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Ricardo Castano-Bernard
Fabrizio Catanese
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Homological Mirror Symmetry and Tropical Geometry

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Introduction

The workshop “Mirror Symmetry and Tropical Geometry” took place in Cetraro, Italy on July 2–8, 2011. The idea was to bring together mathematicians and physicists who worked on both topics or in related areas.

Homological Mirror Symmetry, here abbreviated as HMS, is the area of mathematics revolving around several categorical equivalences connecting symplectic and holomorphic (or algebraic) geometry. This mathematical approach to Mirror Symmetry goes back to the work of Maxim Kontsevich (1993). Further developments of Kontsevich’s program was the subject of many talks at the workshop. This theme is therefore present in several papers of this volume.

Works related to Homological Mirror Symmetry include the paper on HMS for Landau–Ginzburg models by H. Ruddat, the paper of N. Sibilla on HMS for curves, the paper of Kontsevich and Y. Soibelman on complex integrable systems, and the paper by D. Favero, F. Haiden, and L. Katzarkov on the phantom categories which appear in HMS. The variety of methods ranging from homological algebra to delicate questions of symplectic topology and algebraic geometry illustrates the complexity of the subject.

The second topic of the workshop was Tropical Geometry. Roughly speaking, Tropical Geometry studies piecewise-linear objects which appear as certain “degenerations” of the corresponding algebro-geometric objects. The relationship of Tropical Geometry with Mirror Symmetry goes back to the work by Kontsevich and Y. Soibelman (2000) where methods of non-archimedean geometry (in particular, tropical curves) were used for the purposes of Homological Mirror Symmetry. Combined with the subsequent work of Mikhalkin on a “tropical” approach to Gromov–Witten theory and with the work of Gross, Siebert, and several others, Tropical Geometry has become a useful tool for people working in Mirror Symmetry.

On the other hand, “tropical” analogs of many notions of classical symplectic and algebraic geometry are interesting and nontrivial objects by themselves. The paper by G. Mikhalkin and I. Zharkov, which is devoted to the tropical analog of the intermediate Jacobian, is a good illustration of this statement. Methods of tropical geometry are also used in the paper by Kontsevich and Y. Soibelman devoted to the study of Donaldson–Thomas invariants and corresponding wall-crossing formulas.

The volume also contains several papers which are related to the main topics of the workshop in an indirect way. For example, the paper by S. Guillermou and P. Schapira is devoted to the application of the microlocal theory of sheaves developed by the second author jointly with M. Kashiwara to the displaceability problem in symplectic topology. It should be compared with attempts of several mathematicians to describe the Fukaya category (one of the main objects on the "symplectic" side of Homological Mirror Symmetry) in terms of constructible sheaves and corresponding dg-categories.

Two papers are devoted to various aspects of the moduli stacks of bundles. In the paper by O. Ben-Bassat and E. Gasparim the stack of vector bundles on a formal neighborhood of a rational curve in a surface is studied. In the paper by A. Soibelman the "very good" property introduced by Beilinson and Drinfeld in their work on the Geometric Langlands Program is generalized to the case of arbitrary parabolic bundles on a curve and then applied to the additive Deligne–Simpson problem.

A. Neitzke gives a nice review of his joint work with D. Gaiotto and G. Moore on the construction of hyperkähler metrics. Their approach is based on the thermodynamical Bethe Ansatz-type integral equation proposed by them, as well as on the "Kontsevich–Soibelman wall-crossing formulas". There are many interesting and nontrivial analogies between the paper by Neitzke and the paper by Kontsevich and Y. Soibelman in this volume.

S. Gukov and P. Sulkowski propose a way to quantize spectral curves. Then they discuss the relationship of arising "quantum spectral curves" with the topological recursion of Eynard–Orantin as well as with other topics such as A-polynomials of knots.

The paper by M. Kapranov, O. Schiffmann, and E. Vasserot is devoted to the Hall algebra of the "compactified $Spec(\mathbf{Z})$" interpreted as a curve. The "category of vector bundles" on such an object is described in Arakelov terms, as the category of metrized lattices. The (spherical) Hall algebra of this category is a shuffle algebra, similar to Hall algebras of the corresponding categories for "usual" curves. The relations in the algebra are described in terms for the (full) zeta-function.

We believe that the present volume represents a rather complete update about the state of the art in the field, and we hope that it shall become an important reference for graduate students and researchers who want to enter this exciting new field. Papers in this volume represent a tiny portion of the variety of topics discussed at the workshop. In order to give to the reader an idea about the latter we finish the Introduction with the list of talks presented at the Cetraro workshop.

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Lectures

- Mina Aganagic (Berkeley): Knot Homology from Refined Chern–Simons Theory.
- Fedor Bogomolov (NYU): On rationality of the fields of invariants of linear actions for connected nonsemisimple algebraic groups (based on my joint work with Christian Boehning and Hans-Christian Graf von Bohtmer).
- Fabrizio Catanese (Bayreuth): Special Galois coverings and the singular set of the moduli space of curves
- Alexander Efimov (Steklov Institute): Cohomological Hall algebra and Kac's conjecture
- Vladimir Fock (Strasbourg): Integrable systems, dimers and cluster varieties. (Joint work with A Marshakov)
- Kenji Fukaya (Kyoto): Homological Mirror symmetry of toric manifolds
- Alexander Goncharov (Yale): Dimers and cluster integrability
- Mark Gross (UCSD): Examples of stable log maps and tropical geometry
- Sergei Gukov (Caltech): From hyperholomorphic sheaves to quantum group invariants via Langlands duality
- Ilia Itenberg (Strasbourg): Topology of real tropical hypersurfaces
- Mikhail Kapranov (Yale): Arithmetic Hall algebras
- Ludmil Katzarkov (Miami and Vienna): Degenerations and wall crossings
- Viatcheslav Kharlamov (Strasbourg): Anti-symplectic involutions on rational symplectic 4-manifolds
- Maxim Kontsevich (IHES): Integrable systems and canonical bases
- Andrei Losev (ITEP): Homotopical beta-function of the instantonic sigma-model and bosonic string Einstein equation on schemes
- Diego Matessi (U of Milan): Conifold transitions and tropical geometry
- David Morrison (UCSB): Mirror symmetry and non-complete-intersection Calabi–Yau manifolds
- Andy Neitzke (UT Austin): A 2d-4d wall-crossing formula
- Nikita Nekrasov (IHES): Surprises with four dimensional N=2 gauge theories
- Dimitri Orlov (Steklov): Mirror symmetry, B-branes and strange Arnold duality
- Tony Pantev (UPenn): Mirror symmetry and mixed Hodge structures

- Pierre Schapira (Paris VI): Microlocal theory of sheaves and symplectic topology: results and open problems
- Bernd Siebert (Hamburg): Logarithmic Gromov–Witten invariants
- Yan Soibelman (Kansas State): Integrable systems and wall-crossing formulas
- Jake Solomon (Hebrew University): Entropy of Lagrangian submanifolds
- Piotr Sulkowski (Caltech): Quantum curves and topological recursion
- Valerio Toledano Laredo (Northeastern): Yangians, quantum loop algebras and trigonometric connections.
- Ilia Zharkov (Kansas State): Tropical Homology
- Anton Zorich (Rennes): Degeneration of flat versus hyperbolic metric on Riemann surfaces, determinant of Laplacian, and Lyapunov exponents of the Hodge bundle (in collaboration with A. Eskin and M. Kontsevich)

Other Contributions:

- Oren Ben-Bassat (University of Haifa): Deformations of Open Surfaces and their Stacks of Bundles
- Colin Diemer (U of Miami): Circuit Relations and the Secondary Stack
- David Favero (UPenn): Graded matrix factorizations, functor categories, and orbit categories
- Gabriel Kerr (U of Miami): Circuit Relation in the Symplectic Mapping Class Group
- Helge Ruddat (UC Berkeley): Mirror Symmetry partners via vanishing cycles
- Nick Sheridan (MIT): On the Homological Mirror Symmetry conjecture for pairs of pants
- Nicolo Sibilla (Northwestern): Mirror symmetry in dimension one and Fourier–Mukai transform
- Alexander Soibelman (UNC Chapel Hill): The very good property for moduli of parabolic bundles and the additive Deligne–Simpson problem

Contents

Moduli Stacks of Bundles on Local Surfaces

Oren Ben-Bassat and Elizabeth Gasparim

Abstract We give an explicit groupoid presentation of certain stacks of vector bundles on formal neighborhoods of rational curves inside algebraic surfaces. The presentation involves a Möbius type action of an automorphism group on a space of extensions.

1 Introduction

A fundamental question in algebraic geometry is to understand how rational maps on a variety X affect the moduli of vector bundles on X, that is: suppose X and Y birationally equivalent, then what is the relation between the various moduli of vector bundles on X and Y? Here we focus on the case of surfaces, in which case rational maps are obtained by blowing up (possibly singular) points. Suppose $\pi\colon Y \to X$ is the blow up of a point x in X, with $\ell = \pi^{-1}(x)$. Considering pullbacks, one can then study the relative situation of the moduli of vector bundles on X mapping into the moduli of vector bundles on Y. Since π is an isomorphism outside ℓ clearly the heart of the question lies in the geometry of moduli of bundles on a small neighborhood of ℓ. This question was addressed from the point of view of moduli spaces of equivalence classes of vector bundles in [15] for the case when x is a smooth point, and the geometry of the local moduli was used to prove the Atiyah–Jones conjecture for rational surfaces. In this paper we consider the

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moduli *stacks* of vector bundles in formal neighborhoods of ℓ, and give explicit groupoid presentations of such moduli stacks. The stacky point of view, besides clarifying several delicate issues about the local moduli also has the advantage that it generalises to the case of singular surfaces, where ℓ is a line with self-intersection $\ell^2 = -k < -1$. We develop the study of stacks of bundles on (completions of the) local surfaces $Z_k = \mathrm{Tot}(\mathcal{O}(-k))$ and give presentations of certain stacks of rank 2 bundles over these surfaces. The most interesting aspect of these presentations is the "Möbius" transformation (17) discussed in Sect. 2.3.

2 Local Surfaces and Vector Bundles on Them

Notation 1. *In this paper we will work with (associative, commutative, unital) $\mathbb{C}$-algebras. Therefore, affine scheme will mean the spectrum of such an algebra, and all varieties, schemes, and formal schemes are considered over $\mathbb{C}$. We will work over the site of affine schemes or $\mathbb{C}$-algebras with the faithfully flat topology. The schemes we will consider are quasi-compact and quasi-separated. For any positive integer k, we have the algebraic variety*

$$Z_k = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-k)) = Spec_{\mathbb{P}^1}\left(\bigoplus_{i=0}^{\infty} \mathcal{O}_{\mathbb{P}^1}(ik)\right)$$

and $\ell \cong \mathbb{P}^1$ its zero section, so that $\ell^2 = -k$. Let I_ℓ be the sheaf of $\mathcal{O}_{Z_k}$ ideals defining ℓ. We write $Z_k^{(n)}$ for the n^{th} infinitesimal neighborhood of ℓ and $\widehat{Z_k} = Z_k^{(\infty)}$ for the formal neighborhood of ℓ in Z_k. $\widehat{Z_k} = (\ell, lim_n \mathcal{O}_{Z_k}/I_\ell^n)$ is the formal scheme given as the formal completion of Z_k along ℓ. It is a (an inductive or direct) limit in the category of ringed spaces over $\mathbb{P}^1$. There is a presentation

$$Z_k = \left(U \bigsqcup V\right)/\sim,$$

where we will always use the charts $U = \mathbb{C}^2$ with coordinates (z, u), and $V = \mathbb{C}^2$ with coordinates (ξ, v), with $U \cap V = (\mathbb{C} - \{0\}) \times \mathbb{C}$ where the equivalence relation $\sim$ is given by the change of coordinates $(\xi, v) = (z^{-1}, z^k u)$. Note that the zero section ℓ is given in these coordinates by $u = 0$ in the U-chart and $v = 0$ in the V-chart. It is easy to see that $I_\ell \cong \mathcal{O}(k)$. In fact, I_ℓ is the line bundle associated to the divisor $-\ell$ and since $u = \xi^k v$,

$$div(u) = \ell + kf$$

where f is the fiber defined by $\xi = 0$. We similarly have

$$U^{(n)} = Spec(\mathbb{C}[z, u]/(u^{n+1}))$$

and

$$V^{(n)} = Spec(\mathbb{C}[\xi, v]/(v^{n+1})).$$

As above, we have

$$Z_k^{(n)} = \left(U^{(n)} \bigsqcup V^{(n)}\right) / \sim$$

and

$$Z_k^{(\infty)} = \widehat{Z_k} = \left(\hat{U} \bigsqcup \hat{V}\right) / \sim$$

where $\hat{U}$ and $\hat{V}$ are the formal scheme completions of U and V along ℓ.

Remark 1. Unless we explicitly state that n is finite, in each usage of the spaces $Z_k^{(n)}$ we are including the case that $n = \infty$.

These presentations are helpful for describing vector bundles. For instance by the answer to Serre's famous question (proved by Seshadri [22] and in further generality by Quillen [20] and Suslin [23]), $U = Spec(\mathbb{C}[z, u])$ has no non-trivial vector bundles; similarly this is true for $U^{(n)}$ and $\hat{U}$ by Theorem 7 of [10]. In contrast, vector bundles on $Z_k^{(n)}$ were studied on [1–3, 14]. All the schemes we have mentioned up until now are Noetherian and $\widehat{Z_k}$ is a Noetherian formal scheme. If T is an affine scheme such that $Pic(T)$ is trivial then

$$\mathrm{Pic}(\widehat{Z_k} \times T) \simeq \mathrm{Pic}(Z_k^{(n)} \times T) \simeq \mathrm{Pic}(\mathbb{P}^1 \times T) \simeq \mathrm{Pic}(\mathbb{P}^1) \simeq \mathbb{Z};$$

we will use the symbol $\mathscr{O}(j)$ for the line bundle with first Chern class j coming from $\mathbb{P}^1$ in any of these spaces. If E is a rank 2 vector bundle of first Chern class zero on $Z_k^{(n)}$ then the splitting type $j \geq 0$ of E is the integer such that the restriction of E to ℓ is isomorphic to $\mathscr{O}(j) \oplus \mathscr{O}(-j)$. For a vector bundle on $Z_k^{(n)} \times T$ we say that it has constant splitting type j if its splitting type is j over every $t \in T(\mathbb{C})$.

For our explicit presentations of stacks, we will need the following basic results about rank 2 bundles on $Z_k^{(n)}$, which we generalize from [13].

Lemma 1. *Let S be any scheme over $\mathbb{C}$ and E a rank 2 vector bundle on $Z_k^{(n)} \times S$ of constant splitting type $j \geq 0$. Then for any $s \in S(\mathbb{C})$ there is an open subscheme T of S containing s and such that the restriction of E to $Z_k^{(n)} \times T$ has the structure of an extension*

$$0 \to \mathscr{O}(-j) \to E|_{Z_k^{(n)} \times T} \to \mathscr{O}(j) \to 0.$$

Proof. By [12], Theorem 3.3 $E|_{Z_k^{(n)} \times \{s\}}$ can be written as an algebraic extension

$$0 \to \mathscr{O}(-j) \to E|_{Z_k^{(n)} \times \{s\}} \to \mathscr{O}(j) \to 0$$

where $j > 0$. Consider the leftmost injective map as a nowhere vanishing element of the space of global sections $H^0(Z_k^{(n)} \times \{s\}, E|_{Z_k^{(n)} \times \{s\}} \otimes \mathcal{O}(j))$. The pushforward $\pi_{S*}(E|_{\ell \times S} \otimes \mathcal{O}(j))$ is a vector bundle on S and we have chosen a non-zero point in the fiber over s. Choose T' open in S and containing s and an extension of the above section to an element of

$$H^0(T', (\pi_{S*}(E|_{\ell \times S} \otimes \mathcal{O}(j)))|_{T'}) = H^0(\ell \times T', E|_{\ell \times T'} \otimes \mathcal{O}(j))$$

such that this chosen global extension does not vanish on $\ell \times T'$ and hence does not vanish on $Z_k^{(n)} \times T'$, and passes through our chosen element of the fiber over s. This gives us an injective map of constant rank leading to a short exact sequence on $Z_k^{(n)} \times T'$

$$0 \to \mathcal{O}(-j) \to E_{Z_k^{(n)} \times T'} \to L \to 0$$

where L is a line bundle on $Z_k^{(n)} \times T'$ isomorphic to $\mathcal{O}(j)$ over every geometric point of T'. By the see–saw principle there is a T open in T' and containing S such that the restriction of L to $Z_k^{(n)} \times T'$ is isomorphic to $\mathcal{O}(j)$. Therefore the restriction of the above short exact sequence to $Z_k^{(n)} \times T$ gives the desired result.

Remark 2. An alternate approach to the Lemma 1 is to start with any vector bundle which has nowhere zero map of $\mathcal{O}(-j)$ to E over $\ell \times T$ for some affine scheme T and use the fact that $H^1(\ell \times T, I^m_{\ell \times T}) = 0$ for $m > 0$ to extend this map order by order to a map over $Z_k^{(n)} \times T$ which must be nowhere zero.

Lemma 2. *Let T be an affine scheme and E an algebraic extension of $\mathcal{O}_{Z_k^{(n)} \times T}$ modules*

$$0 \to \mathcal{O}(-j) \to E \to \mathcal{O}(j) \to 0,$$

over $Z_k^{(n)} \times T$ which splits over $\ell \times T$ for $j \geq 0$ then, in the chosen coordinates E can be described by a transition matrix of the form

$$\begin{pmatrix} z^j & p \\ 0 & z^{-j} \end{pmatrix}$$

on $(U^{(n)} \cap V^{(n)}) \times T$, where

$$p = \sum_{i=1}^{min(\lfloor (2j-2)/k \rfloor, n)} \sum_{l=ki-j+1}^{j-1} p_{i,l} z^l u^i. \tag{1}$$

and $p_{i.l} \in \mathcal{O}(T)$.

Proof. A Čech cohomology calculation (performed in Theorem 3.3 of [12]) shows that

$$\mathrm{Ext}^1_{Z_k^{(n)}}(\mathscr{O}(j),\mathscr{O}(-j)) = \frac{\mathbb{C}[z,z^{-1},u]/(u^{n+1})}{z^{-j}\,\mathbb{C}[z^{-1},z^k u]/((z^k u)^{n+1}) + z^j\,\mathbb{C}[z,u]/(u^{n+1})}$$

by flat base change for the diagram

$$\begin{array}{ccc} Z_k^{(n)} \times T & \xrightarrow{\pi_T} & T \\ \downarrow{\scriptstyle \pi_{Z_k^{(n)}}} & & \downarrow \\ Z_k^{(n)} & \longrightarrow & \{\cdot\} \end{array}$$

and the Leray spectral sequence for $\pi_{Z_k^{(n)}}$ we have

$$\begin{aligned} \mathrm{Ext}^1_{Z_k^{(n)}\times T}(\pi^*_{Z_k^{(n)}}\mathscr{O}(j),\pi^*_{Z_k^{(n)}}\mathscr{O}(-j)) &= H^1(Z_k^{(n)}\times T,\pi^*_{Z_k^{(n)}}(\mathscr{O}(-2j))) \\ &= H^0(T,R^1\pi_{T*}\pi^*_{Z_k^{(n)}}(\mathscr{O}(-2j))) \\ &= H^0(T,\mathscr{O}_T\otimes H^1(Z_k^{(n)},\mathscr{O}(-2j))) \\ &= H^0(T,\mathscr{O}_T\otimes \ Ext^1_{Z_k^{(n)}}(\mathscr{O}(j),\mathscr{O}(-j))). \quad (2) \end{aligned}$$

Remark 3. As a consequence of the above two Lemmas 2 and 1, we see that any rank 2 vector bundle on $Z_k^{(n)} \times T$ (or $\widehat{Z_k} \times T$) takes a special form locally on T and in this form it is clearly the restriction (completion) of a vector bundle on Z_k. The theorem on formal functions implies then that

$$\widehat{\mathrm{Ext}^i_{Z_k\times T}(V,W)} \cong \mathrm{Ext}^i_{\widehat{Z_k}\times T}(V,W).$$

Notation 2. *Let*

$$N_{j,k}^{(n)} = \{(i,l)|ki-j+1\le l\le j-1 \ \ and \ \ 1\le i\le min(\lfloor(2j-2)/k\rfloor,n)\}.$$

Consider the algebraic variety over $\mathbb{C}$

$$W_{j,k}^{(n)} = Spec\Big(\mathbb{C}[\ \ p_{i,l} \ \ | \ \ (i,l)\in N_{j,k}^{(n)}]\Big). \quad (3)$$

For any fixed j,k it remains finite dimensional even for $n=\infty$. If we pass to the $\mathbb{C}$ points then we get

$$W_{j,k}^{(n)}(\mathbb{C}) = \{p\in \mathrm{Ext}^1_{Z_k^{(n)}}(\mathscr{O}(j),\mathscr{O}(-j)) \ \ | \ \ p|_\ell = 0\}.$$

Let

$$R_{j,k}^{(n)} = \bigoplus_{i=1}^{\lfloor (2j-2)/k \rfloor} \bigoplus_{l=ki-j+1}^{j-1} \mathbb{C} z^l u^i \subset \mathscr{O}(U^{(n)} \cap V^{(n)}). \tag{4}$$

of course $R_{j,k}^{(n)}$ is the set of $\mathbb{C}$ points of $W_{j,k}^{(n)}$ but we distinguish them because of the different notions of automorphisms of $R_{j,k}^{(n)}$ and $W_{j,k}^{(n)}$.

Remark 4. Note that in our chosen form of transition matrix from Eq. (1) we have explicitly chosen $p \in R_{j,k}^{(n)}$.

Definition 1. Consider the open cover $\{U^{(n)} \times W_{j,k}^{(n)}, V^{(n)} \times W_{j,k}^{(n)}\}$ of $Z_k^{(n)} \times W_{j,k}^{(n)}$. We define $\mathbb{E}$, sometimes called the big bundle, to be the bundle

$$\begin{array}{c} \mathbb{E} \\ \downarrow \\ Z_k^{(n)} \times W_{j,k}^{(n)} \end{array}$$

on $Z_k \times W_{j,k}^{(n)}$ defined by transition matrix

$$\begin{pmatrix} z^j & p \\ 0 & z^{-j} \end{pmatrix} \in H^0((U^{(n)} \cap V^{(n)}) \times W_{j,k}^{(n)}, \mathscr{A}ut(\mathscr{O}^{\oplus 2})).$$

Let T be an affine scheme and p a morphism from T to $W_{j,k}^{(n)}$. We denote by E_p the bundle (also described in Lemma 2) given by the pullback $(\mathrm{id}_{Z_k^{(n)}}, p)^* \mathbb{E}$ of $\mathbb{E}$ via the map

$$Z_k^{(n)} \times T \overset{(\mathrm{id}_{Z_k^{(n)}}, p)}{\to} Z_k^{(n)} \times W_{j,k}^{(n)}.$$

Lemma 3 ([4, Thm. 4.9]). *On the first formal neighborhood $Z_k^{(1)}$, two bundles E and E' with transition matrices*

$$\begin{pmatrix} z^j & p_1 \\ 0 & z^{-j} \end{pmatrix} \quad and \quad \begin{pmatrix} z^j & p'_1 \\ 0 & z^{-j} \end{pmatrix}$$

respectively are isomorphic if and only if $p'_1 = \lambda p_1$ for some $\lambda \in \mathbb{C}^\times$.

Remark 5. It follows from this lemma that the coarse moduli space of bundles on $Z_k^{(1)}$ coming from non-trivial extensions of $\mathscr{O}(j)$ by $\mathscr{O}(-j)$ is isomorphic to $\mathbb{P}^{2j-k-2}$.

Example 1. On higher infinitesimal neighborhoods we need to consider far more relations among extension classes then just projectivisation to obtain the moduli of bundles. The simplest of such examples occurs in the case when $k = 1$ and $j = 2$, so that our extension classes have the form

$$p = (p_{1,0} + p_{1,1}z)u + p_{2,1}zu^2.$$

The set of equivalence classes of vector bundles is then $\mathbb{C}^3/\sim$ where the equivalence relation is generated by

$$(p_{1,0}, p_{1,1}, p_{2,1}) \sim (\lambda p_{1,0}, \lambda p_{1,1}, \lambda p'_{2,1}) \;\; if \;\; (p_{1,0}, p_{1,1}) \neq (0,0), \;\; \lambda \neq 0,$$
$$(0,0,p_{2,1}) \sim (0,0,\lambda p_{2,1}), \;\; \lambda \neq 0.$$

Note that $p'_{2,1}$ is does not depend on p, and that the quotient topology makes the entire space the only open neighborhood of the split bundle, which is the image of the origin in $\mathbb{C}^3$.

2.1 Stacks of Vector Bundles

We now define the stack of bundles $\mathfrak{M}_j(Z_k^{(n)})$, the main object we seek to understand in this article.

Definition 2.

$$\mathfrak{M}_j(Z_k^{(n)})\colon \text{Schemes} \to \text{Groupoids}$$

given by

$$T \mapsto \text{Hom}(T, \mathfrak{M}_j(Z_k^{(n)}))$$

where

$$\text{ob}(\text{Hom}(T, \mathfrak{M}_j(Z_k^{(n)})) = \{\text{rank 2 vector bundles on } Z_k^{(n)} \times T \text{ which have splitting type j and first Chern class 0 for every restriction to } Z_k^{(n)} \times \{t\}, t \in T(\mathbb{C})\} \quad (5)$$

and

$$\text{mor}(\text{Hom}(T, \mathfrak{M}_j(Z_k^{(n)}))(V_1, V_2) = \text{Isom}(V_1, V_2).$$

This is a stack [17] with respect to the faithfully flat topology on schemes ($\mathbb{C}$-algebras). Notice that there is automatically a universal bundle $\mathscr{E}$ over $Z_k^{(n)} \times \mathfrak{M}_j(Z_k^{(n)})$. We can similarly define the stack $\mathfrak{M}_j(\widehat{Z_k})$. We similarly have the stacks $\mathfrak{M}(Z_k^{(n)})$ of bundles where we drop the condition on splitting type.

There is an inverse (or projective) system of stacks of finite type over $\mathbb{C}$:

$$\cdots \to \mathfrak{M}_j(Z_k^{(3)}) \to \mathfrak{M}_j(Z_k^{(2)}) \to \mathfrak{M}_j(Z_k^{(1)}) \to \mathfrak{M}_j(Z_k^{(0)}) = \mathfrak{M}_j(\mathbb{P}^1) \tag{6}$$

whose inverse limit in the category of algebraic stacks is $\mathfrak{M}_j(\widehat{Z_k})$. Alternatively we can consider the inverse system $\mathfrak{M}_j(Z_k^{(\bullet)})$ to be an pro-stack of pro-finite type. This type of approximation is studied in [21]. It seems difficult to compute invariants of the stacks $\mathfrak{M}_j(Z_k^{(n)})$ using only the definition above so we will find a more explicit description below.

2.2 The Structure of Vector Bundle Isomorphisms

Consider the bundles E_p defined in Definition 1. There is a exact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}(E_p, E_{p'}) \to \mathrm{End}(\mathscr{O}(-j) \oplus \mathscr{O}(j)) \\ &\longrightarrow \mathrm{Ext}^1(\mathscr{O}(-j) \oplus \mathscr{O}(j), \mathscr{O}(-j) \oplus \mathscr{O}(j)) \to \mathrm{Ext}^1(E_p, E_{p'}) \to 0. \end{aligned} \tag{7}$$

We now explain the structure of isomorphisms between families of bundles coming from extensions by constructing an explicit splitting for the first non-trivial map in this sequence. If the bundles E_p and $E_{p'}$ on $Z_k^{(n)} \times T$, given by maps

$$p, p' : T \to R_{j,k}^{(n)}$$

are isomorphic (see Eq. (4)) then necessarily they have the same splitting type, and in such case we can represent them by transition matrices on

$$(U^{(n)} \cap V^{(n)}) \times T$$

by $\begin{pmatrix} z^j & p \\ 0 & z^{-j} \end{pmatrix}$ and $\begin{pmatrix} z^j & p' \\ 0 & z^{-j} \end{pmatrix}$ respectively. An isomorphism between E_p and $E_{p'}$ is given by a pair of invertible matrices

$$A = \begin{pmatrix} a_U & b_U \\ c_U & d_U \end{pmatrix}$$

regular on $U^{(n)} \times T$ and

$$B = \begin{pmatrix} a_V & b_V \\ c_V & d_V \end{pmatrix}$$

regular on $V^{(n)} \times T$, such that:

$$B \begin{pmatrix} z^j & p \\ 0 & z^{-j} \end{pmatrix} = \begin{pmatrix} z^j & p' \\ 0 & z^{-j} \end{pmatrix} A, \tag{8}$$

or equivalently

$$\begin{aligned} B &= \begin{pmatrix} z^j & p' \\ 0 & z^{-j} \end{pmatrix} A \begin{pmatrix} z^{-j} & -p \\ 0 & z^j \end{pmatrix} \\ &= \begin{pmatrix} a_U + z^{-j} p' c_U & z^{2j} b_U + z^j (p' d_U - a_U p) - pp' c_U \\ z^{-2j} c_U & d_U - z^{-j} p c_U \end{pmatrix}. \end{aligned} \tag{9}$$

Definition 3. We use the notation Y^+ to denote the terms in $Y \in \mathscr{O}((U^{(n)} \cap V^{(n)}) \times T)$ that are not regular on $V^{(n)} \times T$ and $Y^{+,\geq 2j}$ denotes the terms in Y that are not regular on $V^{(n)} \times T$ and have power of z greater than or equal to $2j$.

Lemma 4. *Suppose that* $j > 0$. *Then any isomorphism* (A, B) *between* E_p *and* $E_{p'}$ *on* $Z_k^{(n)} \times T$ *has the form*

$$(A, B) = (M_U, M_V) + (\Phi_U(M), \Phi_V(M)) \tag{10}$$

where

$$\begin{aligned} M &= (M_U, M_V) \in \mathrm{Aut}_{Z_k^{(n)} \times T}(\mathscr{O}(j) \oplus \mathscr{O}(-j)). \\ M_U &= \begin{pmatrix} \underline{a} & \underline{b}_U \\ \underline{c}_U & \underline{d} \end{pmatrix} \end{aligned}$$

and

$$\Phi_U(M) = \begin{pmatrix} -(z^{-j} p' \underline{c}_U)^+ & -z^{-2j} \left(z^j (p' \underline{d} - \underline{a} p) - pp' \underline{c}_U\right)^{+,\geq 2j} \\ 0 & (z^{-j} p \underline{c}_U)^+ \end{pmatrix}$$

depends only on p, p' *and* M *and satisfies*

$$[p' \underline{d} - \underline{a} p - z^{-j} pp' \underline{c}_U] = 0 \in \mathrm{Ext}^1_{Z_k^{(n)} \times T}(\mathscr{O}(j), \mathscr{O}(-j)). \tag{11}$$

Proof. First suppose that such an isomorphism exists, between E_p and $E_{p'}$. Then we have

$$\begin{pmatrix} z^j & p' \\ 0 & z^{-j} \end{pmatrix} A - B \begin{pmatrix} z^j & p \\ 0 & z^{-j} \end{pmatrix} = 0. \tag{12}$$

The left hand side comes out to be

$$\begin{pmatrix} p'c_U + (a_U - a_V)z^j & d_U p' - a_V p + z^j b_U - z^{-j} b_V \\ c_U z^{-j} - c_V z^j & z^{-j}(d_U - d_V) - c_V p \end{pmatrix}. \tag{13}$$

The lower left corner of (13) implies first of all that c must be a section $\underline{c}$ of $\mathcal{O}(2j)$. We need to arrange for the vanishing of all terms in (13). Therefore, we need to solve the equations:

$$\begin{aligned} a_U - a_V &= -z^{-j} p' \underline{c}_U \\ z^j b_U - z^{-j} b_V &= -d_U p' + a_V p \\ d_U - d_V &= z^j \underline{c}_V p. \end{aligned}$$

Becasue $H^1(Z_k^{(n)} \times T, \mathcal{O})$ vanishes, the first and third equations have solutions which are unique up to global functions. Let

$$a_U = \underline{a} - (z^{-j} p' \underline{c}_U)^+$$

and

$$d_U = \underline{d} + (z^j \underline{c}_V p)^+.$$

These solve the first and third equation. If we substitute into the second equation, it reads

$$\begin{aligned} z^j b_U - z^{-j} b_V &= -(z^j \underline{c}_V p)^+ p' + (-(z^{-j} p' \underline{c}_U)^+ + z^{-j} p' \underline{c}_U) p - \underline{d} p' + \underline{a} p \\ &= -\underline{d} p' + \underline{a} p + z^{-j} pp' \underline{c}_U. \end{aligned} \tag{14}$$

This implies that

$$[p'\underline{d} - \underline{a}p - z^{-j} pp' \underline{c}_U] = 0 \in \operatorname{Ext}^1_{Z_k^{(n)} \times T}(\mathcal{O}(j), \mathcal{O}(-j)).$$

Conversely, suppose that these conditions are satisfied by some $\underline{a}$, $\underline{d}$, $\underline{c}$, p, and p', let us record the general form of an element of $\operatorname{Isom}_{Z_k^{(n)} \times T}(E_p, E_{p'})$. It remains only to determine the expression for b_U. By the assumptions we already know that

$$\left(z^j (p'\underline{d} - \underline{a}p) - pp' \underline{c}_U\right)^{+,<2j}$$

is regular on $V^{(n)} \times T$. Hence

$$b_U = \underline{b}_U - z^{-2j} \left(z^j (p'\underline{d} - \underline{a}p) - pp'\underline{c}_U\right)^{+,\geq 2j}.$$

Finally, since u divides p and p', we know that A is invertible if and only if M_U is and therefore the isomorphism (A, B) is invertible if and only if the automorphism M is invertible.

Remark 6. We conclude that the expression of the element (A, B) of $\mathrm{Hom}(E_p, E_{p'})$ under the decomposition (43)

$$\mathrm{Hom}(E_p, E_{p'}) = \mathrm{Hom}(\mathcal{O}(j), \mathcal{O}(-j)) \oplus \phi(\ker(d_1^{1,-1})) \oplus \psi(\ker(d_2^{0,0}))$$

from the appendix is satisfied if we take $\underline{b} \in \mathrm{Hom}(\mathcal{O}(j), \mathcal{O}(-j))$,

$$\psi_U(c) = \begin{pmatrix} -(z^{-j} p'\underline{c}_U)^+ & z^{-2j} (pp'\underline{c}_U)^{+,\geq 2j} \\ \underline{c}_U & (z^{-j} p\underline{c}_U)^+ \end{pmatrix}$$

and

$$\phi_U(\underline{a}, \underline{d}) = \begin{pmatrix} \underline{a} & -z^{-2j} \left(z^j (p'\underline{d} - \underline{a}p)\right)^{+,\geq 2j} \\ 0 & \underline{d} \end{pmatrix}.$$

2.3 Bundle Isomorphism Viewed as an Equivalence Relation

Although we have worked out the structure of the space of isomorphisms between two given bundles, this does not yet give a criterion for when two bundles are isomorphic nor does it provide any understanding of the equivalence relation on $W_{j,k}^{(n)}$ given by isomorphisms of vector bundles. We show that there are algebraic groups $G_{j,k}^{(n)}$ acting on $W_{j,k}^{(n)}$ so that the orbits of this action are identified with the equivalence classes. This action (17) takes on the familiar form of a Möbius transformation. Lange studied in [16] (see also Drézet [11]) the question of universal bundles over the projectivized space of extensions. In a specific example we study here a more difficult problem, the difference being that we do not remove the origin and we consider all vector bundle isomorphisms, not just those that correspond to scaling the extension. First we need to define the structure of a scheme on the sets $\mathrm{Aut}_{Z_k^{(n)}}(\mathcal{O}(j) \oplus \mathcal{O}(-j))$ for n finite.

Definition 4. Consider the functors from schemes to sets given by

$$T \mapsto \mathrm{Aut}_{Z_k^{(n)} \times T}(\mathcal{O}(j) \oplus \mathcal{O}(-j)).$$

These functors are $\mathbb{C}$-groups (sheaves of groups in the faithfully flat topology on schemes) and are easily seen to be representable by reduced schemes. These schemes are in fact affine, being defined inside the finite dimensional affine space

$$\mathbb{E}\mathrm{nd}_{Z_k^{(n)}}(\mathscr{O}(j) \oplus \mathscr{O}(-j))$$

defined with coordinates as in Remark 8 as the complement of the pre-image of 0 by the morphism

$$det_0 : \mathbb{E}\mathrm{nd}_{Z_k^{(n)}}(\mathscr{O}(j) \oplus \mathscr{O}(-j)) \to \mathscr{O}(Z_k^{(n)}) \to Spec(\mathbb{C}[s]).$$

sending s to the restriction of the determinant to ℓ. When we pass to $\mathbb{C}$ points we get the standard determinant followed by restriction to ℓ

$$det_0 : \mathrm{End}_{Z_k^{(n)}}(\mathscr{O}(j) \oplus \mathscr{O}(-j)) \to \mathscr{O}(Z_k^{(n)}) \to \mathscr{O}(Z_k^{(0)}) = \mathbb{C}.$$

We denote these finite dimensional algebraic groups by $G_{j,k}^{(n)}$. These form a directed system of $\mathbb{C}$-spaces (sheaf of sets for the faithfully flat topology on the category of $\mathbb{C}$-algebras) and their direct limit as a $\mathbb{C}$-space (see [9] for this yoga) is representable by an infinite dimensional algebraic variety,

$$\widetilde{G_{j,k}} = G_{j,k}^{(\infty)}$$

which has $\mathrm{Aut}_{Z_k^{(\infty)}}(\mathscr{O}(j) \oplus \mathscr{O}(-j))$ as its underlying set of $\mathbb{C}$-points. In fact, $\widetilde{G_{j,k}}$ is an infinite-dimensional algebraic group. The sequence $G_{j,k}^{(\bullet)}$

$$\cdots \to G_{j,k}^{(3)} \to G_{j,k}^{(2)} \to G_{j,k}^{(1)} \to G_{j,k}^{(0)} = \mathrm{Aut}_{\mathbb{P}^1}(\mathscr{O}(j) \oplus \mathscr{O}(-j)) \tag{15}$$

is an pro-finite-type pro-scheme. We often write elements of $\mathrm{Hom}(T, G_{j,k}^{(n)})$ as matrices.

Consider the following direct sum decomposition of the vector space of functions

$$\mathscr{O}_{Z_k^{(n)}}(U^{(n)} \cap V^{(n)}) = \mathscr{O}_{Z_k^{(n)}}(U^{(n)} \cap V^{(n)})^{\succ} \oplus \mathscr{O}_{Z_k^{(n)}}(U^{(n)} \cap V^{(n)})_{good} \oplus \mathscr{O}_{Z_k^{(n)}}(U^{(n)} \cap V^{(n)})^{\prec}$$

where the sector named "good" corresponds to the terms appearing in Lemma 1, and also

$$z^j \mathscr{O}_{Z_k^{(n)}}(U^{(n)} \cap V^{(n)})^{\prec} \subset \mathscr{O}(V^{(n)})$$

and

$$z^{-j}\mathscr{O}_{Z_k^{(n)}}(U^{(n)}\cap V^{(n)})^{\succ} \subset \mathscr{O}(U^{(n)})$$
$$q - q_{good} = q^{\succ} + q^{\prec}.$$

As in Eq. (8) we write elements of

$$\mathrm{Hom}(T, G_{j,k}^{(n)}) \subset H^0(Z_k^{(n)} \times T, \mathscr{O}^{\oplus 2} \oplus \mathscr{O}(2j) \oplus \mathscr{O}(-2j))$$

in the form

$$g = \begin{pmatrix} \underline{a} & \underline{b} \\ \underline{c} & \underline{d} \end{pmatrix}. \tag{16}$$

with $\underline{b} = (\underline{b}_U, \underline{b}_V)$ and $\underline{b}_U$ holomorphic on $U^{(n)} \times T$, etc. First of all notice that the group $\mathrm{Hom}(T, G_{j,k}^{(n)})$ acts on the functions p on $U^{(n)} \cap V^{(n)} \times T$ which vanish on the zero section by the formula

$$gp = \frac{\underline{a}p - z^j\underline{b}_U}{\underline{d} - z^{-j}p\underline{c}_U}. \tag{17}$$

A special case (where $\underline{b}$ and $\underline{c}$ are taken to be zero) of this action was observed for general varieties and bundles in [11]. For n finite, such functions vanishing on ℓ belong to $u\mathbb{C}[z, z^{-1}][u]/(u^{n+1})$, in the case $n = \infty$ such functions belong to $u\mathbb{C}[z, z^{-1}][[u]]$. The action $p \mapsto gp$ does not preserve the finite dimensional space $R_{j,k}^{(n)}$ which was written in (4). This means that we need to somehow correct the morphism $(g, p) \mapsto gp$. This will happen in the next definition.

Definition 5. Define a morphism

$$G_{j,k}^{(n)} \times R_{j,k}^{(n)} \to R_{j,k}^{(n)}$$

by

$$\begin{aligned}(g, p) \mapsto g \bullet p &= \frac{\underline{a}p - z^j\underline{b}_U}{\underline{d} - z^{-j}p\underline{c}_U} - \left(\frac{\underline{a}p - z^j\underline{b}_U}{\underline{d} - z^{-j}p\underline{c}_U}\right)^{\succ} - \left(\frac{\underline{a}p - z^j\underline{b}_U}{\underline{d} - z^{-j}p\underline{c}_U}\right)^{\prec}. \\ &= \left(\frac{\underline{a}p - z^j\underline{b}_U}{\underline{d} - z^{-j}p\underline{c}_U}\right)_{good}\end{aligned} \tag{18}$$

This morphism will become one of the structure maps of a groupoid (see Eq. (40)). It is not the action of a group.

Consider

$$A_g(p) = \begin{pmatrix} \underline{a} - (z^{-j}p\underline{c}_U)^+ & \underline{b}_U - z^{-2j}\left(z^j((g \bullet p)\underline{d} - \underline{a}p) - p(g \bullet p)\underline{c}_U\right)^{+,\geq 2j} \\ \underline{c}_U & \underline{d} + (z^{-j}\underline{c}_U(g \bullet p))^+ \end{pmatrix} \tag{19}$$

and

$$B_g(p) = \begin{pmatrix} \underline{a} + (z^{-j}p\underline{c}_U)^+ & \underline{b}_V + \left(z^j((g \bullet p)\underline{d} - \underline{a}p) - p(g \bullet p)\underline{c}_U\right)^{+,<2j} \\ \underline{c}_V & \underline{d} - (z^{-j}\underline{c}_U(g \bullet p))^+ \end{pmatrix}. \tag{20}$$

They are regular over $U^{(n)} \times T$ and $V^{(n)} \times T$ respectively because they satisfy $(A_g(p), B_g(p)) = (M_U, M_V)$ from Lemma 4 in the case that $p' = g \bullet p$. That is to say, they satisfy

$$B_g(p)\begin{pmatrix} z^j & p \\ 0 & z^{-j} \end{pmatrix} = \begin{pmatrix} z^j & g \bullet p \\ 0 & z^{-j} \end{pmatrix} A_g(p) \tag{21}$$

and so the pair $(A_g(p), B_g(p))$ provides an isomorphism between E_p and $E_{g \bullet p}$. We have shown the following Lemma.

Lemma 5. *There is a morphism*

$$\begin{aligned} G_{j,k}^{(n)} \times R_{j,k}^{(n)} &\to R_{j,k}^{(n)} \\ (g, p) &\mapsto g \bullet p \end{aligned} \tag{22}$$

such that for two bundles E_p and $E_{p'}$ of constant splitting type j,

$$\begin{aligned} \mathrm{Isom}_{Z_k^{(n)} \times T}(E_p, E_{p'}) &= \{g \in \mathrm{Hom}(T, G_{j,k}^{(n)}) \mid g \bullet p = p'\} \\ &= \{g \in \mathrm{Hom}(T, G_{j,k}^{(n)}) \mid \mathit{11} \text{ is satisfied}\}. \end{aligned} \tag{23}$$

□

Consider the isomorphism

$$(A_{g_1}(g_2 \bullet p)A_{g_2}(p), B_{g_1}(g_2 \bullet p)B_{g_2}(p))$$

between E_p and $E_{g_1 \bullet (g_2 \bullet p)}$. In Lemma 4, we defined an element

$$g_1 \bullet_p g_2 \in G_{j,k}^{(n)}(\mathbb{C})$$

such that this isomorphism equals $(A_{g_1 \bullet_p g_2}, B_{g_1 \bullet_p g_2})$. Similarly, the isomorphism $(A_g(p)^{-1}, B_g(p)^{-1})$ between $E_{g \bullet p}$ and E_p corresponds to a an element

$$g^{(-1)_p} \in G_{j,k}^{(n)}(\mathbb{C}). \tag{24}$$

From here it is clear (since both $A_{e_{G_{j,k}^{(n)}}}(p)$ and $B_{e_{G_{j,k}^{(n)}}}(p)$ are the identity matrix) that

$$g \bullet_p g^{(-1)_p} = e_{G_{j,k}^{(n)}} = g^{(-1)_p} \bullet_p g. \tag{25}$$

Definition 6. Define $g_1 \bullet_p g_2$ and $g^{(-1)_p}$ to be the elements of $G_{j,k}^{(n)}(\mathbb{C})$ corresponding via Lemma 4 to the isomorphisms $(A_{g_1}(g_2 \bullet p)A_{g_2}(p), B_{g_1}(g_2 \bullet p)B_{g_2}(p))$ and $(A_g(p)^{-1}, B_g(p)^{-1})$ described above.

The elements $g_1 \bullet_p g_2$ vary algebraically with g_1 and g_2 and give a morphism of schemes

$$G_{j,k}^{(n)} \times G_{j,k}^{(n)} \times W_{j,k}^{(n)} \to G_{j,k}^{(n)}$$
$$(g_1, g_2, p) \mapsto g_1 \bullet_p g_2.$$

The restriction to $p = 0$ in $W_{j,k}^{(n)}$ gives us back the standard multiplication but in general this structure does depend on p.

Therefore by definition we have

$$B_{g_1}(g_2 \bullet p)B_{g_2}(p) = B_{g_1 \bullet_p g_2}(p). \tag{26}$$

(and also $A_{g_1}(g_2 \bullet p)A_{g_2}(p) = A_{g_1 \bullet_p g_2}(p)$). An immediate consequence of this together with (21) is

$$g_1 \bullet (g_2 \bullet p) = (g_1 \bullet_p g_2) \bullet p, \tag{27}$$

and we also have

$$\begin{aligned} B_{(g_1 \bullet_{(g_3 \bullet p)} g_2) \bullet_p g_3}(p) &= B_{g_1 \bullet_{(g_3 \bullet p)} g_2}(g_3 \bullet p)B_{g_3}(p) \\ &= B_{g_1}(g_2 \bullet (g_3 \bullet p))B_{g_2}(g_3 \bullet p)B_{g_3}(p) \\ &= B_{g_1}(g_2 \bullet (g_3 \bullet p))B_{g_2 \bullet_p g_3}(p) = B_{g_1 \bullet_p (g_2 \bullet_p g_3)}(p) \end{aligned} \tag{28}$$

and similarly for $A_g(p)$. Because every isomorphism (A, B) which takes one of our chosen transition matrices corresponding to a bundle E_p to another transition matrix of the same form corresponds (7) to a unique $g \in \mathrm{Hom}(T, G_{j,k}^{(n)})$ we conclude that

$$(g_1 \bullet_{(g_3 \bullet p)} g_2) \bullet_p g_3 = g_1 \bullet_p (g_2 \bullet_p g_3). \tag{29}$$

This will be used to verify the associativity of the groupoid structure. A direct inspection of (18), (19) and (20) shows that identity matrix $e_{G_{j,k}^{(n)}}$ satisfies

$$e_{G_{j,k}^{(n)}} \bullet p = p \tag{30}$$

for any p and corresponds to the identity map from E_p to itself. Therefore we of course have

$$e_{G^{(n)}_{j,k}} \bullet_p g = g = g \bullet_p e_{G^{(n)}_{j,k}} \tag{31}$$

for any p.

3 An Explicit Groupoid in Schemes

In this section we describe an explicit groupoid in schemes and show that its associated stack is isomorphic to the stack of rank 2 vector bundles of splitting type j and first Chern class 0 on $Z_k^{(n)}$.

3.1 Review of Groupoids in Schemes and Their Sheaf Theory

We begin with a review of the definition of a groupoid in schemes and the notion of a sheaf on a groupoid in schemes. Recall that a groupoid

$$\mathscr{G} = (A, R, s, t, m, e, \iota)$$

in schemes consists of schemes A (the atlas) and R (the relations), morphisms s, t, m, e, ι

$$\begin{gathered} R \underset{s}{\overset{t}{\rightrightarrows}} A, \qquad R \xleftarrow{\;e\;} A \\ R_t \times_{A\ s} R \xrightarrow{\;m\;} R \end{gathered} \tag{32}$$

and

$$R \xrightarrow{\;\iota\;} R$$

which satisfy some conditions which we write below. Here

$$R_t \times_{A\ s} R = \{(r_1, r_2) \in R \times R | t(r_1) = s(r_2)\}.$$

Let p_1, p_2 be the first and second projections

$$R_t \times_{A\ s} R \overset{p_1, p_2}{\longrightarrow} R$$

and let Δ be the diagonal

$$R \times R \xleftarrow{\Delta} R.$$

The morphisms then must satisfy

$$m \circ (m, \mathrm{id}_R) = m \circ (\mathrm{id}_R, m) \tag{33}$$

on all composable elements of $R \times R \times R$,

$$t \circ m = t \circ p_2, \quad s \circ m = s \circ p_1 \tag{34}$$

on all composable elements of $R \times R$

$$m \circ (\iota, \mathrm{id}_R) \circ \Delta = e \circ s, \quad m \circ (\mathrm{id}_R, \iota) \circ \Delta = e \circ s \tag{35}$$

on R, and also

$$m \circ (\mathrm{id}_R, e \circ t) \circ \Delta = \mathrm{id}_R, \quad m \circ (e \circ s, \mathrm{id}_R) \circ \Delta = \mathrm{id}_R \tag{36}$$

on R. Notice that for any scheme S that by taking the set of morphisms of schemes from S into R and A one gets a pair of sets and these naturally form a groupoid in sets using the obvious maps. We denote this groupoid in sets by

$$\mathrm{Hom}(S, \mathscr{G}).$$

A (coherent/locally free of rank r) sheaf of modules on the groupoid consists of a (coherent/locally free of rank r) sheaf $\mathscr{S}$ of $\mathscr{O}_A$ modules on A together with an isomorphism f of sheaves of $\mathscr{O}_R$ modules over R

$$f : s^*\mathscr{S} \to t^*\mathscr{S}$$

which satisfies

$$p_2^* f \circ p_1^* f = m^* f \tag{37}$$

and

$$e^* f = \mathrm{id}. \tag{38}$$

To make sense of this equality, one must use the identities

$$s \circ p_1 = s \circ m, \quad \text{and} \quad t \circ p_2 = t \circ m.$$

3.2 Stacks from Groupoids

Let $\mathscr{G} = (A, R, s, t, m, e, \iota)$ be a groupoid in schemes.

We associate to it a stack $[\mathscr{G}]$ defined as the stack on the fppf site associated to the prestack pre-$[\mathscr{G}]$ which associates to any test scheme T the groupoid in sets

$$\text{pre-}[\mathscr{G}](T) = \mathrm{Hom}(T, \mathscr{G}).$$

Notice that such a morphism consists of a map from maps from T to A, and T to R which satisfy the obvious compatibilities.

Remark 7. In the case that $R = G \times A$ and the groupoid structure is just given by a group action of G on A, we may denote the associated quotient stack by $[A/G]$, leaving the structure implicit.

There is an equivalence [17] of Abelian categories of coherent sheaves which takes vector bundles to vector bundles

$$\mathrm{Coh}(\mathscr{G}) \xrightarrow{\cong} \mathrm{Coh}([\mathscr{G}]). \tag{39}$$

Definition 7. We denote by $[\mathscr{S}]$ the sheaf on $[\mathscr{G}]$ corresponding to a sheaf $\mathscr{S}$ on $\mathscr{G}$ under the equivalence (39) given above.

3.3 Groupoid Presentations for Stacks of Rank 2 Bundles

We define a groupoid in schemes to be called $\mathscr{G}_{j,k}^{(n)}$. The atlas of $\mathscr{G}_{j,k}^{(n)}$ is $W_{j,k}^{(n)}$ and the relations are $G_{j,k}^{(n)} \times W_{j,k}^{(n)}$.

The arrow s is given by the projection

$$G_{j,k}^{(n)} \times W_{j,k}^{(n)} \xrightarrow{s} W_{j,k}^{(n)}.$$

defined by

$$(g, p) \mapsto p.$$

The arrow t is given by the map

$$G_{j,k}^{(n)} \times W_{j,k}^{(n)} \xrightarrow{t} W_{j,k}^{(n)}. \tag{40}$$

defined by

$$(g, p) \mapsto g \bullet p.$$