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Duality Theories for Boolean Algebras with Operators

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Contents

Introduction

There are two natural dualities for Boolean algebras. The first duality, sometimes called the *discrete duality*, is algebraic in nature; it concerns the duality between the category of all sets with mappings between sets as the morphisms, and the category of all complete and atomic Boolean algebras with complete homomorphisms between such algebras as the morphisms. The second duality—the famous Stone duality—is topological in nature and much deeper; it concerns the duality between the category of all Boolean spaces with continuous mappings between such spaces as the morphisms, and the category of all Boolean algebras with homomorphisms between such algebras as the morphisms. Each of these two dualities can be extended to Boolean algebras with normal operators, and in both cases the extensions are non-trivial, illuminating, and have important applications. There is a third duality for Boolean algebras with normal operators—a hybrid between the algebraic and the topological dualities—that seems not to have been notice before, even in the case of Boolean algebras, and although it may seem somewhat less natural than the other two dualities, it has important applications.

In this monograph, we develop these three dualities for Boolean algebras with *normal* operators (from now on called simply *Boolean algebras with operators*, for brevity, the normality of the operators being tacitly assumed).

The first chapter is concerned with algebraic duality. It begins by reviewing the known duality between arbitrary relational structures and arbitrary complete and atomic Boolean algebras with operators—or, what amounts to the same thing, the duality between relational structures and their complex algebras (up to isomorphism, complete and atomic Boolean algebra with operators are just complex algebra of relational structures—see Theorem 1.3 below). This duality dates back to Jónsson-Tarski [21].

The duality between structures is accompanied by a corresponding duality between morphisms. For relational structures $\mathfrak{U}$ and $\mathfrak{V}$ with corresponding dual complex algebras $\mathfrak{Cm}(U)$ and $\mathfrak{Cm}(V)$, every bounded homomorphism from $\mathfrak{U}$ to $\mathfrak{V}$ has a dual that is a complete homomorphism from $\mathfrak{Cm}(V)$ to $\mathfrak{Cm}(U)$, and conversely, every complete homomorphism from $\mathfrak{Cm}(V)$ to $\mathfrak{Cm}(U)$ has a dual that is a bounded homomorphism from $\mathfrak{U}$ to $\mathfrak{V}$. The dual of an epimorphism is a monomorphism and conversely. Furthermore, if the notion of dual morphism is defined carefully, then every bounded homomorphism between relational structures is its own second dual, as is every complete homomorphism between complex algebras. In particular, the correspondence between bounded homomorphisms from $\mathfrak{U}$ to $\mathfrak{V}$ and complete homomorphisms from $\mathfrak{Cm}(V)$ to $\mathfrak{Cm}(U)$ is a (bijective) contravariant functor that maps monomorphisms to epimorphisms, and epimorphisms to monomorphisms. For the special case of frames and modal algebras—that is to say, relational structures with a single binary relation, and complete and atomic Boolean algebras with a single complete unary operator—this duality of morphisms is stated in Thomason [39] with a brief hint of the proof. For the general case of arbitrary relational structures and arbitrary complete and atomic Boolean algebras with complete operators, one half of the duality of morphisms is given in Goldblatt [13] (and repeated in Blackburn-de Rijke-Venema [1] with a credit to Goldblatt). The other half is implicit in Jónsson [19] (and repeated explicitly in [1]), but no proof is provided; we provide a proof. Some related results are formulated at the end of Hansoul [16].

The duality between morphisms is exploited to establish, first of all, a duality between inner subuniverses of a relational structure $\mathfrak{U}$ and complete ideals in the complex algebra of $\mathfrak{U}$; and second of all, a duality between bounded congruence relations on $\mathfrak{U}$ and complete subuniverses of $\mathfrak{Cm}(U)$. One aspect of these dualities is a dual lattice isomorphism from the lattice of inner subuniverse of $\mathfrak{U}$ to the lattice of complete ideals in $\mathfrak{Cm}(U)$, and a dual lattice isomorphism from the lattice of bounded congruence relations on $\mathfrak{U}$ to the lattice of complete inner subuniverses of $\mathfrak{Cm}(U)$. These dualities imply corresponding dualities between the inner substructures of $\mathfrak{U}$ and the complete quotients of $\mathfrak{Cm}(U)$ on the one hand, and between bounded quotients of $\mathfrak{U}$ and complete subalgebras of $\mathfrak{Cm}(U)$ on the other hand. For example, the dual complex algebra of each inner substructure $\mathfrak{V}$ of $\mathfrak{U}$ is isomorphic to the quotient of $\mathfrak{Cm}(U)$ modulo the complete ideal that is the dual of the universe of $\mathfrak{V}$, and vice versa. Similarly, the dual complex algebra of each quotient of $\mathfrak{U}$ modulo a bounded congruence Θ is isomorphic to the complete subalgebra of $\mathfrak{Cm}(U)$ whose universe is the dual of the congruence Θ, and vice versa.

In the final part of Chapter 1, sharper forms are established for two theorems in Goldblatt [13] (see also Thomason [39] and Goldblatt [12] for the special case of frames and modal algebras). First of all, it is shown in [13] that the complex algebra of the disjoint union of a system of relational structures is isomorphic to the (external, or Cartesian) direct product of the corresponding system of complex algebras. The result is strengthened here to show that the complex algebra of the disjoint union is actually equal to (and not just isomorphic to) the *internal* direct product of the system of complex algebras. This strengthened form plays an important role in the next chapter, on topological duality. Secondly, it is shown in [13] (generalizing a result of Monk [26]) that an ultraproduct of a system of complex algebras modulo an ultrafilter D is embeddable into the complex algebra of the ultraproduct of the corresponding system of dual relational structures modulo D. This result is strengthened here to show that the complex algebra of the ultraproduct of relational structures is, up to isomorphism, just the completion of the ultraproduct of the system of complex algebras. In other words, the algebraic dual of an ultraproduct of a system of relational structures is just the completion of the corresponding ultraproduct of the system of dual complex algebras.

The second chapter is concerned with topological duality. It studies the duality between arbitrary Boolean algebras with operators (not just complete and atomic algebras) and arbitrary relational spaces, that is to say, arbitrary relational structures endowed with the topology of a Boolean space under which the fundamental relations of the structure are continuous and clopen. This topological duality was first investigated by Halmos [15] and Goldblatt [11, 12] for the case of Boolean algebras with a single unary operator, and by Goldblatt [13] for arbitrary Boolean algebras with operators. (Both authors build on early work of Jónsson-Tarski [21], but Goldblatt is in fact concerned with the duality between bounded distributive lattices with operators and Priestley spaces, so he also builds on the work of Priestley [30]. The reports [12] are published versions of Goldblatt's doctoral dissertation [11], so we shall always refer to them instead of to [11].)

Hansoul [16] and Sambin-Vaccaro [32] contain related developments. Our approach is different from that of Goldblatt. We show, however, that our approach (which seems closer to the standard treatment of topological algebraic structures) is equivalent to his approach.

The duality between structures is accompanied by a corresponding duality between morphisms. For Boolean algebras with operators $\mathfrak{A}$ and $\mathfrak{B}$, and corresponding dual relational spaces $\mathfrak{U}$ and $\mathfrak{V}$, every homomorphism (not necessarily complete) from $\mathfrak{A}$ to $\mathfrak{B}$ has a dual that is a continuous bounded homomorphism from $\mathfrak{V}$ to $\mathfrak{U}$, and conversely, every continuous bounded homomorphism from $\mathfrak{V}$ to $\mathfrak{U}$ has a dual that is a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. The dual of an epimorphism is a monomorphism and conversely. Furthermore, if the notion of dual morphism is defined carefully, then every homomorphism between Boolean algebras with operators is its own second dual, as is every continuous bounded homomorphism between relational spaces. In particular, the correspondence between homomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$ and continuous bounded homomorphisms from $\mathfrak{V}$ to $\mathfrak{U}$ is a (bijective) contravariant functor that maps monomorphisms to epimorphisms, and epimorphisms to monomorphisms. For the special case of Boolean algebras with a single unary operator and relational spaces with a single binary relation, versions of this duality of morphisms are stated in Halmos [15] and Goldblatt [12]. A general version of the duality of morphisms for arbitrary relational spaces and arbitrary Boolean algebras with operators is given in Goldblatt [13] (and repeated in [1] with a credit to Goldblatt). See also Hansoul [16] and Sambin-Vaccaro [32] for related developments.

This duality is exploited here to show that there is, first of all, a duality between ideals (not necessarily complete) in an arbitrary Boolean algebra with operators $\mathfrak{A}$ and special open subsets of the dual relational space $\mathfrak{U}$; and second of all, a duality between subuniverses (not necessarily complete) of $\mathfrak{A}$ and (bounded) relational congruences on $\mathfrak{U}$. One aspect of these dualities is a lattice isomorphism from the lattice of ideals in $\mathfrak{A}$ to the lattice of special open subsets of $\mathfrak{U}$, and a dual lattice isomorphism from the lattice of subuniverses of $\mathfrak{A}$ to the lattice of relational congruences on $\mathfrak{U}$. These dualities imply corresponding dualities between the structures themselves. For example, if $\mathfrak{U}$ is a relational space and $\mathfrak{A}$ the dual Boolean algebra with operators, then the dual relational space of the quotient of $\mathfrak{A}$ modulo an ideal is, up to homeo-isomorphism, the inner subspace of $\mathfrak{U}$ whose universe is the complement of the special open set that is the dual of the ideal; and vice versa. Similarly, the dual relational space of a subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ is, up to homeo-isomorphism, the quotient of $\mathfrak{U}$ modulo the relational congruence that is the dual of the universe of $\mathfrak{B}$.

Next, we take up the problem of describing the dual relational space $\mathfrak{U}$ of a Boolean algebra with operators $\mathfrak{A}$ that satisfies some completeness condition. It is shown, for example, that $\mathfrak{A}$ is complete as an algebra if and only if $\mathfrak{U}$ is complete as a relational space. This result can be extended to weaker forms of completeness. For example, $\mathfrak{A}$ is countably complete as an algebra if and only if $\mathfrak{U}$ is countably complete as a relational space.

In the final part of Chapter 2, the duality between morphisms is used to describe the dual spaces of direct and subdirect products of systems of Boolean algebras with operators. Consider a disjoint system $(\mathfrak{U}_i : i \in I)$ of relational spaces, and suppose $\mathfrak{U}$ is the union of this system. Let $(\mathfrak{A}_i : i \in I)$ be the corresponding system of dual Boolean algebras with operators, and let $\mathfrak{A}$ be the internal direct product of this system, and $\mathfrak{D}$ the internal weak direct product. If the systems in question are finite (that is to say, if the index set I is finite), then the dual algebra of the relational space $\mathfrak{U}$ is just $\mathfrak{A}$; put another way, the dual of the direct product of finitely many Boolean algebras with operators is the disjoint union of the corresponding dual relational spaces. When the systems in question are infinite, the situation becomes more complicated to describe. The dual of every compactification of $\mathfrak{U}$ is isomorphic (via relativization) to a subalgebra of the direct product $\mathfrak{A}$, and in fact to a subalgebra that includes the weak direct product $\mathfrak{D}$; moreover, every subalgebra of $\mathfrak{A}$ that includes $\mathfrak{D}$ is the isomorphic image (via relativization) of some compactification of $\mathfrak{U}$. Compactifications of $\mathfrak{U}$ are mapped via this duality correspondence to the same subalgebra of $\mathfrak{A}$ if and only if the compactifications are equivalent in the sense that they are homeo-isomorphic over $\mathfrak{U}$, so one may speak with some justification of the dual Boolean algebra with operators of an equivalence class of compactifications of $\mathfrak{U}$. The function that maps each such equivalence class to the isomorphic copy (via relativization) of its dual Boolean algebra with operators is a lattice isomorphism from the lattice of equivalence classes of compactifications of $\mathfrak{U}$ to the lattice of subalgebras of the direct product $\mathfrak{A}$ that include the weak direct product $\mathfrak{D}$.

The preceding lattice isomorphism implies that the union space $\mathfrak{U}$ has a maximum compactification, and the dual Boolean algebra with operators of this maximum compactification is isomorphic (via relativization) to the direct product $\mathfrak{A}$. We prove that this maximum compactification of $\mathfrak{U}$ is just the Stone-Čech compactification of $\mathfrak{U}$. There are several different ways in which this result may be interpreted, and they all turn out to be true.

While the first chapter is concerned with the algebraic duality between relational structures and complete, atomic Boolean algebras with operators (that is to say, complex algebras of relational structures), and the second chapter is concerned with the topological duality between relational spaces and arbitrary Boolean algebras with operators, the third chapter is concerned with a hybrid duality that combines aspects of the algebraic and the topological dualities. Every weakly bounded homomorphism from a relational *structure* $\mathfrak{U}$ into a relational *space* $\mathfrak{V}$ has a dual homomorphism from the Boolean algebra with operators that is the topological dual of $\mathfrak{V}$ to the complete and atomic Boolean algebra with operators that is the algebraic dual of $\mathfrak{U}$—namely the complex algebra $\mathfrak{Cm}(U)$—and vice versa; and each of these morphisms is its own second dual. The epi-mono duality no longer holds: duals of epimorphisms are monomorphisms, but duals of monomorphisms may fail to be epimorphisms.

An arbitrary relational structure $\mathfrak{U}$ may be turned into a *locally compact* relational space (as opposed to a *compact* relational space, as considered in Chapter 2) by endowing $\mathfrak{U}$ with the discrete topology in which every subset of $\mathfrak{U}$ is simultaneously open and closed. We refer to such a discretely topologized relational structure as a *discrete space*. The hybrid duality mentioned above allows us to characterize the dual relational spaces (in the sense of Chapter 2) of all those subalgebras of $\mathfrak{Cm}(U)$ that contain the singleton subsets (or what amounts to the same thing, that contain the finite subsets) of $\mathfrak{U}$: they are just the weak compactifications of the discrete space $\mathfrak{U}$. Weak compactifications of $\mathfrak{U}$ are mapped via this duality correspondence to the same subalgebra of $\mathfrak{Cm}(U)$ if and only if the weak compactifications are equivalent in the sense that they are homeo-isomorphic over $\mathfrak{U}$, so one may speak with some justification of the dual Boolean algebra with operators of an equivalence class of weak compactifications of $\mathfrak{U}$. The function that maps each such equivalence class to the isomorphic copy (via relativization) of its dual Boolean algebra with operators is shown to be a lattice isomorphism from the lattice of equivalence classes of weak compactifications of $\mathfrak{U}$ to the lattice of subalgebras of $\mathfrak{Cm}(U)$ that contain the singleton subsets of $\mathfrak{U}$.

By restricting this result to the special case where there are no operators, we obtain the corollary—apparently new for Boolean algebras—that the function mapping each compactification of a discrete topological space U to the isomorphic copy (via relativization) of its dual Boolean algebra is an isomorphism from the lattice of equivalence classes of compactifications of U to the lattice of subalgebras of $Sb(U)$ (the Boolean algebra of subsets of U) that include the finite-cofinite subalgebra.

The lattice isomorphism mentioned above implies that there is a maximum weak compactification of a discrete space $\mathfrak{U}$, and the dual Boolean algebra with operators of this maximum weak compactification is isomorphic (via relativization) to the complex algebra $\mathfrak{Cm}(U)$. It is shown that this maximum weak compactification is in fact the Stone-Čech weak compactification of $\mathfrak{U}$. There are several ways in which this result may be interpreted, and all of them turn out to be true. This theorem may be viewed as a generalization of the well-known theorem in Boolean algebra that the dual of the Boolean algebra of all subsets of a set U is, up to homeomorphism, the Stone-Čech compactification of the discretely topologized space U.

The notions and results in this work not explicitly credited to others are due to the author. Many of them are extensions to Boolean algebras with operators of known notions and results for Boolean algebras (without operators). This applies in particular to the notions and results in Chapter 2 (see, for example, Chapters 34–38 and 43 in [10], or Chapter 3 in Koppelberg [23]). The notions and results in Chapter 3 appear to be new not only for Boolean algebras with operators, but also for Boolean algebras themselves. As mentioned above, some aspects of the duality theories discussed here have been extended to distributive lattices with operators. In this connection, the reader is referred to Goldblatt [13] and to the work of Mai Gehrke and her collaborators (see, for example, Gehrke [7], where further references to the literature may be found).

An understanding of the basic arithmetic and algebraic theory of Boolean algebras—say, along the lines developed in Chapters 6–8, 11, 12, 14, 17, 18, 20, and 26 of [10]—is assumed in this monograph. In particular, familiarity with fundamental laws of Boolean algebra and the algebraic notions of subuniverse, subalgebra, homomorphism, direct product, congruence, ideal, filter, and atom is helpful. In Chapter 2, some knowledge of basic topology (not too much) is also assumed. In particular, familiarity with the notions of open set, closed set, dense set, closure of a set, compact space, Hausdorff space, quotient topology, subspace topology, and continuous function is helpful. The necessary algebraic and topological notions and results (for the most part, with proofs) can all be found, for example, in [10]. We shall use that work as our standard reference on Boolean algebra and topology.

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Chapter 1
Algebraic Duality

In this chapter we study the algebraic duality that exists between relational structures and complete and atomic Boolean algebras with operators. The duality between the structures and algebras carries with it a corresponding duality between morphisms: every bounded homomorphism between relational structures corresponds to a dual complete homomorphism between the dual algebras, and conversely. The duality between the morphisms implies other dualities as well. Here are some examples. Every inner subuniverse of a relational structure corresponds to a complete ideal in the dual algebra, and vice versa. Every bounded congruence on a relational structure corresponds to a complete subuniverse of the dual algebra, and vice versa. The disjoint union of a system of relational structures corresponds to the direct product of the system of dual algebras, and vice versa.

1.1 Algebraic Duality for Boolean Algebras

In order to motivate the subsequent development, it is helpful to review some important aspects of the algebraic duality that exists between sets and complete, atomic Boolean algebras (without operators). The reader can find a more detailed presentation in [10].

Every set U is naturally correlated with a complete and atomic Boolean algebra, namely the Boolean algebra of all subsets of U. This algebra is denoted by $Sb(U)$ and is called the (*first*) *dual* of the set U. Inversely, every complete and atomic Boolean algebra A is naturally correlated with a set, namely the set of atoms in A. This set is called the (*first*) *dual* of the algebra A. If one starts with a set U, forms

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its dual complete and atomic Boolean algebra A, and then forms the dual set of A, the result is the set V of all singletons of elements in U. Obviously, U and V are isomorphic (in the sense of being bijectively equivalent) via the mapping that takes each element in U to its singleton. Similarly, if one starts with a complete and atomic Boolean algebra A, forms its dual set U of atoms, and then forms the dual complete and atomic Boolean algebra of U, the result is a complete and atomic Boolean algebra B in which the elements are the subsets of the set of atoms in A. The function that maps each element r in A to the set of atoms that are below r is a Boolean isomorphism from A to B. This state of affairs is expressed by saying that each set and each complete and atomic Boolean algebra is isomorphic to its second dual. In practice, every set is identified with its second dual, and every complete and atomic Boolean algebra is also identified with its second dual. This means that every element u in a set U is identified with its singleton $\{u\}$, and every element r in a Boolean algebra A is identified with the set of atoms in A that are below r. Consequently, instead of speaking about U and its second dual V, and A and its second dual B, we may speak just of U and $B = Sb(U)$. This simplifies and clarifies the presentation quite a bit.

The duality between sets and complete and atomic Boolean algebras carries with it a corresponding duality between the functions on sets and the complete Boolean homomorphisms on the algebras. If ϑ is a mapping from a set U into a set V, then there is a natural mapping φ from $Sb(V)$ into $Sb(U)$ that is defined by

$$\varphi(X) = \vartheta^{-1}(X) = \{u \in U : \vartheta(u) \in X\}$$

for subsets X of V, and φ proves to be a complete Boolean homomorphism from $Sb(V)$ into $Sb(U)$. This complete homomorphism is called the (*first*) *dual* of the mapping ϑ. Inversely, if φ is a complete homomorphism from $Sb(V)$ into $Sb(U)$, then there is a natural mapping ϑ from U into V that is defined by

$$\vartheta(u) = r \qquad \text{if and only if} \qquad r \in \bigcap\{X \subseteq V : u \in \varphi(X)\},$$

or, equivalently,

$$\vartheta(u) = r \qquad \text{if and only if} \qquad u \in \varphi(\{r\}),$$

for elements r in U. This mapping is called the (*first*) *dual* of the complete homomorphism φ. If one starts with a complete homomorphism φ from $Sb(V)$ to $Sb(U)$, forms the dual mapping ϑ from U to V, and then forms the dual of ϑ, the result is the original complete homomorphism φ. Similarly, if one starts with a mapping ϑ from U to V, forms the dual complete homomorphism φ from $Sb(V)$ to $Sb(U)$, and then forms the dual of φ, the result is the original mapping ϑ. This state of affairs is expressed by saying that every mapping between sets U and V, and every complete homomorphism between the corresponding Boolean algebras $Sb(V)$ and $Sb(U)$ is its own second dual. A mapping ϑ from U to V is one-to-one or onto if and only if the dual complete homomorphism from $Sb(V)$ to $Sb(U)$ is onto or one-to-one respectively. Finally, if ϑ maps a set U to a set V, and δ maps V to a set W, and if φ and ψ are the respective duals of ϑ and δ, then the dual of the composition $\delta \circ \vartheta$ is just the composition $\varphi \circ \psi$. The category of sets with mappings as morphisms is therefore dually equivalent to the category of complete and atomic Boolean algebras with complete homomorphisms as morphisms.

1.2 Boolean Algebras with Operators

We begin our development by reviewing the basic definitions, notation, and terminology that we shall use. In general, we shall use the set-theoretical definition of a natural number n as the set of its predecessors,

$$n = \{0, \dots, n-1\}.$$

Consequently, the phrase "for $i < n$" means "for $i = 0. \dots, n-1$", and it is equivalent to the phrase "for $i \in n$".

A Boolean algebra is an algebra of the form

$$(A\,,\,+\,,\,-),$$

where A is a non-empty set, $+$ is a binary operation on A called *addition*, and $-$ is a unary operation on A called *complement*. Other common Boolean operations, distinguished constants, and relations are defined for A in the usual way. For instance, binary operations $\cdot$ of *multiplication* and $\ominus$ of *symmetric difference* are defined on A by

$$r \cdot s = -(-r + -s) \qquad \text{and} \qquad r \ominus s = (r \cdot -s) + (-r \cdot s)$$

for elements r and s in A, distinguished constants *zero* and *one* are defined in A by

$$0 = -(r + -r) \qquad \text{and} \qquad 1 = r + -r$$

(where r is an arbitrary element in A), and a *partial order* $\leq$ is defined on A by

$$r \leq s \qquad \text{if and only if} \qquad r + s = s.$$

The class of Boolean algebra may be axiomatized in several different ways. We assume that the reader is familiar with some axiomatization and with the basic laws of Boolean algebra that are a consequence of this axiomatization (see, for example, Chapters 2 and 6–8 in [10]). In general, we shall rarely cite specific Boolean laws in our proofs, but rather shall simply say that a certain conclusion follows "by Boolean algebra".

The *supremum* or *sum* of a set X of elements in a Boolean algebra is the least upper bound r of X in the sense that r is above every element in X (in the sense of the partial order defined above), and every other upper bound of X is above r. If the supremum of X exists, then we shall denote it by $\sum X$. Notice that the supremum of the empty set is 0, since 0 is obviously an upper bound, and in fact the least upper bound, of the empty set. Dually, the *infimum* or *product* of a set X of elements is the greatest lower bound s of X in the sense that s is a lower bound of X, and every other lower bound of X is below s. If the infimum of X exists, then we shall denote it by $\prod X$. Notice that the infimum of the empty set is 1, since 1 is obviously a lower bound, and in fact the greatest lower bound, of the empty set. A Boolean algebra is called *complete* if the supremum and infimum of every set of elements in the algebra exists. A necessary and sufficient condition for the algebra to be complete is that the supremum of every subset exists. An *atom* in the algebra is defined to be a minimal non-zero element (in the sense of the defined partial order), and the algebra is called *atomic* if every non-zero element is above an atom.

A *field of sets* is a Boolean algebra in which the universe consists of some (but not necessarily all) subsets of a set U, and the basic operations are the set-theoretic ones of forming unions and complements with respect to U. The set-theoretic operations of union, intersection, and complement are denoted respectively by $\cup$, $\cap$, and $\sim$.

An *ideal* in a Boolean algebra with universe A is a subset M of A that contains 0, that is *closed under addition* in the sense that $r + s$ belongs to M whenever r and s are both in M, and that contains $r \cdot s$ whenever r is in M and s in A. The first condition is equivalent to the condition that M be non-empty; the last condition is equivalent to the condition that M be *downward closed* in the sense that if r is in M

and if $s \leq r$, then s is in M. An ideal M is said to be *maximal* if it is a *proper* ideal—that is to say, if M is different from A—and if A is the only ideal that properly includes M. The *Maximal Ideal Theorem* for Boolean algebras says that every proper ideal can be extended to a maximal ideal (see, for example, Theorem 12 in [10]).

Dually, a *filter* in the Boolean algebra is a subset N of A that contains 1, that is *closed under multiplication* in the sense that $r \cdot s$ belongs to N whenever r and s are both in N, and that contains $r + s$ whenever r is in N and s in A. The first condition is equivalent to the condition that N be non-empty; the last condition is equivalent to the condition that N be *upward closed* in the sense that if r is in N and if $r \leq s$, then s is in N. A filter N is said to be *maximal* if it is proper and if the only filter in the algebra that properly includes N is the improper filter. A maximal Boolean filter is called an *ultrafilter*. A proper filter N is an ultrafilter just in case, for every element r in the algebra, N contains either r or $-r$. A subset X of the Boolean algebra has the *finite meet property* if the product of any finite number of elements in X is non-zero. For fields of sets, the terminology *finite intersection property* is often employed. As is well known, every subset with the finite meet property can be extended to an ultrafilter (see Exercise 12 in Chapter 20 of [10]).

For an ideal M in a Boolean algebra A, the set of complements

$$-M = \{-r : r \in M\}$$

is a filter in A, and conversely, for a filter N in A, the set of complements $-N$ is an ideal in A. The set $-M$ is called the *dual* (*Boolean*) *filter* of M, and $-N$ is called the *dual* (*Boolean*) *ideal* of N. An ideal or a filter is proper or maximal if and only if its dual is proper or maximal respectively. In fact, the function that maps each ideal to its dual filter is a lattice isomorphism from the lattice of ideals in A to the lattice of filters in A (see, for example, pp. 168–169 in [10]).

We turn now to the notion of a Boolean algebra with operators, and related notions, which were introduced and studied for the first time by Jónsson and Tarski in [21]. An operation f of *rank* n—that is to say, an operation of n arguments—on the universe A of a Boolean algebra is said to be *distributive* if it is distributive over addition in each argument in the sense that for each index $i < n$ and for each sequence

$$r_0, \ldots, r_{i-1}, r_{i+1}, \ldots, r_{n-1}, s, t$$

of elements in A,

$$f(r_0,\ldots,r_{i-1},s+t,r_{i+1},\ldots,r_{n-1})$$
$$= f(r_0,\ldots,r_{i-1},s,r_{i+1},\ldots,r_{n-1})$$
$$+ f(r_0,\ldots,r_{i-1},t,r_{i+1},\ldots,r_{n-1}).$$

For example, a binary operation $\circ$ on A is distributive if

$$r\circ(s+t) = (r\circ s)+(r\circ t) \qquad \text{and} \qquad (s+t)\circ r = (s\circ r)+(t\circ r)$$

for all r, s, and t in A. A distributive operation on A is called an *operator* (on A). Such an operator f always has the following *general distributivity property*: for any sequence $X_0,\ldots,X_{n-1}$ of finite, non-empty subsets of A, writing $t_i = \sum X_i$, we have

$$f(t_0,\ldots,t_{n-1}) = \sum\{f(r_0,\ldots,r_{n-1}) : r_i \in X_i \text{ for } i<n\}. \tag{1}$$

The proof by induction is straightforward and is left to the reader (see Theorem 1.6 in [21]). An operator f of rank n is always *monotone* in the sense that for all sequences $r_0,\ldots,r_{n-1}$ and $s_0,\ldots,s_{n-1}$ of elements in A, if $r_i \le s_i$ for each $i<n$, then

$$f(r_0,\ldots,r_{n-1}) \le f(s_0,\ldots,s_{n-1}).$$

This is an immediate consequence of the general distributivity property and the definition of $\le$. Note: operations of rank 0 are identified with individual constants in A.

An operation f of rank n on the universe of a Boolean algebra A is said to be *completely distributive* if it is completely distributive over addition in each argument in the sense that for each index $i<n$, for each sequence $r_0,\ldots,r_{i-1},r_{i+1},\ldots,r_{n-1}$ of elements in A, and for each (finite or infinite) subset X of A, if the supremum $t=\sum X$ exists, then the supremum

$$\sum\{f(r_0,\ldots,r_{i-1},s,r_{i+1},\ldots,r_{n-1}) : s\in X\}$$

exists and

$$f(r_0,\ldots,r_{i-1},t,r_{i+1},\ldots,r_{n-1})$$
$$= \sum\{f(r_0,\ldots,r_{i-1},s,r_{i+1},\ldots,r_{n-1}) : s\in X\}.$$

For example, a binary operation $\circ$ on A is completely distributive if for every subset X of A, if the supremum $t = \sum X$ exists, then the suprema

$$\sum\{r \circ s : s \in X\} \qquad \text{and} \qquad \sum\{s \circ r : s \in X\}$$

both exist, and

$$r \circ t = \sum\{r \circ s : s \in X\} \qquad \text{and} \qquad t \circ r = \sum\{s \circ r : s \in X\}$$

for all r in A. A completely distributive operation on A is called a *complete operator*. Such an operator f always has the following *general complete distributivity property*: for any sequence $X_0, \dots, X_{n-1}$ of (finite or infinite) subsets of A, if the sum $t_i = \sum X_i$ exists in A for each $i < n$, then the sum on the right side of (1) exists and the equation in (1) holds. The proof involves a straight-forward induction on the rank n of the complete operator. For example, in the case of a binary complete operator $\circ$, we have

$$\begin{aligned} t_0 \circ t_1 &= \sum\{r \circ t_1 : r \in X_0\} \\ &= \sum\{\sum\{r \circ s : s \in X_1\} : r \in X_0\} \\ &= \sum\{r \circ s : r \in X_0 \text{ and } s \in X_1\}. \end{aligned}$$

In [21], a weaker notion of complete distributivity is used in which the set X is always required to be non-empty. We shall refer to operators with this weaker property as *quasi-completely distributive*, or simply *quasi-complete*. Such an operator f always satisfies a restricted version of the general complete distributivity property in which each of the sets in the sequence $X_0, \dots, X_{n-1}$ is assumed to be non-empty (see Theorem 1.6 in [21]).

An operation f on A of rank n is called *normal* if it always assumes the value 0 when at least one of its arguments is 0, that is to say, if for each $i < n$ and for each sequence

$$r_0, \dots, r_{i-1}, r_{i+1}, \dots, r_{n-1} \tag{2}$$

of elements in A,

$$f(r_0, \dots, r_{i-1}, 0, r_{i+1}, \dots, r_{n-1}) = 0.$$

Notice that a quasi-complete operator f is complete if and only if it is normal. Indeed, if (2) is a sequence of elements in A, and if a set X is empty, then

$$t = \sum X = \sum \varnothing = 0$$

and

$$\sum\{f(r_0,\ldots,r_{i-1},s,r_{i+1},\ldots,r_{n-1}) : s \in X\} = \sum\varnothing = 0$$

(where $\varnothing$ denotes the empty set), so that

$$f(r_0,\ldots,r_{i-1},t,r_{i+1},\ldots,r_{n-1}) \\ = \sum\{f(r_0,\ldots,r_{i-1},s,r_{i+1},\ldots,r_{n-1}) : s \in X\}$$

if and only if

$$f(r_0,\ldots,r_{i-1},0,r_{i+1},\ldots,r_{n-1}) = 0.$$

A *Boolean algebra with operators* is an algebra of the form

$$\mathfrak{A} = (A\,,\,+\,,\,-\,,\,f_\xi)_{\xi\in\Xi}, \tag{3}$$

where $(A\,,\,+\,,\,-)$ is a Boolean algebra—called the *Boolean part* of $\mathfrak{A}$—and f_ξ is an operator on A for each ξ in Ξ. The algebra is called *normal* if each of its operators is normal. It follows from the remark at the end of the preceding paragraph that a Boolean algebra with complete operators is automatically normal. In this monograph, *all Boolean algebras with operators are assumed to be normal*, so we shall not bother to repeat this hypothesis. In other word, when we speak of a Boolean algebra with operators, it will always be understood that the algebra is assumed to be normal. We use upper case German fraktur letters from the beginning of the alphabet to refer to Boolean algebras with operators, and corresponding italic letters to refer to the universes of these algebras. For example, A is assumed to be the universe of a given algebra $\mathfrak{A}$. Such an algebra is said to possess a certain Boolean property if the Boolean part of $\mathfrak{A}$ possesses this property. For instance, $\mathfrak{A}$ is said to be atomic if the Boolean part of $\mathfrak{A}$ is atomic. *An exception is made, however, for the notion of completeness*: $\mathfrak{A}$ is said to be *complete* if the Boolean part of $\mathfrak{A}$ is complete *and* if each of the operators is complete, that is to say, completely distributive.

The *similarity type* of a Boolean algebra with operators is the sequence of ranks of its fundamental operations. If the algebra has the form (3), then the similarity type is $(2, 1, n_\xi)_{\xi\in\Xi}$, where n_ξ is the rank of the operator f_ξ. Throughout this paper, *all Boolean algebras with operators will be assumed to have the same arbitrary but fixed similarity type*. In proofs, *we will always deal with one exemplary operator* $\circ$

that is assumed to be binary. This will free the proofs from excessive notation and hopefully make the main ideas of the proofs clearer to the reader. The passage from the case of a binary operator to the case of an operator of arbitrary rank n is always straightforward, and can safely be left to the reader. Occasionally, the case of a constant—an operator of rank 0—must be treated somewhat differently, and in these cases we will point out the differences.

Another simplification of notation may be helpful as well. In order to distinguish carefully between the operations of different Boolean algebras with operators, say $\mathfrak{A}$ and $\mathfrak{B}$, one should employ different notations to distinguish the fundamental operations of the two algebras, for example, by using superscripts to write

$$\mathfrak{A} = (A\,,\,+^{\mathfrak{A}}\,,\,-^{\mathfrak{A}}\,,\,f_\xi^{\mathfrak{A}})_{\xi\in\Xi} \quad \text{and} \quad \mathfrak{B} = (B\,,\,+^{\mathfrak{B}}\,,\,-^{\mathfrak{B}}\,,\,f_\xi^{\mathfrak{B}})_{\xi\in\Xi}.$$

In practice, the context usually makes clear when the operation symbols in question refer to the operations of $\mathfrak{A}$ and when they refer to the operations of $\mathfrak{B}$; so we shall omit such superscripts when no confusion can arise.

We shall need the following theorem, which characterizes when a bijection between the sets of atoms of two complete and atomic Boolean algebras with operators can be extended to an isomorphism.

Theorem 1.1. *Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are complete and atomic Boolean algebras with operators. A bijection φ from the set of atoms in $\mathfrak{A}$ to the set of atoms in $\mathfrak{B}$ can be extended to an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ if and only if the condition*

$$t \le f(r_0, \ldots, r_{n-1}) \quad \textit{if and only if} \quad \varphi(t) \le f(\varphi(r_0), \ldots, \varphi(r_{n-1}))$$

is satisfied for each operator f of rank n and for each sequence of atoms $r_0, \ldots, r_{n-1}, t$ in $\mathfrak{A}$.

Proof. The necessity of the condition is obvious. To establish its sufficiency, suppose that the condition holds. Write X for the set of all atoms in $\mathfrak{A}$, and for each element u in $\mathfrak{A}$, write X_u for the set of atoms in $\mathfrak{A}$ that are below u. In both $\mathfrak{A}$ and $\mathfrak{B}$, each element is the supremum of the set of atoms that it dominates, distinct elements are the suprema of distinct sets of atoms, and every set of atoms has a supremum. One consequence of this observation and the assumption that φ is a bijection between the sets of atoms is that the suprema

$$\sum\{\varphi(t) : t \in X_u\} \qquad \text{and} \qquad \sum\{\varphi(t) : t \in X \sim X_u\}$$

are disjoint and sum to the unit in $\mathfrak{B}$, so

$$\textstyle\sum\{\varphi(t) : t \in X \sim X_u\} = -\bigl(\sum\{\varphi(t) : t \in X_u\}\bigr). \tag{1}$$

A second consequence is that the function ψ from $\mathfrak{A}$ to $\mathfrak{B}$ defined by

$$\psi(u) = \textstyle\sum\{\varphi(t) : t \in X_u\} \tag{2}$$

for each element u in $\mathfrak{A}$ is a bijection from the universe of $\mathfrak{A}$ to the universe of $\mathfrak{B}$.

If Y is an arbitrary set of elements in $\mathfrak{A}$, then the set of atoms below the sum $u = \sum Y$ is the union, over all v in Y, of the set of atoms below v, that is to say,

$$X_u = \textstyle\bigcup_{v\in Y} X_v, \tag{3}$$

so

$$\begin{aligned}\psi(u) = \textstyle\sum\{\varphi(t) : t \in X_u\} &= \textstyle\sum\{\varphi(t) : t \in \bigcup_{v\in Y} X_v\}\\ &= \textstyle\sum_{v\in Y}\sum\{\varphi(t) : t \in X_v\} = \sum_{v\in Y}\psi(v)\end{aligned}$$

by (2) and (3), the general associative law for Boolean addition, and (2) (with v in place of u). Consequently, ψ preserves arbitrary sums. The set of atoms below a complement $-u$ is the complement in X of the set of atoms below u, that is to say,

$$X_{-u} = X \sim X_u, \tag{4}$$

so

$$\begin{aligned}\psi(-u) = \textstyle\sum\{\varphi(t) : t \in X_{-u}\} &= \textstyle\sum\{\varphi(t) : t \in X \sim X_u\}\\ &= -\bigl(\textstyle\sum\{\varphi(t) : t \in X_u\}\bigr) = -\psi(u),\end{aligned}$$

by (2) (with $-u$ in place of u), (4), (1), and (2). Consequently, ψ preserves complements and is therefore a Boolean isomorphism.

It remains to check that ψ preserves each operator. Consider the case of a binary operator $\circ$. Let u and v be elements in $\mathfrak{A}$, and write $w = u \circ v$. It is to be shown that $\psi(w) = \psi(u) \circ \psi(v)$. Because $\mathfrak{A}$ is atomic, we have

$$u = \textstyle\sum X_u, \qquad v = \sum X_v, \qquad w = \sum X_w. \tag{5}$$