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Fractals, Wavelets, and their Applications

Contributions from the International Conference and Workshop on Fractals and Wavelets

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Preface

Fractal geometry is a young field. It was initially developed in the 1980s, driven by the motivation to model rough phenomena in nature, and by new opportunities of computer visualization. Towards the end of the 1980s, wavelets were introduced for the needs of signal and image processing. Today, the field of fractals and wavelets has grown into a respected mathematical discipline with specific concepts and techniques, and with plenty of applications inside and outside mathematics.

In November 2013 a workshop and the first International Conference on Fractals and Wavelets in India took place at Rajagiri School of Engineering and Technology, Kochi, Kerala.

In the workshop, from November 9 to 12, leading experts from all over the world gave comprehensive survey lectures on the state of the art in their areas. In the International Conference from November 13 to 16, new research results were presented by mathematicians from ten countries. There were more than 100 participants from India, revealing that research in fractals and wavelets has taken root at many Indian universities, with an emphasis on applications to engineering, medicine, Internet traffic, hydrology, and other fields.

This volume contains all invited lectures of the workshop as well as selected contributions to the conference. Providing readable surveys, it can be used as a reference book for those who want to start work in the field. It documents the present state of research in the area, both in India and abroad, and can help to develop cooperation among widely scattered groups.

The organizers of the conference would like to thank the management of Rajagiri School of Engineering and Technology, Cochin, Kerala, India for the inspiration and support provided to conduct the conference.

The organizers acknowledge the financial support given by the International Centre for Theoretical Physics, the International Mathematical Union, the International Council for Industrial and Applied Mathematics, the National Board for Higher Mathematics India, the Department of Science & Technology India, the Defence Research & Development Organisation India, the Indian National Science Academy, the Kerala State Council for Science Technology & Environment, and The South Indian Bank Limited.

Contents

Introduction

This book is divided into three parts: Fractal Theory, Wavelet Theory, and Applications. Each part begins with survey papers written for a general audience, followed by surveys on more advanced topics and by contributed papers presenting recent results.

In the first part, C. Bandt gives an introduction to basic fractal concepts and methods, followed by an introduction to self-similar sets. Self-similar sets are generated by similitudes and form the most simple class of fractals. Another important class, known from appealing computer visualizations, is generated by iteration of polynomials, rational functions, and entire functions of a complex variable. S. Sutherland presents the theory of Julia and Fatou sets, and R.L. Devaney discusses their topological intricacies.

Four other invited lectures provide new concepts and techniques which were developed by the authors. The concept of fractal homeomorphism is introduced by M. Barnsley, B. Harding, and M. Rypka. Dimension results on self-affine sets and measures are simplified by K. Simon by introducing the concept of almost self-affine set. M. Urbański treats the more complicated class of self-affine sets over an infinite alphabet, and A. Tetenov studies projection and rigidity properties of fractal curves in n-dimensional space.

The contributed lectures of Part I deal with new three-dimensional fractals, projections of Mandelbrot percolation sets, and approaches to fractals in more general topological spaces.

Wavelet Theory and fractal functions are studied in Part II. Roughly speaking, wavelets are basis functions with self-similarity properties which ensure an efficient coding of signals and images. General bases in Hilbert spaces called frames are the fundamental concept here. O. Christensen gives an introduction to frame theory. In a second lecture with Hong Oh Kim and Rae Young Kim he presents recent trends and open problems in the field. P.R. Massopust introduces a new class of fractal functions, using the new concept of a local iterated function system.

The contributed talks of Part II concern a variety of different constructions of fractal functions and wavelets with good approximation properties, such as preservation of convexity.

Part III starts with an invited lecture of N. Cohen, the inventor of fractal antennas. Taking examples from his field, he discusses the problems and difficulties which arise on the way before new inventions can be implemented into practice.

The contributed lectures in this part deal with application to cancer detection and brain signal analysis, chemical engineering and hydrology, Internet traffic, image processing, and tomography. They illustrate the rapid development and wide range of applied fractal research in India.

Part I
Fractal Theory

Introduction to Fractals

Christoph Bandt

Abstract This non-technical introduction tries to place fractal geometry into the development of contemporary mathematics. Fractals were introduced by Mandelbrot to model irregular phenomena in nature. Many of them were known before as mathematical counterexamples. The essential model assumption is self-similarity which makes it possible to describe fractals by parameters which are called dimensions or exponents. Most fractals are constructed from dynamical systems. Measures and probability theory play an important part in the study of fractals.

Keywords Fractal • Self-similarity • Box dimension

1 Mandelbrot's Vision of Fractals

1.1 Potential Applications

Benoit B. Mandelbrot coined the term "fractal" and created fractal geometry with his groundbreaking monography "The Fractal Geometry of Nature" [5] in 1982. He begins this work with some words which have become famous: "Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line."

Workshop on Fractals and Wavelets at Rajagiri School, Kochi, India, 9 Nov 2013.

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Mandelbrot clearly saw the need of modelling irregular phenomena in science and economy, since he had worked on this field for many years. He knew that traditional methods do not work—a view which the majority of his colleagues did not share. But his main message was positive: there are mathematical concepts which can be applied to all of these phenomena.

Meanwhile fractal geometry is established as a mathematical area with deep theorems and exciting intrinsic problems. Nevertheless, we must not forget that *it is the diversity of potential applications which makes our field so attractive.*

1.2 Mandelbrot's Way

As a rule, new ideas and their inventors do not get accepted right away. Actually, Mandelbrot got his first tenure professorship at the age of 75. In his autobiography [6] which appeared after his death in 2010, he characterizes himself as a "scientific maverick." He was born in 1924 in the Jewish quarter of the Polish capital which became known as Warsaw ghetto during the Nazi occupation in World War II. Many relatives, all neighbors and friends of his childhood were killed by the Germans. Fortunately, his family emigrated to France before the war. When Nazi occupation came to France, Mandelbrot had to cover his identity and live under continuous threat for several years.

After the war, Mandelbrot became a mathematics student, proved his exceptional geometrical talent, and gained scholarships at elite universities in France and the USA. But then, instead of joining mainstream research, he became interested in various obscure phenomena and strange applied problems. Mandelbrot had to struggle 25 years to get recognized. In 1975, he wrote the initial French version of his book which collected his views and results. Physicists started to accept and apply his ideas. And then, within few years, fractals became very popular, and Mandelbrot got a lot of honors.

2 Fractals in Contemporary Mathematics

Before we come to details, let me point out some personal views. In my opinion, fractals form one important facet in the development of twenty-first century mathematics.

2.1 Classical Mathematics

Classical mathematics, centered around analysis, was triggered by applications in astronomy, physics, and engineering, by problems with a moderate amount of data. Some ingenious ideas of Gauss are difficult to comprehend even today, and the

Riemann hypothesis is still not solved—but most questions with impact outside mathematics had a relatively simple structure. Up to the middle of the twentieth century we had no calculators. All numerical calculations were done by hand, with the help of tables. Nevertheless, classical mathematics was a driving force for the development of all our achievements in science and technology: mechanical watches, cars, railways, the atom bomb, the first computers, and the first studies on global change. Mathematics also played a leading role in education. It was considered necessary to understand the modern world.

2.2 New Challenges

Now we live in a world which is extremely complex and difficult to understand. Nobody can oversee all structures he/she is involved in. Computers and huge amounts of data are virtually everywhere. Life sciences, economy, and climate research investigate processes of incredible complexity. They cannot do anything without advanced mathematical methods. At the same time, public reputation of mathematics has shrunk, and mathematical communication and education are in a worldwide crisis.

In my mind, the basic challenge of today's mathematical education is to give people an orientation in a complex world. People must find strategies to comprehend and influence their environment. We must remain masters of computers and not become their slaves. We have to decide the essential things and leave routine work to the machines.

Mathematics has the chance to shape the future, in research as well as in education. Tremendous efforts are required to meet this challenge. Communication practice and curricula must be thoroughly revised and changed. We need classical mathematics as well as new concepts and techniques.

Fractals and networks are among the new concepts. It is no surprise that both are strongly connected with computers. Complex networks became a research field around 2000 in connection with the fast development of the world wide web. The study of fractals was greatly enhanced by the development of computer graphics facilities in the 1980s. Mandelbrot was a long-term fellow at IBM, the leading computer enterprise in the time of mainframe computers.

3 Self-similarity

3.1 Fractal Symmetry

There is no precise mathematical definition of the word "fractal". Most experts agree that negative properties like "something very strange" or "very irregular," or Mandelbrot's first definition "sets with non-integer dimension" are not helpful. The essential property is self-similarity: *small parts and big parts of the figure* *look similar.* When we see a piece of the figure, we cannot conclude where we are, nor can we say something on the size of the piece. This property is also called *scale-invariance.* The structure of a fractal is nearly the same at every corner, on large scale and on small scale.

Self-similarity is a kind of symmetry which simplifies the analysis of fractals. Every kind of symmetry simplifies problems. To determine the volume of a solid, for instance, we need a triple integral, but for a solid of revolution, we need only a single integral. The real line, one of the fundamental sets in mathematics, is symmetric with respect to translations and reflection. It is also self-similar in the above sense, so it can be called a fractal. The same holds for $\mathbb{R}^n$.

3.2 The Benefit of Self-similarity

In order to study the whole structure of a fractal, it is enough to study small pieces, because the structure is everywhere similar. An everyday example is the process of distribution of public money. When you know how this works in small scale, in a town or university, then you can become a minister—since the mechanisms and problems in the government are similar, only at larger scale.

When we consider a cloud, a mountain scenery, a satellite image of a coastline, a tree or lightning, we can agree that self-similarity is present, at least to some extent. So the definition should apply to reality. However, self-similarity is not a property of nature. It is a *model assumption,* like the concept of a straight line or a circle. In practice, lines are never infinite and they are never straight, but calculations with lines have been successful. When we find sufficiently simple theoretical classes of self-similar figures, we can try to use them as models of reality.

In the sequel we shall consider different relations between small and big pieces of certain sets which lead to different classes of fractals with a rigorous definition. For introductory reading, we recommend the classical treatments which convey fascination in fractals: Mandelbrot's original work [5], Barnsley's well-illustrated textbook [1], and Falconer's mathematical treatment [3]. One may also consult the introduction by Peitgen et al. [7], Schroeder's view of a physicist [9], and Edgar's collection of seminal papers [2]. The web also contains a lot of stimulating material.

4 The Cantor Set

4.1 The Topological Viewpoint

The Cantor set is the basic example of a fractal. It comes in different disguise, see Fig. 1. The topology of $\mathbb{R}^n$ characterizes it as an uncountable compact (closed and bounded) set without isolated points which is totally disconnected—there is

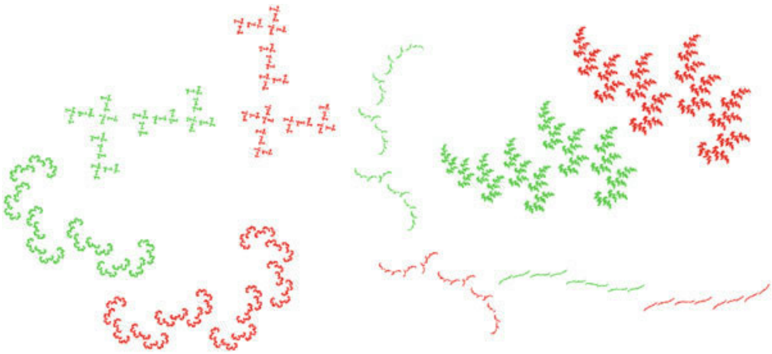

Fig. 1 Self-similar Cantor sets

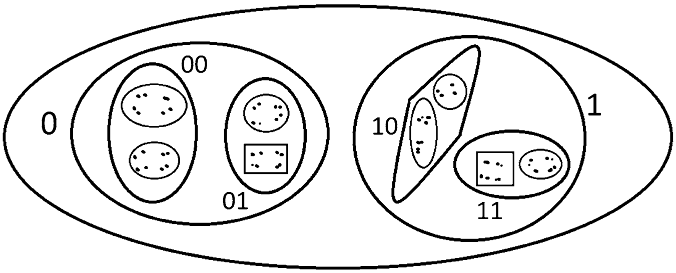

Fig. 2 The mathematical picture of the Cantor set

no continuous curve which connects points within the set. The last property is emphasized by Mandelbrot's name *Cantor dust*.

In some of the examples of Fig. 1, self-similarity is obvious to the eye. In a more abstract way, self-similarity can be introduced to the Cantor set C in many ways. We divide C into two closed and disjoint subsets C_0 and C_1. In the plane, this is done by enclosing the pieces into domains D_0, D_1 bounded by curves which do not intersect C, as indicated in Fig. 2. Next, we divide the set C_0 into two closed disjoint sets C_{00} and C_{01}, and C_1 into two closed disjoint subsets C_{10}, C_{11}. Then we do the same with the new sets $C_w = C_{w_1 w_2}$ where $w_1, w_2 \in \{0, 1\}$, and so on. It is clear that all these subsets are Cantor sets again. From the viewpoint of topology, the pieces C_w are equal to C.

In mathematical constructions of a Cantor set C, one starts with a surrounding set D, which may be an interval, rectangle, or ball, and continues with surrounding sets D_w, so that D_{w0} and D_{w1} are disjoint for every word w. This abstract construction of a Cantor set is shown in Fig. 2.

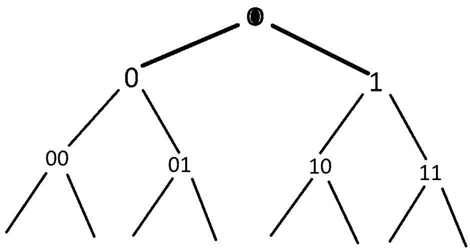

Fig. 3 The binary tree—another picture of the Cantor set

4.2 Algebraic Description of the Cantor Set

We consider the alphabet $A = \{0, 1\}$ with two letters. Each piece C_w and its surrounding set D_w is given by a word $w = w_1 w_2 \dots w_n \in A^n$ of some length n. The set of all words of the alphabet addresses the pieces of C which we constructed, and their surrounding sets.

Next consider a point $x \in C$. It is contained in some D_{w_1}, then in some $D_{w_1 w_2}$, in some $D_{w_1 w_2 w_3}$ and so on. These sets are nested: $D \supset D_{w_1} \supset D_{w_1 w_2} \supset \dots$ If we construct the D_w in such a way that their diameter tends to zero with $n \to \infty$ (this is not difficult to arrange), then x will be the intersection of the nested sequence:

$$\{x\} = \bigcap_{n=1}^{\infty} D_{w_1 w_2 \dots w_n} \qquad \text{for some sequence } s = w_1 w_2 \dots$$

Thus each point x of the Cantor set corresponds to a unique sequence s in the alphabet A. This is the abstract concept of a Cantor set:

C is the set of all sequences $s = w_1 w_2 \dots$ over some finite alphabet A.

4.3 The Binary Tree

The binary tree is another strong picture of the Cantor set (Fig. 3). Each node of the tree is denoted by a word $w = w_1 \dots w_n$, and is connected to its parent $w_1 \dots w_{n-1}$ as well as to its children $w0$ and $w1$. The root of the tree is denoted by the empty word $\bullet$. The points of C correspond to infinite non-intersecting paths starting in $\bullet$, which can be written in the form $s = w_1 w_2 \dots$ Such trees appear in the analysis of algorithms and in programming. Of course this tree is usually modified by assigning varying numbers of children to the nodes or by identifying certain words, as indicated in Fig. 4. The structure of many algorithms, as well as the structure of languages, the structure of human thinking and society, show some self-similarity—certainly not as regular as our figures.

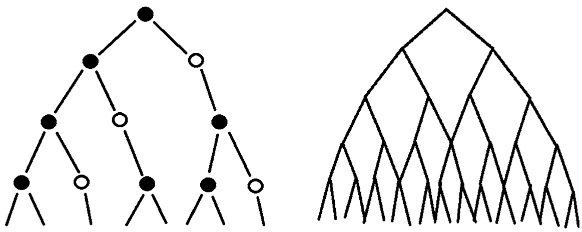

Fig. 4 Two regular modifications of the binary tree. How many nodes does level n contain?

4.4 Description of Self-similarity

How is the whole set C related to C_0, or any other piece C_w? The sequences of points in C_0 are those which start with 0. When we assign to each sequence $s = s_1 s_2 \ldots$ the sequence $0s = 0s_1 s_2 \ldots$ then we get a one-to-one correspondence between C and C_0. More generally, for every word $w = w_1 \ldots w_n$ we have a function $f_w(s) = ws = w_1 \ldots w_n s_1 s_2 \ldots$ which maps C onto C_w in a one-to-one way. It is not difficult to show that this function is a homeomorphism with respect to the product topology, or to the usual topology of sets in Fig. 1. *Adding letters in front of a sequence will lead us from the whole set to subsets.*

4.5 The Number System

When you feel uncomfortable with the use of an alphabet, think of our decimal number system. Each real number between 0 and 1 has a decimal expansion $x = 0.a_1 a_2 \ldots$ Here the alphabet is $A = \{0, 1, \ldots 9\}$. One instance of the Cantor set is defined by those decimal numbers which involve only digits 0 and 9. It is too tiny to draw, try it! Cantor's original middle-third set from 1888 takes all numbers with digits 0 and 2 in the ternary expansion:

$$C = \left\{ x \in [0,1] \;\middle|\; x = \textstyle\sum_{k=1}^{\infty} a_k 3^{-k} \text{ with } a_k \in \{0,2\} \right\}. \tag{1}$$

4.6 The Interval

If we do not exclude digits, we get the unit interval, with ten basic pieces for the decimal system and two pieces for the binary system. We can call it C, but it is not a Cantor set, it is connected. The reason is that some points have two addresses, for instance $0.1000\ldots = 0.0999\ldots$ in the decimal system and $\frac{1}{2} = 0.1000\ldots = 0.0111\ldots$ in the binary system. Thus the two pieces C_0 and C_1 have a common point. Moreover, the pieces C_{w0} and C_{w1} are also connected since in the binary system $0.w_1 \dots w_n 0111 \dots = 0.w_1 \dots w_n 1000 \dots$ So the self-similarity is preserved, and the unit interval [0, 1] *is a fractal*.

5 Some Fractal Curves

5.1 A Nowhere Differentiable Curve

As modifications of [0, 1], we obtain some fractals which were known as mathematical counterexamples around 1900. Von Koch suggested in 1904 to lift the middle third of an interval $[a, b]$ instead of deleting it. He replaces the middle third of the interval by two intervals of the same length with a common endpoint c, and repeats this procedure with all small intervals again and again. The result is a *continuous curve K which does not possess a tangent in any of its points.*

5.2 A Proof with Self-similarity

We assume there is a tangent in some point $x \in K$ and derive a contradiction. For each $\varepsilon > 0$ there must be a little piece K_w which is inside the double cone around the tangent line with vertex x and angle $\pm\varepsilon$. But since the piece K_w is geometrically similar to K (see Sect. 6), the same must hold for K and a line through some point $y \in K$. Since y is inside the triangle $T = \triangle abc$, every side of the triangle is seen from y under an angle of at least 30°. If a double cone through y contains K, then at least two of the vertices a, b, c are on one side of the cone. Thus the opening angle of the cone is at least 30° which contradicts the assumption for $\varepsilon < 15°$. We proved that K has no tangent.

Mandelbrot did not consider the non-differentiability of K as a bad property. On the contrary, he recommended Koch's curve as a model for coastlines. They have (almost) infinite length when we measure them precisely enough.

The construction of K can also be rephrased as a "decreasing set construction." We delete from the triangle T a maximal equilateral triangle, delete from each of the remaining triangles again an equilateral triangle, etc. Now we can also delete isosceles triangles with a smaller base b and will get wilder Koch curves, cf. Fig. 5.

5.3 A Plane-Filling Curve

One can ask what happens in the limit $b \to 0$. We still have a curve, but with a lot of double points. This curve will cover the whole triangle T which is now right-angled. Such plane-filling continuous curves were constructed by Peano and by Hilbert around 1890, and they were quite disturbing for the mathematical concept of dimension. Mandelbrot turned the property into the positive: look here, this is a

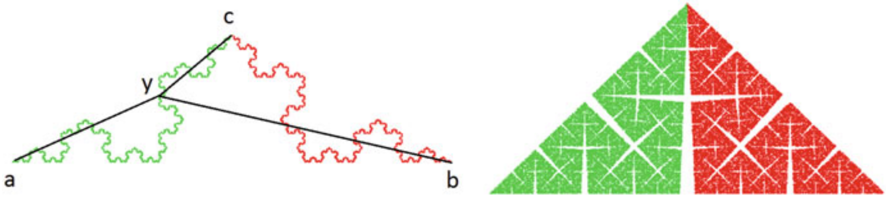

Fig. 5 The Koch curve and an almost plane-filling modification

good model for the system of human blood circulation. This must consist of vessels, but must be space-filling. Because whenever you hurt you anywhere, blood will come out.

5.4 A Simple Curve with Positive Area

There is another variation due to Knopp 1915: we can consider triangle bases b_n which decrease with the level n of construction, in such a way that the sum of the triangle areas which we cut out at level n is half as large as at level $n-1$. In that case we are left with a proper continuous curve K with no double points. But since the sum of deleted areas is less than the area of T, the curve has positive area! A more complicated proof for the existence of such curves was given by Osgood in 1905.

5.5 Different Types of Self-similarity

The pieces of Koch's curve are all geometrically similar to each other. So we call it a self-similar set. The right-angled triangle which comes as the limit Peano curve is also a self-similar set. Knopp's curve with positive area is not self-similar, but there are continuous bijective maps between the pieces K_w and K, and we also consider it a fractal. The Koch and Knopp curves are homeomorphic to the unit interval—but not the Peano curve, due to double points. So the topological relation between pieces and the whole is the same, but the metric properties differ. As a matter of fact, the fractal dimension of the curve with deleted base b is ... (see Sect. 7).

5.6 The Graph of Brownian Motion

To conclude this section, we mention a very important curve construction which introduces *random self-similarity*. We assume that we have a device which yields independent random numbers with standard normal distribution. Any mathematical software on your computer will do, even Excel. Those numbers are between -5

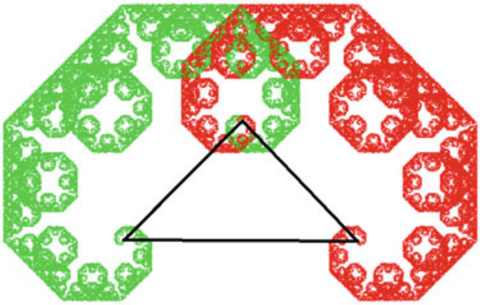

Fig. 6 The Lévy curve and its basic triangle

and 5. Our basic interval I will start at $(0,0)$ and end at $(1,z)$ where z is the first random number. The midpoint of the interval is $(\frac{1}{2},\frac{z}{2})$. Now the midpoint is shifted by $z_\bullet/\sqrt{2}$ in vertical direction, up or down, depending on the random number $z_\bullet$. Next, we shift the midpoints of the resulting intervals I_0 and I_1 by $z_0/2$ and $z_1/2$ in vertical direction, getting four intervals $I_{w_1w_2}$. We proceed by induction: on level n the midpoints intervals I_w are moved up or down by $z_w/\sqrt{2}^{n+1}$. This random construction will converge, with probability one, to the graph of a continuous function f. Of course we get different functions for different random numbers, as shown in Fig. 7 below. The construction is self-affine, not self-similar, since in each step, horizontal direction shrinks by the factor $\frac{1}{2}$ and vertical direction by $1/\sqrt{2}$.

5.7 *Lévy and His Curve*

This is the midpoint displacement algorithm for Brownian motion, the most fundamental stochastic process which was suggested as a model for the financial market by Bachelier in 1905 [5]. The construction was known to the great probabilist Paul Lévy in the 1940s. Mandelbrot considered himself as a student of Lévy: he "came closest to being my mentor" [5, p. 398]. Incidentally, Lévy also discovered a plane-filling curve which is obtained when we repeatedly replace an interval I_w by two intervals I_{w0}, I_{w1} which form an isosceles triangle with right angle over I_w (Fig. 6). This curve is self-similar and it is plane-filling, which is rather difficult to prove. As a young man, Lévy reported this result to a meeting of the French academy in 1912. Probably he was discouraged by the reaction of the audience since he published his work only in 1937 [2]. Thus the dimension of the Lévy curve is 2. The dimension of its *boundary* was found to be 1.955 by several authors only around 2000, which confirms the fact that the interior of Fig. 6 is very fragmented.

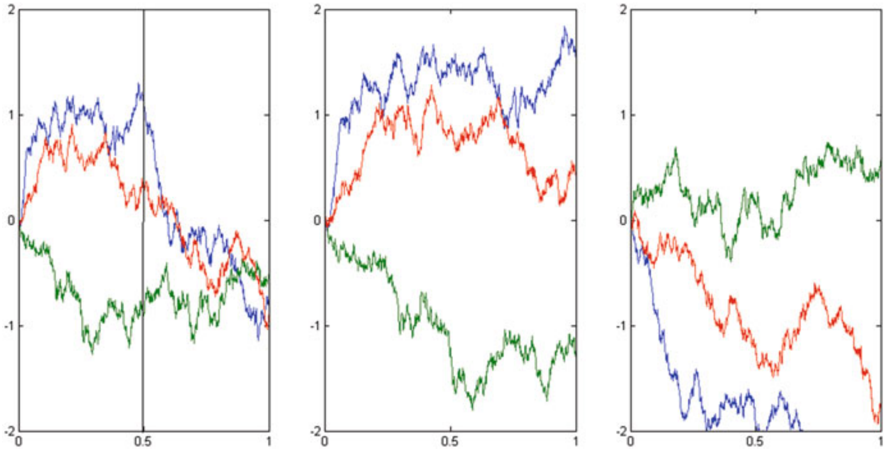

Fig. 7 Illustration for random self-similarity: three graphs of Brownian motion on [0, 1] and their rescaled *left* and *right* parts

5.8 Random Self-similarity

For the graphs of Brownian motion constructed in Sect. 5.6 and illustrated in Fig. 7, self-similarity is more complicated. *All pieces are realizations of the same random process, after rescaling*. Rescaling here means that the graph of $f(x)$ with $x \in [a, b]$ is replaced by the graph of $\frac{1}{\sqrt{b-a}} \cdot [f(a+x(b-a)) - f(a)]$ for $x \in [0, 1]$. The rescaled piece is a realization of Brownian motion on [0, 1], probably not the one with which we started. This holds for the graphs of the function over arbitrary intervals $[a, b]$, not only for the dyadic construction intervals. It is this random kind of relation what we observe in clouds and mountains. Random constructions are much more realistic models of nature than the Koch curve, but their study is also much more difficult. Two-dimensional midpoint displacement constructions were used to model mountain scenery in [5] and in various computer games.

6 Fractal Constructions by Mappings

6.1 Hutchinson's Equation

Self-similarity can be understood best when it is defined by mappings. The middle-third set 1 of Cantor is transformed into its left piece C_0 by the map $f_0(x) = \frac{x}{3}$, and into its right piece C_1 by $f_1(x) = \frac{x+2}{3}$. Both maps are similarity maps, so C is self-similar, and can be characterized as solution of the equation

$$C = f_0(C) \cup f_1(C) . \tag{2}$$

Since f_0 and f_1 map the whole set onto a subset, it is natural to assume that they are contractive maps. That means, the distance of the images of two points x, y is strictly smaller than the distance of the points themselves. A bit more rigorously, f is a contraction if there is a number $r < 1$ such that

$$|f(x) - f(y)| \leq r \cdot |x - y| \, .$$

Hutchinson [4] proved that for any two contractions f_0, f_1 on $\mathbb{R}^n$ there is a unique compact non-empty set C which fulfils Eq. (2). Thus we have a lot of self-similar sets, one for any choice of two contractions. We can also take three or more, but we stick to the simplest case.

6.2 The IFS Algorithm

Barnsley [1] came up with a computer construction for C. Take one point c of C, for instance the fixed point of f_0. Then the images $c_i = f_i(c)$ must also be in C, because of Eq. (2). The same holds for $f_w = f_{w_1} \cdots f_{w_n}(c)$ for each 0-1-word $w = w_1 \ldots_n$. Since the pieces C_w have diameter smaller r^n times the diameter of C, the collection of these 2^n points for some moderate n, say 15, will be a perfect computer approximation of the fractal C. This requires five lines of code.

A random algorithm for approximating C starts with $c_0 = c$, and takes independent random numbers $z_n \in \{0, 1\}$ for $n = 1, 2, \ldots$ (this is like coin-tossing: head is 1 and tail is 0). Then define

$$c_n = f_0(c_{n-1}) \text{if } z_n = 0 \text{ and } c_n = f_1(c_{n-1}) \text{ if } z_n = 1 \, .$$

This requires only two lines of code, and with high probability, it will soon generate points in all C_w. As a rule, already 30,000 points give a reasonable picture for the eye. Barnsley suggested the name “iterated function system,” or IFS, for such algorithms.

6.3 An Exercise in Complex Numbers

Let us model a tree T in the complex plane, with a stem S from 0 to $2i$ and two branches from $2i$ to $3i + 1$ and $3i - 1$. The mappings from the stem to the branches will be $f_0(z) = \frac{z}{2} \cdot (1 - i) + 2i$ and $f_1(z) = \frac{z}{2} \cdot (1 + i) + 2i$. When we turn on the IFS algorithm, we get the Lévy curve! That is too much, so we shall later decrease the ratios of the f_i, replacing $\frac{1}{2}$ by 0.4, say. First we have to care for the branches which are not drawn by the IFS algorithm. Only the leaves of the tree form the self-similar set. One has to add a third map, $f_2(z) = \frac{1}{3} \cdot \operatorname{Im} z$ to obtain stem and

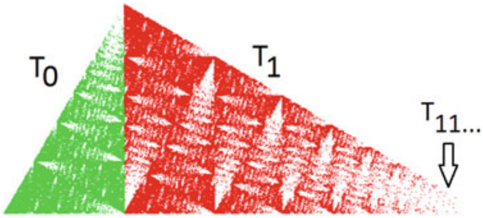

Fig. 8 A fractal measure

branches. The tree is not strictly self-similar, since f_2 is not a similarity map, but it fulfils the equation $T = S \cup f_0(T) \cup f_1(T)$ (cf. [1]). Much more natural trees were constructed by Prusinkiewicz [8] with the related concept of L-systems.

6.4 Fractal Measures

In our first experiment with random IFS in 1987, we tried to draw a triangle with vertices 0, 1, and $(1+\sqrt{3}i)/4$. Any right-angled triangle is self-similar, as you know from high-school. The mappings have the form $f_i(z) = a_i\bar{z} + b_i$. The random IFS with 500,000 points yields the surprising Fig. 8.

After a while, we found that the picture is correct. The IFS will indeed fill the whole triangle, if it runs long enough. But the picture shows what today is called a *fractal measure or multifractal*, with an uneven distribution of the points. The areas of the right piece T_1 of the triangle is three times larger than the area of the left piece T_0, but the number of IFS points in both pieces is the same. What is worse, the area of n-th level pieces $T_{11..1}$ and $T_{00..0}$ are related by the factor 3^n and still both get the same number of IFS points. With nonlinear maps, we get even more impressive examples of fractal measures. Fractal measures have become a separate area of fractal geometry. Actually, measure theory is the mathematical toolbox which has most often been used by mathematicians in the field. Hausdorff defined measures of fractional dimension already in 1918 (cf. [2]), which can be considered as the starting point of fractal geometry. It took more than 10 years before colleagues started to understand Hausdorff's ingenious idea.

6.5 Dynamical Systems

This is another key concept connected with fractals. A dynamical system consists of a set X with some mathematical structure and a mapping $g : X \to X$ which preserves that structure. For a Cantor set C fulfilling Eq. (2), the mapping $g : C \to C$ can be defined as inverse of f_0 and f_1,

$$g(c) = f_0^{-1}(c) \text{ for } c \in f_0(C) \text{ and } g(c) = f_1^{-1}(c) \text{ for } c \in f_1(C)\,.$$

If the f_i are contracting, then g is expanding. The fractals best-known to the public are the Julia sets of complex quadratic maps $g(z) = z^2 + b$ with various constants $b \in \mathbb{C}$. Here f_0, f_1 can be considered as the quadratic roots $f_i(z) = \pm\sqrt{z-b}$, the two inverse branches of g, and the Julia set as the solution of (2). *The Mandelbrot set, that well-known logo of fractal geometry, is the set of all constants b for which the Julia set is not a Cantor set.* The situation is somewhat involved, however. Among others, the f_i are not contractive.

6.6 Attractors

There are important fractals which are generated by a single map g which provides the self-similarity, but in such a way that there is no obvious structure of pieces. One example is the Hénon map on $X = \mathbb{R}^2$ given by the simple formula $h(x, y) = (y+1-ax^2, bx)$ [7, Sect. 12.1]. The fractal which is obtained by repeated application of such a map is called an *attractor of the dynamical system* (X, g). The structure of the Hénon attractors, even for the standard values $a = 1.4, b = 0.3$, is not yet mathematically understood although famous mathematicians have tried their best. Also some properties of the Mandelbrot set are not yet resolved. Even in the unit interval, it is not exactly known for which parameters r between 3.5 and 4 the quadratic mappings $g(x) = rx(1-x)$ have Cantor set or interval attractors. There are lots of open mathematical problems in this field.

7 Dimensions and Exponents

7.1 The Concept of an Exponent

When we adopt the topological viewpoint, there is only one Cantor set, up to topological equivalence. This viewpoint is now too general. We want to study metric properties, we want to distinguish thick and thin Cantor sets in Fig. 1.

How can we describe, measure and classify fractals? There is one important principle: size does not matter. Geometrically similar sets are considered to be equal, only shape is important. The type of self-similarity is studied: how much does the structure change if we pass to smaller pieces? Even though we are more specific than topologists, our parameters will be more general than those of Euclidean geometry. They are called dimensions or exponents.

The paradigm of classical mathematics is the differential equation. Give me the equation, give me the initial values, and I tell you all details about the system for all times up to infinity. This paradigm is not valid anymore. Even for rather simple differential equations, the tiniest deviation from the initial conditions can completely change the development of the system.

More importantly, nobody wants to care about every detail when the system is complicated. And it is not good to care about too many details of a complex system, because the system can organize itself when the essential parameters are properly regulated. This can be seen in everyday life, for instance in the education of small children. We need a robust but not too detailed description. For fractals, exponents are the appropriate parameters.

7.2 Box Dimension

Probably the simplest exponent is box dimension of a fractal F in the plane. Draw a mesh of squares with side length s, and count the number $N(s)$ of squares which intersect F. If you want, you can do this several times, shifting and rotating the mesh, and take $N(s)$ as average. The number $N(s)$ itself, however, is not interesting since size does not matter.

The trick is to do this for different s and *study the function* $s \mapsto N(s)$. When F is a line segment, or a rectifiable curve, then $N(s) \approx \frac{k}{s}$ for some constant k. When F is a rectangle, or, more general, F contains interior points, then $N(s) \approx \frac{k}{s^2}$ for some constant k. Thus a general assumption will be

$$N(s) \approx \frac{k}{s^\beta} \text{ or, equivalently } \log N(s) \approx \log k - \beta \log s \tag{3}$$

for some number $\beta \in [0, 2]$ which is called the *box dimension of* F. Since rectifiable curves have dimension 1 and open sets have dimension 2, and the empty set has dimension 0, the name is justified.

7.3 How to Continue?

If you are a theoretical mathematician, you will now look for examples where the approach does not work, will define box dimension as a limit—or better, upper and lower limit so that it always exists and then find classes of sets where upper limit and lower limit coincides. You can also read Hausdorff's beautiful old paper in [2] which presents a mathematically clean dimension concept, using arbitrary sets instead of boxes, and infinite coverings.

If you are a physicist or more applied mathematician, you will look for nice model sets where you can try the method numerically, by determining concrete values $N(s)$, drawing points $(s, N(s))$ into a logarithmic plot and calculating β from a linear regression.

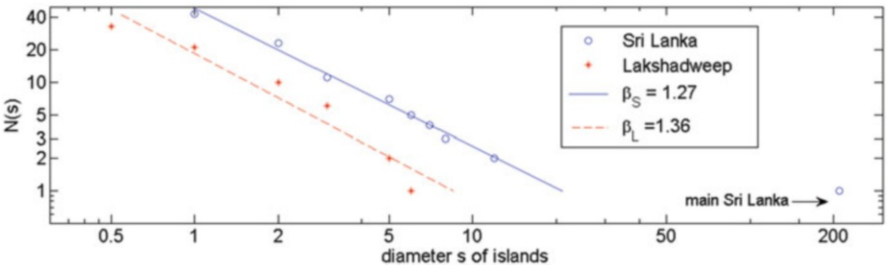

Fig. 9 Exponent of disconnectedness for groups of islands

7.4 *An Illustrative Example*

We conclude by presenting a small example in the physicists' way. We shall not count boxes, but connected components of the fractal F. The number $N(s)$ will be the number of connected components of F with diameter larger than or equal to s. The corresponding β in (3) is some measure of fragmentation of F which could be called *exponent of disconnectedness*. Connected fractals will have $\beta = 0$. Note that β is not defined when F is a Cantor set.

Here we take two maps, from Sri Lanka and from the Lakshadweep islands, and count the number of islands according to diameter (length). Probably my count is not very accurate. You can improve it. For Sri Lanka, I found islands with diameter 210, 12, 8, 7, 6, two times 5 mm, four times 3 mm, twelve times 2 mm and twenty times 1 mm. One millimeter is about 1.65 km in reality, but for the exponent this is not relevant. For Lakshadweep islands, I got diameters 6, 5, four times 3 mm, four times 2 mm, eleven times 1 mm and twelve times $\frac{1}{2}$ mm. Here 1 mm is almost 2 km, but as we said, size does not matter. We draw the values $N(s)$ into the logarithmic plot of Fig. 9 and determine the two regression lines corresponding to Eq. (3).

It turns out that Sri Lanka does not provide a good linear approximation, because of the big main island. If we drop that point, we get a regression line with slope -1.27. The Lakshadweep islands have no mainland, so they have a more fractal appearance. The exponent is $\beta \approx 1.35$, only slightly larger than 1.27. The line does not approximate too well, perhaps due to inaccurate counting. When we neglect the mainland of Sri Lanka, the degree of fragmentation for the two groups of islands is more or less the same.

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