. (4) to find the multipliers of local conservation laws.
3. For each set of multipliers, find the corresponding fluxes $\Phi^i[U] = \Phi^i(x, U, \partial U, \ldots, \partial^r U)$ satisfying the identity

$$\Lambda_\sigma[U] R^\sigma[U] \equiv \mathrm{D}_i \Phi^i[U]. \tag{5}$$

4. Consequently, one obtains the local CL

$$\mathrm{D}_i \Phi^i[u] = \mathrm{D}_i \Phi^i(x, u, \partial u, \ldots, \partial^r u) = 0$$

   with fluxes $\Phi^i[u]$ holding for any solution of the PDE system (2).

The fluxes $\Phi^i[U] = \Phi^i(x, U, \partial U, \ldots, \partial^r U)$ in (5) can be found in the following ways:

- Directly manipulate the left-hand side of (5) to obtain the right-hand side divergence form.

- Treat the fluxes as unknowns in expression (5). Expand the right-hand side to set up a linear set of PDEs for the fluxes. Solve this linear set of PDEs.
- If one is unable to perform either of the first two ways successfully, then one can formally obtain the fluxes through use of an integral (homotopy) formula that appears in [11].

Example 1 Nonlinear Telegraph System

Consider the nonlinear telegraph system

$$\begin{aligned} R_1[u,v] &= v_t - (u^2+1)u_x - u = 0 \\ R_2[u,v] &= u_t - v_x = 0. \end{aligned} \tag{6}$$

We seek local CL multipliers of the form

$$\Lambda_1 = \xi[U,V] = \xi(x,t,U,V), \qquad \Lambda_2 = \varphi[U,V] = \varphi(x,t,U,V) \tag{7}$$

for the nonlinear telegraph system (6). In terms of the Euler operators

$$\mathrm{E}_U = \frac{\partial}{\partial U} - \mathrm{D}_x \frac{\partial}{\partial U_x} - \mathrm{D}_t \frac{\partial}{\partial U_t}, \quad \mathrm{E}_V = \frac{\partial}{\partial V} - \mathrm{D}_x \frac{\partial}{\partial V_x} - \mathrm{D}_t \frac{\partial}{\partial V_t}$$

the multipliers (7) yield a local CL of the nonlinear telegraph system (6) if and only if the determining equations

$$\begin{aligned} \mathrm{E}_U(\xi[U,V]R_1[U,V] + \varphi[U,V]R_2[U,V]) &\equiv 0 \\ \mathrm{E}_V(\xi[U,V]R_1[U,V] + \varphi[U,V]R_2[U,V]) &\equiv 0 \end{aligned} \tag{8}$$

hold for arbitrary differentiable functions $U(x,t)$, $V(x,t)$. It is straightforward to show that the Eq. (8) hold if and only if

$$\begin{aligned} \varphi_V - \xi_U &= 0 \\ \varphi_U - (U^2+1)\xi_V &= 0 \\ \varphi_x - \xi_t - U\xi_V &= 0 \\ (U^2+1)\xi_x - \varphi_t - U\xi_U - \xi &= 0. \end{aligned} \tag{9}$$

The five linearly independent solutions [41] of the linear determining system (9) are given by

$$\begin{aligned} &(\xi_1,\varphi_1) = (0,1), \qquad (\xi_2,\varphi_2) = (t, x - \tfrac{1}{2}t^2), \qquad (\xi_3,\varphi_3) = (1,-t) \\ &(\xi_4,\varphi_4) = (\mathrm{e}^{x+\frac{1}{2}U^2+V}, U\mathrm{e}^{x+\frac{1}{2}U^2+V}), \quad (\xi_5,\varphi_5) = (\mathrm{e}^{x+\frac{1}{2}U^2-V}, -U\mathrm{e}^{x+\frac{1}{2}U^2-V}). \end{aligned}$$

Correspondingly, through manipulation, one obtains the following five local conservation laws [41]

$$\begin{aligned}
\mathrm{D}_t u + \mathrm{D}_x[-v] &= 0\\
\mathrm{D}_t[(x - \tfrac{1}{2}t^2)u + tv] + \mathrm{D}_x[(\tfrac{1}{2}t^2 - x)v - t(\tfrac{1}{3}u^3 + u)] &= 0\\
\mathrm{D}_t[v - tu] + \mathrm{D}_x[tv - (\tfrac{1}{3}u^3 + u)] &= 0\\
\mathrm{D}_t[\mathrm{e}^{x+\frac{1}{2}u^2+v}] + \mathrm{D}_x[-u\mathrm{e}^{x+\frac{1}{2}u^2+v}] &= 0\\
\mathrm{D}_t[\mathrm{e}^{x+\frac{1}{2}u^2-v}] + \mathrm{D}_x[u\mathrm{e}^{x+\frac{1}{2}u^2-v}] &= 0.
\end{aligned}$$

Example 2 KdV Equation

As a second example, consider again the KdV equation [10]

$$R[u] = u_t + uu_x + u_{xxx} = 0. \tag{10}$$

It is convenient to also write (10) as

$$u_t = g[u] = -(uu_x + u_{xxx}). \tag{11}$$

Due to the evolutionary form of the KdV equation (10), it follows that all local CL multipliers are of the form $\Lambda[U] = \Lambda(t, x, U, \partial_x U, \ldots, \partial_x^l U)$, $l = 1, 2, \ldots$. Then $\mathrm{E}_U(\Lambda[U](U_t + UU_x + U_{xxx})) \equiv 0$ if and only if

$$\begin{aligned}
&-\mathrm{D}_t\Lambda - U\mathrm{D}_x\Lambda - \mathrm{D}_x^3\Lambda + (U_t + UU_x + U_{xxx})\Lambda_U\\
&-\mathrm{D}_x((U_t + UU_x + U_{xxx})\Lambda_{\partial_x U}) + \cdots\\
&+(-1)^l\mathrm{D}_x^l((U_t + UU_x + U_{xxx})\Lambda_{\partial_x^l U}) \equiv 0.
\end{aligned} \tag{12}$$

Note that the linear determining Eq. (12) is of the form

$$\alpha_1 + \alpha_2 U_t + \alpha_3\partial_x U_t + \cdots + \alpha_{l+2}\partial_x^l U_t \equiv 0 \tag{13}$$

where in Eq. (13) each coefficient α_i depends at most on t, x, U and x-derivatives of U. Since $U(x, t)$ is an arbitrary function in Eq. (13), it follows that each of the terms $U_t, \partial_x U_t, \ldots, \partial_x^l U_t$ must be treated as independent variables in (13). Hence $\alpha_i = 0$, $i = 1, \ldots, l + 2$. Thus Eq. (13) splits into an overdetermined linear system of $l + 2$ determining equations for the local multipliers $\Lambda(t, x, U, \partial_x U, \ldots, \partial_x^l U)$, given by

$$\tilde{\mathrm{D}}_t\Lambda + U\mathrm{D}_x\Lambda + \mathrm{D}_x^3\Lambda = 0 \tag{14}$$

$$\sum_{k=1}^{l}(-\mathrm{D}_x)^k \Lambda_{\partial_x^k U} = 0 \tag{15}$$

$$(1-(-1)^q)\Lambda_{\partial_x^q U} + \sum_{k=q+1}^{l}\frac{k!}{q!(k-q)!}(-\mathrm{D}_x)^{k-q}\Lambda_{\partial_x^k U} = 0, \quad q = 1, \dots, l-1 \tag{16}$$

$$(1-(-1)^l)\Lambda_{\partial_x^l U} = 0 \tag{17}$$

where $\tilde{\mathrm{D}}_t = \frac{\partial}{\partial t} + g[U]\frac{\partial}{\partial U} + (g[U])_x\frac{\partial}{\partial U_x} + \cdots$ is the total derivative operator restricted to the KdV equation, with $g[U] = -(UU_x + U_{xxx})$.

Now we seek local CL multipliers of the form $\Lambda[U] = \Lambda(x, t, U)$. Then the determining Eqs. (15)–(17) are satisfied and the determining Eq. (14) becomes

$$\begin{aligned}(\Lambda_t + U\Lambda_x + \Lambda_{xxx}) + 3\Lambda_{xxU}U_x + 3\Lambda_{xUU}U_x^2 \\ + \Lambda_{UUU}U_x^3 + 3\Lambda_{xU}U_{xx} + 3\Lambda_{UU}U_xU_{xx} = 0.\end{aligned} \tag{18}$$

Equation (18) holds for arbitrary values of x, t, U, U_x and U_{xx}. Hence Eq. (18) splits into six equations. Their solution yields the three local CL multipliers $\Lambda_1 = 1$, $\Lambda_2 = U$, $\Lambda_3 = tU - x$. In turn, after simple manipulations, these three multipliers yield the divergence expressions

$$\begin{aligned}U_t + UU_x + U_{xxx} &\equiv \mathrm{D}_t U + \mathrm{D}_x(\tfrac{1}{2}U^2 + U_{xx})\\ U(U_t + UU_x + U_{xxx}) &\equiv \mathrm{D}_t(\tfrac{1}{2}U^2) + \mathrm{D}_x(\tfrac{1}{3}U^3 + UU_{xx} - \tfrac{1}{2}U_x^2)\\ (tU - x)(U_t + UU_x + U_{xxx}) &\equiv \mathrm{D}_t(\tfrac{1}{2}tU^2 - xU)\\ &\quad + \mathrm{D}_x(-\tfrac{1}{2}xU^2 + tUU_{xx} - \tfrac{1}{2}tU_x^2 - xU_{xx} + U_x).\end{aligned}$$

Thus the corresponding local conservation laws for the KdV Eq. (10) are given by

$$\begin{aligned}\mathrm{D}_t u + \mathrm{D}_x(\tfrac{1}{2}u^2 + u_{xx}) = 0\\ \mathrm{D}_t(\tfrac{1}{2}u^2) + \mathrm{D}_x(\tfrac{1}{3}u^3 + uu_{xx} - \tfrac{1}{2}u_x^2) = 0\\ \mathrm{D}_t(\tfrac{1}{2}tu^2 - xu) + \mathrm{D}_x(-\tfrac{1}{2}xu^2 + tuu_{xx} - \tfrac{1}{2}tu_x^2 - xu_{xx} + u_x) = 0.\end{aligned}$$

One can show that there is only one additional local CL multiplier of the form $\Lambda[U] = \Lambda(x, t, U, U_x, U_{xx})$, given by

$$\Lambda_4 = U_{xx} + \tfrac{1}{2}U^2.$$

Moreover, one can show that in terms of the recursion operator

$$\mathbf{R}^*[U] = \mathrm{D}_x^2 + \tfrac{1}{3}U + \tfrac{1}{3}\mathrm{D}_x^{-1} \circ U \circ \mathrm{D}_x$$

the KdV equation has an infinite sequence of local CL multipliers given by

$$\Lambda_{2n} = (\mathbf{R}^*[U])^n U, \qquad n = 1, 2, \ldots .$$

General Expression Relating Local CL Multipliers and Solutions of Adjoint Equations.

Consider a given PDE system (2). Let $R^\sigma[U] = R^\sigma(x, U, \partial U, \ldots, \partial^k U)$, $\sigma = 1, \ldots, N$, where $U(x) = (U^1(x), \ldots, U^m(x))$ is *arbitrary* and $U(x) = u(x)$ solves the PDE system (2).

In terms of m *arbitrary* functions $V(x) = (V^1(x), \ldots, V^m(x))$, the *linearizing operator* $\mathrm{L}[U]$ associated with the PDE system (2) is given by

$$\mathrm{L}^\sigma_\rho[U]V^\rho \equiv \left[\frac{\partial R^\sigma[U]}{\partial U^\rho} + \frac{\partial R^\sigma[U]}{\partial U^\rho_i}\mathrm{D}_i + \cdots + \frac{\partial R^\sigma[U]}{\partial U^\rho_{i_1\ldots i_k}}\mathrm{D}_{i_1}\cdots \mathrm{D}_{i_k}\right]V^\rho,$$
$$\sigma = 1, \ldots, N$$

and, in terms of N *arbitrary* functions $W(x) = (W_1(x), \ldots, W_N(x))$, the *adjoint operator* $\mathrm{L}^*[U]$ associated with the PDE system (2) is given by

$$\mathrm{L}^{*\sigma}_{\ \rho}[U]W_\sigma \equiv \frac{\partial R^\sigma[U]}{\partial U^\rho}W_\sigma - \mathrm{D}_i\left(\frac{\partial R^\sigma[U]}{\partial U^\rho_i}W_\sigma\right) + \cdots$$
$$+ (-1)^k \mathrm{D}_{i_1}\cdots \mathrm{D}_{i_k}\left(\frac{\partial R^\sigma[U]}{\partial U^\rho_{i_1\cdots i_k}}W_\sigma\right), \qquad \rho = 1, \ldots, m.$$

In particular, $W_\sigma \mathrm{L}^\sigma_\rho[U]V^\rho - V^\rho \mathrm{L}^{*\sigma}_{\ \rho}[U]W_\sigma$ is a divergence expression.

Let

$$W_\sigma = \Lambda_\sigma[U] = \Lambda_\sigma(x, U, \partial U, \ldots, \partial^l U), \quad \sigma = 1, \ldots, N.$$

By direct calculation, in terms of Euler operators, one can show that

$$\mathrm{E}_{U^\rho}(\Lambda_\sigma[U]R^\sigma[U]) \equiv \mathrm{L}^{*\sigma}_{\ \rho}[U]\Lambda_\sigma[U] + F_\rho(R[U]) \tag{19}$$

with

$$F_\rho(R[U]) = \frac{\partial \Lambda_\sigma[U]}{\partial U^\rho}R^\sigma[U] - \mathrm{D}_i\left(\frac{\partial \Lambda_\sigma[U]}{\partial U^\rho_i}R^\sigma[U]\right) + \cdots$$
$$+ (-1)^l \mathrm{D}_{i_1}\cdots \mathrm{D}_{i_l}\left(\frac{\partial \Lambda_\sigma[U]}{\partial U^\rho_{i_1\cdots i_l}}R^\sigma[U]\right), \qquad \rho = 1, \ldots, m. \tag{20}$$

From (19), it follows that $\{\Lambda_\sigma(x, U, \partial U, \dots, \partial^l U)\}_{\sigma=1}^N$ yields a set of local CL multipliers for the PDE system (2) if and only if the right-hand side of (19) vanishes for arbitrary $U(x)$. Moreover, since the expressions (20) vanish on any solution $U(x) = u(x)$ of $\mathbf{R}\{x; u\}$ (2), it follows that every set of local CL multipliers $\{\Lambda_\sigma(x, U, \partial U, \dots, \partial^l U)\}_{\sigma=1}^N$ of the PDE system (2) must be a solution of its adjoint system of PDEs, which is the adjoint of its linearizing system of PDEs, when $U(x) = u(x)$ is a solution of $\mathbf{R}\{x; u\}$ (2), i.e.,

$$\mathrm{L}^{*\sigma}_{\rho}[u]\Lambda_\sigma[u] = 0, \qquad \rho = 1, \dots, m. \tag{21}$$

The proof of the following theorem follows directly from expression (19).

Theorem 5 *Consider a given PDE system (2). A set of functions* $\{\Lambda_\sigma(x, U, \partial U, \dots, \partial^l U)\}_{\sigma=1}^N$ *yields a set of local CL multipliers for PDE system (2) if and only if the identities*

$$\mathrm{L}^{*\sigma}_{\rho}[U]\Lambda_\sigma[U] + \frac{\partial \Lambda_\sigma[U]}{\partial U^\rho} R^\sigma[U] - \mathrm{D}_i\left(\frac{\partial \Lambda_\sigma[U]}{\partial U_i^\rho} R^\sigma[U]\right) + \cdots$$
$$+ (-1)^l \mathrm{D}_{i_1} \cdots \mathrm{D}_{i_l}\left(\frac{\partial \Lambda_\sigma[U]}{\partial U^\rho_{i_1 \cdots i_l}} R^\sigma[U]\right) \equiv 0, \qquad \rho = 1, \dots, m$$

hold for m arbitrary functions $U(x) = (U^1(x), \dots, U^m(x))$ *in terms of the components* $\{\mathrm{L}^{*\sigma}_{\rho}[U]\}$ *of the adjoint operator of the linearizing operator (Fréchet derivative) for the given PDE system (2).*

The derivation leading to Eq. (21) can be summarized in terms of the following theorem.

Theorem 6 *Consider a given PDE system (2). Suppose one has a set of local CL multipliers* $\{\Lambda_\sigma(x, U, \partial U, \dots, \partial^l U)\}_{\sigma=1}^N$ *for the PDE system (2). Let* $\{\mathrm{L}^{*\sigma}_{\rho}[U]\}$ *be the components of the adjoint operator of the linearizing operator (Fréchet derivative) for the PDE system (2) and let* $U(x) = u(x) = (u^1(x), \dots, u^m(x))$ *be any solution of the PDE system (2). Then* $\mathrm{L}^{*\sigma}_{\rho}[u]\Lambda_\sigma[u] = 0$.

The Situation When the Linearizing Operator is Self-adjoint

Definition 3 Let $\mathrm{L}[U]$, with its components $\mathrm{L}^{\sigma}_{\rho}[U]$, be the linearizing operator associated with a PDE system $\mathbf{R}\{x; u\}$ (2). The adjoint operator of $\mathrm{L}[U]$ is $\mathrm{L}^*[U]$, with components $\mathrm{L}^{*\sigma}_{\rho}[U]$. $\mathrm{L}[U]$ is a *self-adjoint* operator if and only if $\mathrm{L}[U] \equiv \mathrm{L}^*[U]$, i.e., $\mathrm{L}^{\sigma}_{\rho}[U] \equiv \mathrm{L}^{*\sigma}_{\rho}[U]$, $\sigma, \rho = 1, \dots, m$.

One can show that a given PDE system, *as written*, has a variational formulation if and only if its associated linearizing operator is self-adjoint [8, 42, 43].

If the linearizing operator associated with a given PDE system is self-adjoint, then each set of local CL multipliers yields a local symmetry of the given PDE system. In particular, one has the following theorem.

Theorem 7 *Consider a given PDE system* $\mathbf{R}\{x; u\}$ *(2) with* $N = m$*, i.e., the number of dependent variables appearing in PDE system (2) is the same as the number of equations in PDE system (2). Suppose the associated linearizing operator* $\mathrm{L}[U]$ *for PDE system (2) is self-adjoint. Let* $\{\Lambda_\sigma(x, U, \partial U, \ldots, \partial^l U)\}_{\sigma=1}^m$ *be a set of local CL multipliers for (2). Let*

$$\eta^\sigma(x, u, \partial u, \ldots, \partial^l u) = \Lambda_\sigma(x, u, \partial u, \ldots, \partial^l u), \quad \sigma = 1, \ldots, m$$

where $U(x) = u(x)$ *is any solution of the PDE system (2). Then*

$$\eta^\sigma(x, u, \partial u, \ldots, \partial^l u)\frac{\partial}{\partial u^\sigma} \tag{22}$$

is a local symmetry of $\mathbf{R}\{x; u\}$.

Proof Since the hypothesis of Theorem 6 is satisfied with $\mathrm{L}[U] = \mathrm{L}^*[U]$, from the equations of this theorem it follows that in terms of the components of the associated linearizing operator $\mathrm{L}[U]$, one has

$$\mathrm{L}_\rho^\sigma[u]\Lambda_\sigma(x, u, \partial u, \ldots, \partial^l u) = 0, \qquad \rho = 1, \ldots, m \tag{23}$$

where $u = \theta(x)$ is any solution of the given PDE system (2). But the set of Eq. (23) is the set of determining equations for a local symmetry $\Lambda_\sigma(x, u, \partial u, \ldots, \partial^l u)\frac{\partial}{\partial u^\sigma}$ of PDE system (2). Hence (22) is a local symmetry of PDE system (2). □

The converse of Theorem 7 is false. In particular, suppose $\eta^\sigma(x, u, \partial u, \ldots, \partial^l u)\frac{\partial}{\partial u^\sigma}$ is a local symmetry of a PDE system $\mathbf{R}\{x; u\}$ (2) with a self-adjoint linearizing operator $\mathrm{L}[U]$. Let $\Lambda_\sigma(x, U, \partial U, \ldots, \partial^l U) = \eta^\sigma(x, U, \partial U, \ldots, \partial^l U)$, $\sigma = 1, \ldots, m$, where $U(x) = (U^1(x), \ldots, U^m(x))$ is arbitrary. Then it does not necessarily follow that $\{\Lambda_\sigma(x, U, \partial U, \ldots, \partial^l U)\}_{\sigma=1}^m$ is a set of local CL multipliers of $\mathbf{R}\{x; u\}$. This can be seen as follows: In the self-adjoint case, the set of local symmetry determining equations is a subset of the set of local multiplier determining equations. Here *each* local symmetry yields a set of local CL multipliers if and only *each* solution of the set of local symmetry determining equations also solves the remaining set of local multiplier determining equations.

To illustrate the situation, consider the following example of a nonlinear PDE whose linearizing operator is self-adjoint but the PDE has a point symmetry that does not yield a multiplier for a local CL

$$u_{tt} - u(uu_x)_x = 0. \tag{24}$$

It is easy to see that the PDE (24) has the scaling point symmetry $x \to \alpha x$, $u \to \alpha u$, corresponding to the infinitesimal generator

$$\mathrm{X} = (u - xu_x)\frac{\partial}{\partial u}\cdot \tag{25}$$

The self-adjoint linearizing operator associated with PDE (24) is given by

$$\mathrm{L}[U] = \mathrm{D}_t^2 - U^2\mathrm{D}_x^2 - 2UU_x\mathrm{D}_x - 2UU_{xx} - U_x^2.$$

The determining equation for the local CL multipliers $\Lambda(t, x, U, U_t, U_x)$ of the PDE (24) is an identity holding for all values of the variables t, x, U, U_t, U_x, U_{tt}, U_{tx}, U_{xx}, U_{ttt}, U_{ttx}, U_{txx}, U_{xxx}, and splits into a system of two equations consisting of

$$\tilde{\mathrm{D}}_t^2\Lambda - U^2\mathrm{D}_x^2\Lambda - 2UU_x\mathrm{D}_x\Lambda - (2UU_{xx} + U_x^2)\Lambda = 0 \tag{26}$$

and

$$2\Lambda_U + \tilde{\mathrm{D}}_t\Lambda_{U_t} - \mathrm{D}_x\Lambda_{U_t} = 0 \tag{27}$$

in terms of the "restricted" total derivative operator $\tilde{\mathrm{D}}_t = \frac{\partial}{\partial t} + U_t\frac{\partial}{\partial U} + U_{tx}\frac{\partial}{\partial U_x} + g[U]\frac{\partial}{\partial U_t} + U_{txx}\frac{\partial}{\partial U_{xx}} + \mathrm{D}_t(g[U])\frac{\partial}{\partial U_{tt}}$ where $g[U] = U(UU_x)_x$.

Equation (26) is the determining equation for $\Lambda(t, x, u, u_t, u_x)\frac{\partial}{\partial u}$ to be a contact symmetry of the given PDE (24). If the contact symmetry satisfies the second determining Eq. (27) then it yields a local CL multiplier $\Lambda(t, x, U, U_t, U_x)$ of PDE (24). It is easy to check that the scaling symmetry (25) obviously satisfies the contact symmetry determining Eq. (26) but does not satisfy the second determining Eq. (27) when $u(x, t)$ is replaced by an arbitrary function $U(x, t)$. Hence the scaling symmetry (25) does not yield a local conservation law of PDE (24).

3.3 Noether's Theorem

In 1918, Emmy Noether presented her celebrated procedure (*Noether's theorem*) to find local CLs for a DE system that admits a variational principle.

When a given DE system admits a variational principle, then the extremals of the associated action functional yield the given DE system (the *Euler-Lagrange equations*). In this case, Noether showed that if a one-parameter local transformation leaves invariant the action functional (action integral), then one obtains the fluxes of a local CL through an explicit formula that involves the infinitesimals of the local transformation and the Lagrangian (Lagrangian density) of the action functional.

3.3.1 Euler-Lagrange Equations

Consider a functional $J[U]$ in terms of n independent variables $x = (x^1, \ldots, x^n)$ and m arbitrary functions $U = (U^1(x), \ldots, U^m(x))$ and their partial derivatives to order k, defined on a domain Ω

$$J[U] = \int_{\Omega} L[U]dx = \int_{\Omega} L(x, U, \partial U, \ldots, \partial^k U)dx. \tag{28}$$

In (28), the function $L[U] = L(x, U, \partial U, \ldots, \partial^k U)$ is called a *Lagrangian* and the functional $J[U]$ is called an *action integral*.

Consider an infinitesimal change $U(x) \to U(x) + \varepsilon v(x)$ where $v(x)$ is any function such that $v(x)$ and its derivatives to order $k-1$ vanish on the boundary $\partial\Omega$ of the domain Ω. The corresponding infinitesimal change (variation) in the Lagrangian $L[U]$ is given by

$$\begin{aligned} \delta L &= L(x, U + \varepsilon v, \partial U + \varepsilon \partial v, \ldots, \partial^k U + \varepsilon \partial^k v) - L(x, U, \partial U, \ldots, \partial^k U) \\ &= \varepsilon \left(\frac{\partial L[U]}{\partial U^i} v^i + \frac{\partial L[U]}{\partial U^i_j} v^i_j + \cdots + \frac{\partial L[U]}{\partial U^i_{j_1 \cdots j_k}} v^i_{j_1 \cdots j_k} \right) + O(\varepsilon^2). \end{aligned} \tag{29}$$

Let

$$\begin{aligned} W^l[U, v] = v^i &\left(\frac{\partial L[U]}{\partial U^i_l} + \cdots + (-1)^{k-1} \mathrm{D}_{j_1} \cdots \mathrm{D}_{j_{k-1}} \frac{\partial L[U]}{\partial U^i_{l j_1 \cdots j_{k-1}}} \right) \\ &+ v^i_{j_1} \left(\frac{\partial L[U]}{\partial U^i_{j_1 l}} + \cdots + (-1)^{k-2} \mathrm{D}_{j_2} \cdots \mathrm{D}_{j_{k-1}} \frac{\partial L[U]}{\partial U^i_{j_1 l j_2 \cdots j_{k-1}}} \right) \\ &+ \cdots + v^i_{j_1 \cdots j_{k-1}} \frac{\partial L[U]}{\partial U^i_{j_1 j_2 \cdots j_{k-1} l}}. \end{aligned} \tag{30}$$

After repeatedly using integration by parts, one can show that

$$\delta L = \varepsilon (v^i \mathrm{E}_{U^i}(L[U]) + \mathrm{D}_l W^l[U, v]) + O(\varepsilon^2) \tag{31}$$

where E_{U^i} is the Euler operator with respect to U^i. The corresponding variation in the action integral $J[U]$ is given by

$$\begin{aligned} \delta J = J[U + \varepsilon v] - J[U] &= \int_{\Omega} \delta L dx \\ &= \varepsilon \int_{\Omega} (v^i \mathrm{E}_{U^i}(L[U]) + \mathrm{D}_l W^l[U, v]) dx + O(\varepsilon^2) \\ &= \varepsilon \Big(\int_{\Omega} v^i \mathrm{E}_{U^i}(L[U]) dx + \int_{\partial\Omega} W^l[U, v] n^l d\sigma \Big) + O(\varepsilon^2). \end{aligned} \tag{32}$$

Hence if $U(x) = u(x)$ extremizes the action integral $J[U]$, then the $O(\varepsilon)$ term in δJ must vanish. Thus $\int_{\Omega} v^i \mathrm{E}_{u^i}(L[u]) \mathrm{d}x = 0$ for an *arbitrary* function $v(x)$ defined on the domain Ω. Hence, if $U(x) = u(x)$ extremizes the action integral $J[U]$, then

$u(x)$ must satisfy the PDE system

$$\mathrm{E}_{u^i}(L[u]) = \frac{\partial L[u]}{\partial u^i} + \cdots + (-1)^k \mathrm{D}_{j_1} \cdots \mathrm{D}_{j_k} \frac{\partial L[u]}{\partial u^i_{j_1 \cdots j_k}} = 0, \quad i = 1, \ldots, m. \quad (33)$$

The Eq. (33) are called the *Euler-Lagrange equations* satisfied by an extremum $U(x) = u(x)$ of the action integral $J[U]$. Thus the following theorem has been proved.

Theorem 8 *If a smooth function $U(x) = u(x)$ is an extremum of an action integral (28), then $u(x)$ satisfies the Euler-Lagrange equations (33).*

3.3.2 Standard Formulation of Noether's Theorem

Definition 4 In the *standard formulation of Noether's theorem*, the action integral (28) is invariant under the one-parameter Lie group of point transformations

$$\begin{aligned} (x^*)^i &= x^i + \varepsilon \xi^i(x, U) + O(\varepsilon^2), \qquad i = 1, \ldots, n \\ (U^*)^\mu &= U^\mu + \varepsilon \eta^\mu(x, U) + O(\varepsilon^2), \qquad \mu = 1, \ldots, m \end{aligned} \quad (34)$$

with infinitesimal generator $\mathrm{X} = \xi^i(x, U)\frac{\partial}{\partial x^i} + \eta^\mu(x, U)\frac{\partial}{\partial U^\mu}$, if and only if $\int_{\Omega^*} L[U^*]dx^* = \int_\Omega L[U]dx$ where Ω^* is the image of Ω under the Lie group of point transformations (34).

The *Jacobian* of the one parameter Lie group of point transformations (34) is given by $\mathrm{J} = \det(\mathrm{D}_i(x^*)^j) = 1 + \varepsilon \mathrm{D}_i \xi^i(x, U) + O(\varepsilon^2)$. Then $\mathrm{d}x^* = \mathrm{J}\mathrm{d}x$. Moreover, $L[U^*] = \mathrm{e}^{\varepsilon \mathrm{X}} L[U]$ in terms of the infinitesimal generator X. Consequently, in the standard formulation of Noether's theorem, X is a point symmetry of $J[U]$ if and only if

$$0 = \int_\Omega (\mathrm{J}\mathrm{e}^{\varepsilon \mathrm{X}} - 1) L[U] dx = \varepsilon \int_\Omega (L[U]\mathrm{D}_i \xi^i(x, U) + \mathrm{X}^{(k)} L[U]) dx + O(\varepsilon^2) \quad (35)$$

holds for arbitrary $U(x)$ where $\mathrm{X}^{(k)}$ is the k-th extension (prolongation) of the infinitesimal generator X. Hence, if X is a point symmetry of $J[U]$, then the $O(\varepsilon)$ term in (35) must vanish. Thus $L[U]\mathrm{D}_i \xi^i(x, U) + \mathrm{X}^{(k)} L[U] \equiv 0$.

The one-parameter Lie group of point transformations (34) with infinitesimal generator X is equivalent to the one-parameter family of transformations in evolutionary form given by

$$\begin{aligned} (x^*)^i &= x^i, \qquad i = 1, \ldots, n \\ (U^*)^\mu &= U^\mu + \varepsilon[\eta^\mu(x, U) - U^\mu_i \xi^i(x, U)] + O(\varepsilon^2), \qquad \mu = 1, \ldots, m \end{aligned} \quad (36)$$

with k-th extended infinitesimal generator $\hat{\mathrm{X}}^{(k)} = \hat{\eta}^{\mu}[U]\frac{\partial}{\partial U^{\mu}} + \cdots$. Under transformation (36), $U(x) \to U(x) + \varepsilon v(x)$ has components $v^{\mu}(x) = \hat{\eta}^{\mu}[U] = \eta^{\mu}(x, U) - U_i^{\mu}\xi^i(x, U)$. Hence $\delta L = \varepsilon\hat{\mathrm{X}}^{(k)}L[U] + O(\varepsilon^2)$. Thus

$$\int_{\Omega} \delta L dx = \varepsilon \int_{\Omega} \hat{\mathrm{X}}^{(k)} L[U] dx + O(\varepsilon^2). \tag{37}$$

Consequently, after setting $v^{\mu}(x) = \hat{\eta}^{\mu}[U] = \eta^{\mu}(x, U) - U_i^{\mu}\xi^i(x, U)$, and comparing expressions (32) and (37), it follows that

$$\hat{\mathrm{X}}^{(k)} L[U] \equiv \hat{\eta}^{\mu}[U]\mathrm{E}_{U^{\mu}}(L[U]) + \mathrm{D}_l W^l[U, \hat{\eta}[U]]. \tag{38}$$

By direct calculation, one can show the following.

Lemma 1 *Let $F[U] = F(x, U, \partial U, \ldots, \partial^k U)$ be an arbitrary function of its arguments. Then, in terms of the extended infinitesimal generators $\mathrm{X}^{(k)}$ and $\hat{\mathrm{X}}^{(k)}$, one has the identity*

$$\mathrm{X}^{(k)} F[U] + F[U]\mathrm{D}_i\xi^i(x, U) \equiv \hat{\mathrm{X}}^{(k)} F[U] + \mathrm{D}_i(F[U]\xi^i(x, U)). \tag{39}$$

Theorem 9 Standard formulation of Noether's theorem. *Suppose a given PDE system is derivable from a variational principle, i.e., the given PDE system is a set of Euler-Lagrange equations (33) whose solutions $u(x)$ are extrema $U(x) = u(x)$ of an action integral $J[U]$ with Lagrangian $L[U]$. Suppose the one-parameter Lie group of point transformations (34) with infinitesimal generator* X *leaves invariant $J[U]$. Then*

1. *The identity*

$$\hat{\eta}^{\mu}[U]\mathrm{E}_{U^{\mu}}(L[U]) \equiv -\mathrm{D}_i(\xi^i(x, U)L[U] + W^i[U, \hat{\eta}[U]]) \tag{40}$$

   holds for arbitrary functions $U(x)$, i.e., $\{\hat{\eta}[U]\}_{\mu=1}^m$ is a set of local CL multipliers of the Euler-Lagrange system (33).
2. *The local conservation law*

$$\mathrm{D}_i(\xi^i(x, u)L[u] + W^i[u, \hat{\eta}[u]]) = 0 \tag{41}$$

   holds for any solution $u = \theta(x)$ of the Euler-Lagrange system (33).

Proof Let $F[U] = L[U]$ in the identity in Lemma 1. Then the identity

$$\hat{\mathrm{X}}^{(k)} L[U] + \mathrm{D}_i(L[U]\xi^i(x, U)) \equiv 0 \tag{42}$$

holds for arbitrary functions $U(x)$. Substitution for $\hat{\mathrm{X}}^{(k)}L[U]$ in (42) through (38) yields the identity (40). If $U(x) = u(x)$ solves the Euler-Lagrange system (33),