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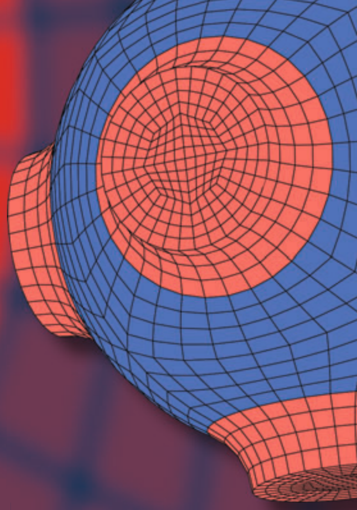

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Strength of Materials and Theory of Elasticity in 19th Century Italy

A Brief Account of the History of Mechanics of Solids and Structures

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A Brief Account of the History of Mechanics of Solids and Structures

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Preface

In 1877 Giovanni Curioni, Professor in the *Scuola d'applicazione per gl'ingegneri* (School of Application for Engineers) in Turin, chose the name *Scienza delle costruzioni* for his course of mechanics applied to civil and mechanical constructions.

The choice reflected a change that had occurred in the teaching of structural disciplines in Italy, following the establishment of schools of application for engineers by Casati's reform of 1859. On the model of the École polytechnique, the image of the purely technical engineer was replaced by that of the 'scientific engineer', inserting into the teaching both 'sublime mathematics' and modern theories of elasticity. Similarly, the art of construction was to be replaced by the science of construction. The *Scienza delle costruzioni* came to represent a synthesis of theoretical studies of continuum mechanics, carried out primarily by French scholars of elasticity, and the mechanics of structures, which had begun to develop in Italian and German schools. In this respect it was an approach without equivalence in Europe, where the contents of continuum mechanics and mechanics of structures were, and still today are, taught in two different disciplines.

In the 1960s of the twentieth century, the locution *Scienza delle costruzioni* took a different sense for various reasons. Meanwhile, the discipline established by Curioni was divided into two branches, respectively, called *Scienza delle costruzioni* and *Tecnica delle costruzioni*, relegating this last to applicative aspects. Then technological developments required the study of materials with more complex behavior than the linear elastic one; there was a need for protection from phenomena of fatigue and fracture, and dynamic analysis became important for industrial applications (vibrations) and civil incidents (wind, earthquakes). Finally, introduction of modern structural codes on the one hand made obsolete the sophisticated manual calculation techniques developed between the late 1800s and early 1900s, on the other hand it necessitated a greater knowledge of the theoretical aspects, especially of continuum mechanics. This necessity to deepen the theory inevitably led a to drift toward mathematical physics in some scholars.

All this makes problematic a modern definition of *Scienza delle costruzioni*. To overcome this difficulty, in our work we decided to use the term *Scienza delle costruzioni* with a fairly wide sense, to indicate the theoretical part of construction engineering. We considered Italy and the nineteenth century for two reasons. Italy, to account for the lack of knowledge of developments in the discipline in this country, which is in any case a major European nation. The nineteenth century, because it is one in which most problems of design of structures were born and reached maturity, although the focus was concentrated on materials with linear elastic behavior and external static actions.

The existing texts on the history of *Scienza delle costruzioni*, among which one of the most complete in our opinion is that by Stephen Prokofievich Timoshenko, *History of Strength of Materials*, focus on French, German, and English schools, largely neglecting the Italian. Moreover, Edoardo Benvenuto's text, *An Introduction to the History of Structural Mechanics*, which is very attentive to the Italian contributions, largely neglects the nineteenth century. Only recently, Clifford Ambrose Truesdell, mathematician and historian of mechanics, in his *Classical Field Theories of Mechanics* highlighted the important contributions of Italian scientists, dusting off the names of Piola, Betti, Beltrami, Lauricella, Cerruti, Cesaro, Volterra, Castigliano, and so on.

The present book deals largely with the theoretical foundations of the discipline, starting from the origin of the modern theory of elasticity and framing the Italian situation in Europe, examining and commenting on foreign authors who have had a key role in the development of mechanics of continuous bodies and structures and graphic calculation techniques. With this in mind, we have mentioned only those issues most 'applicative', which have not seen important contributions by Italian scholars. For example, we have not mentioned any studies on plates that were brought forward especially in France and Germany and which provided fundamental insights into more general aspects of continuum mechanics. Consider, for instance, the works on plates by Kirchhoff, Saint Venant, Sophie Germain, and the early studies on dynamic stresses in elastic bodies by Saint Venant, Navier, Cauchy, Poncelet. Finally, we have not mentioned any of the experimental works carried out especially in England and Germany, including also some important ones from a theoretical point of view about the strength and fracture of materials.

The book is intended as a work of historical research, because most of the contents are either original or refer to our contributions published in journals. It is directed to all those graduates in scientific disciplines who want to deepen the development of Italian mathematical physics in the nineteenth century. It is directed to engineers, but also architects, who want to have a more comprehensive and critical vision of the discipline they have studied for years. Of course, we hope it will be helpful to scholars of the history of mechanics as well.

We would like to thank Raffaele Pisano and Annamaria Pau for reading drafts of the book and for their suggestions.

Editorial Considerations

Figures related to quotations are all redrawn to allow better comprehension. They are, however, as much as possible close to the original ones. Symbols of formulas are always those of the authors, except cases easily identifiable. Translations of texts from French, Latin, German, and Italian are as much as possible close to the original texts. For Latin, a critical transcription has been preferred where some shortenings are resolved, ‘v’ is modified to ‘u’ and vice versa where necessary, ij to ii, following the modern rule; moreover, the use of accents is avoided. Titles of books and papers are always reproduced in the original spelling. For the name of the different characters the spelling of their native language is used, excepting for the ancient Greeks, for which the English spelling is assumed, and some medieval people, for which the Latin spelling is assumed, following the common use.

Through the text, we searched to avoid modern terms and expressions as much as possible while referring to ‘old’ theories. In some cases, however, we transgressed this resolution for the sake of simplicity. This concerns the use, for instance, of terms like *field*, *balance*, and *energy* even in the period they were not used or were used differently from today. The same holds good for expressions like, for instance, *principle of virtual work*, that was common only since the nineteenth century.

Danilo Capecchi
Giuseppe Ruta

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Chapter 1
The Theory of Elasticity in the 19th Century

Abstract Until 1820 there was a limited knowledge about the elastic behavior of materials: one had an inadequate theory of bending, a wrong theory of torsion, the definition of Young's modulus. Studies were made on one-dimensional elements such as beams and bars, and two-dimensional, such as thin plates (see for instance the work of Marie Sophie Germain). These activities started the studies on three-dimensional elastic solids that led to the theory of elasticity of three-dimensional continua becoming one of the most studied theories of mathematical physics in the 19th century. In a few years most of the unresolved problems on beams and plates were placed in the archives. In this chapter we report briefly a summary on three-dimensional solids, focusing on the theory of constitutive relationships, which is the part of the theory of elasticity of greatest physical content and which has been the object of major debate. A comparison of studies in Italy and those in the rest of Europe is referenced.

1.1 Theory of Elasticity and Continuum Mechanics

The theory of elasticity has ancient origins. Historians of science, pressed by the need to provide an *a quo* date, normally refer to the *Lectures de potentia restitutiva* by Robert Hooke in 1678 [78]. One can debate this date, but for the moment we accept it because a historically accurate reconstruction of the early days of the theory of elasticity is out of our purpose; we limit ourselves only to pointing out that Hooke should divide the honor of the primeval introduction with at least Edme Mariotte [95]. Hooke and Mariotte studied problems classified as engineering: the displacement of the point of a beam, its curvature, the deformation of a spring, etc.

Explanations *per causas* of elasticity can be traced back to the *Quaestio* 31 of Isaac Newton's *Opticks* of 1704 [117], in which the corpuscular constitution of matter is discussed. Many alternative conceptions were developed in the 18th century, especially with reference to the concept of ether; for a few details we refer to the literature [7]. In the early years of the 19th century the theory of elasticity was intimately connected to some corpuscular theories, such as that of Laplace [88][1],

[1] vol. 4, pp. 349, 350.

D. Capecchi and G. Ruta, *Strength of Materials and Theory of Elasticity in 19th Century Italy*, Advanced Structured Materials 52,
DOI 10.1007/978-3-319-05524-4_1

[68] who refined the approach of Newton, and considered the matter consisting of small bodies, with extension and mass, or that of Ruggero Boscovich [12] according to which matter is based on unextended centers of force endowed with mass. The masses are attracted with forces depending on their mutual distance; repulsive at short distance, attractive at a greater distance, as illustrated in Fig. 1.1.

It should be said that it was not just engineering that influenced the development of the theory of elasticity; an even superficial historical analysis shows that such researches were also linked to the attempt to provide a mechanistic interpretation of nature. According to this interpretation every physical phenomenon must be explained by particle mechanics: matter has a discrete structure and space is filled with fine particles with uniform properties, which form the *ether*. All the physical phenomena propagate in space by a particle of ether to its immediate neighbor by means of impacts or forces of attraction or repulsion. This point of view allows one to overcome the difficulties of the concept of action at a distance: In which way, asked the physicists of the time, can two bodies interact, for instance attract each other, without the action of an intervening medium? Any physical phenomenon corresponds to a state of stress in the ether, propagated by contact.

With the beginning of the 19th century the need was felt to quantitatively characterize the elastic behavior of bodies and the mathematical theory of elasticity was born. Its introduction was thought to be crucial for an accurate description of the physical world, in particular to better understand the phenomenon of propagation of light waves through the air. The choices of physicists were strongly influenced by mathematics in vogue at that time, that is the differential and integral calculus, hereinafter *Calculus*. It presupposed the mathematics of continuum and therefore was difficult to fit into the discrete particle model, which had become dominant.

Most scientists adopted a compromise approach that today can be interpreted as a technique of homogenization. The material bodies, with a fine corpuscular structure, are associated with a mathematical continuum C, as may be a solid of Euclidean geometry. The variables of displacement are represented by a sufficiently regular function **u** defined in C, that assumes significant values only for those points P of C that are also positions of particles. The derivatives of the function **u** with respect to the variables of space and time also have meaning only for the points P. The internal forces exchanged between particles, at the beginning thought of as concentrated, are represented by distributed mean values that are attributed to all the points of

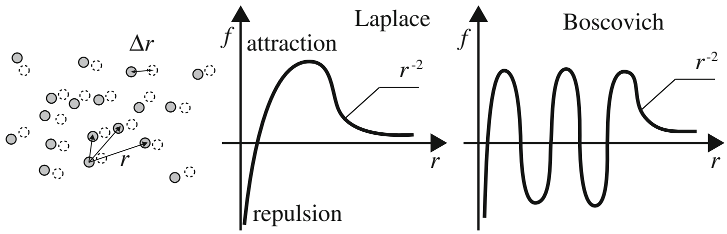

Fig. 1.1 Molecular model: force f between two molecules as a function of their distance r

C, thus becoming stresses σ. Other scientists gave up the corpuscular physical model considering it only in the background. They founded their theories directly on the continuum, whose points had now all ‘physical’ meaning. On the continuum are defined both the displacements and the stresses, as had already been done in the 18th century by Euler and Lagrange for fluids. Some scientists oscillated between the two approaches, among them Augustin Cauchy (1760–1848) (but the Italian Gabrio Piola (1794–1850) was in a similar position [19]) who, while studying the distribution of internal forces of solids, systematized mathematical analysis, dealing with the different conceptions of infinite and infinitesimal, of discrete and continuum. His oscillations in mathematical analysis were reflected in his studies on the constitution of matter [56, 57].

In the following we present in some detail and sense of history what we have just outlined above, speaking of the various corpuscular approaches and continuum approach, referring primarily to the relationship between the internal force and displacement, or between stress and strain, that is the constitutive law. Other problems of the theory of elasticity, always in the context of continua, will be mentioned later, to finally devote several sections to the elasticity theory of discrete systems in general and to the structures formed by beams in particular.

1.1.1 The Classical Molecular Model

The theories of elasticity of the early 19th century were based on different corpuscular assumptions, introduced almost simultaneously by Fresnel, Cauchy and Navier [25, 27, 70, 114]. French scientists adopted the single word *molecule* for particles, which lived long in European scientific literature, often flanked by *atom*, without the two terms necessarily had different meanings, at least until the studies of the chemical constitution of matter advanced and the terms atom and molecule assumed precise technical meanings which differentiate the areas of application.

Augustin Jean Fresnel studied the propagation of light through the ether, imagined as a set of material points that exchange elastic forces. In a work of 1820 he obtained very interesting results, as for instance the theorem:

> As long as small displacements are concerned and whatever the law of the forces that the molecules of the medium exert on each other, the movement of a molecule in any direction produces a repulsive force equal in magnitude and direction to the resultant of the three repulsive forces generated by three rectangular displacement of this molecule equal to the static components of the first [small] displacement [70].[2] (A.1.1)

This theorem about the force that rises among the molecules, ‘nearly self evident in its statement’, was presented by Cauchy in an appendix of his famous paper on stress [26],[3] where an explicit reference to Fresnel was made.

[2] pp. 344–345. Our translation.

[3] Addition, pp. 79–81.

The first systematic work on the equilibrium and the motion of three-dimensional elastic bodies was however due to Navier, who in 1821 read before the Académie des sciences de Paris an important memoir published only in 1827 [114].

Navier, referring explicitly to Lagrange's *Méchanique analitique* [83], wrote the equations of local equilibrium of forces acting on an elastic body, thought of as an aggregate of particles that attract or repel each other with an elastic force variable linearly with their mutual displacements:

> One considers a solid body as an assemblage of material molecules placed at a very small distance. These molecules exert two opposite actions on each other, that is a proper attractive force and a repulsive force due to the principle of heat. Between one molecule M and any other M' of the neighboring molecules there is an attraction P which is the difference of these two forces. In the natural state of the body all the forces P are zero or reciprocally destroy, because the molecule M is at rest. When the body changes its shape, the force P takes a different value Π and there is equilibrium between all the forces Π and the forces applied to the body, by which the change of the shape of the body is produced [114].[4] (A.1.2)

Let X, Y, Z be the external forces per unit of volume, ϵ a constant (to use a modern term it is the second Lamé constant) and x, y, z the displacement of the generic point P having initial coordinates a, b, c, then the equilibrium equations obtained by Navier are [114][5]:

$$\begin{aligned}
-X &= \epsilon\left(3\frac{d^2x}{da^2}+\frac{d^2x}{db^2}+\frac{d^2x}{dc^2}+2\frac{d^2y}{da\,db}+2\frac{d^2z}{da\,dc}\right)\\
-Y &= \epsilon\left(\frac{d^2y}{db^2}+3\frac{d^2y}{da^2}+\frac{d^2y}{dc^2}+2\frac{d^2x}{da\,db}+2\frac{d^2z}{db\,dc}\right)\\
-Z &= \epsilon\left(\frac{d^2z}{db^2}+\frac{d^2z}{dc^2}+3\frac{d^2z}{da^2}+2\frac{d^2x}{da\,dc}+2\frac{d^2y}{db\,dc}\right).
\end{aligned} \tag{1.1}$$

Navier obtained these equations with the use of the principle of virtual work [114].[6] He followed the approach, already mentioned, common to all French scientists of the 19th century, by considering the body as discrete when he wanted to study the equilibrium, while as continuous when he came to describe the geometry and obtained simple mathematical relationships, replacing the summations with integrals.[7] Note that in the work of Navier the concept of stress, which was crucial to the mechanics of structures developed later, was not present.

In the academic French world the molecular model of Navier became dominant because of the influence of the teaching of Laplace. On October 1st, 1827 Poisson and Cauchy presented to the Académie des sciences de Paris two memoirs similar

[4] pp. 375–376. Our translation.

[5] p. 384.

[6] p. 384.

[7] The difficulty of replacing summations with integrals has been the subject of many comments of French scholars, especially Poisson and Cauchy.

to each other, where Navier's molecular model was adopted [116].[8] Poisson gave decisive contributions in this field. In two other papers read at the Académie des sciences de Paris on April 14th, 1828 [127] and on October 12th, 1829 [128] he expressed its assumptions:

> The molecules of all bodies are subject to their mutual attraction and repulsion due to heat. According that the first of these two forces is greater or less than the second, the result is an attractive or repulsive force between two molecules, but in both cases, the resultant is a function of the distance from a molecule to the other whose law is unknown to us; we only know that this function decreases in a very fast manner, and becomes insensible as soon as the distance has acquired a significant magnitude. However, we assume that the radius of activity of the molecules is very large compared to the intervals between them, and we assume, moreover, that the rapid decrease of the action takes place only when the distance became the sum of a very large number of these intervals [127].[9] (A.1.3)

and introduced the concept of *stress*:

> Let M be a point in the inner part of the body, at a sensible distance from the surface [Fig. 1.2a]. Let us consider a plane through this point, dividing the body into two parts, which we will suppose horizontal [...]. Let us denote by A the upper part and A′ the lower part, in which we will include the material points belonging to the plane itself. From the point M considered as a center let us draw a sphere including a very large amount of molecules, yet the radius of which is in any case negligible with respect to the radius of the molecular activity. Let ω be the area of its horizontal section; over this section let us raise a vertical cylinder, the height of which is at least the same as the radius of molecular activity; let us call B this cylinder; the force of the molecules of A′ over those of B, divided by ω, will be the *pressure* exerted by A′ over A, with respect to the unity of surface and relative to the point M [129].[10] (A.1.4)

For isotropic materials Cauchy [30][11] and Poisson [127] obtained relations close to those by Navier. This is for instance the expression given by Poisson:

$$
\begin{aligned}
X-\frac{d^2u}{dt^2}+a^2\left(\frac{d^2u}{dy^2}+\frac{2}{3}\frac{d^2v}{dydx}+\frac{2}{3}\frac{d^2w}{dzdx}+\frac{1}{3}\frac{d^2u}{dx^2}+\frac{1}{3}\frac{d^2u}{dz^2}\right)&=0,\\
Y-\frac{d^2v}{dt^2}+a^2\left(\frac{d^2v}{dy^2}+\frac{2}{3}\frac{d^2u}{dxdy}+\frac{2}{3}\frac{d^2w}{dzdy}+\frac{1}{3}\frac{d^2v}{dx^2}+\frac{1}{3}\frac{d^2v}{dz^2}\right)&=0,\\
Z-\frac{d^2w}{dt^2}+a^2\left(\frac{d^2w}{dz^2}+\frac{2}{3}\frac{d^2u}{dxdz}+\frac{2}{3}\frac{d^2v}{dydz}+\frac{1}{3}\frac{d^2w}{dx^2}+\frac{1}{3}\frac{d^2w}{dy^2}\right)&=0,
\end{aligned}
\tag{1.2}
$$

where X, Y, Z are the forces per unit of mass and a a constant of elasticity [127].[12]

---

[8] pp. CLV, CLIX. The memoir of Cauchy appeared first with the title *Mémoire sur l'équilibre et le mouvement d'un système de points materiels sollecités par forces d'attraction ou de répulsion mutuelle* [30]. That of Poisson appeared with the title *Note sur les vibrations des corps sonores* [126].

[9] pp. 368–369. Our translation.

[10] p. 29. Our translation. Stress was indicated by French scientists by *pressure* or *tension*.

[11] pp. 250–251.

[12] p. 403.

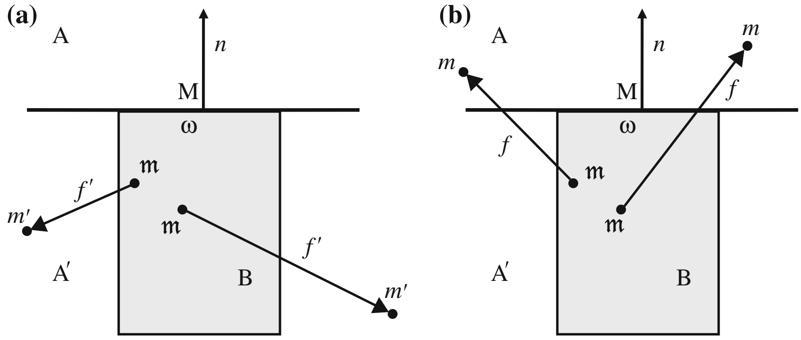

Fig. 1.2 Stress according to Poisson (**a**) and Cauchy (**b**)

In the following, we show in some detail the main features of the classical molecular model, along with its origin, trying to grasp its strengths and weaknesses. The focus is on the constitutive stress–strain relations because here one can see better the consequences of the assumptions about the molecular model. Reference is made to the work by Cauchy of 1828 [29, 30],[13] among the most complete and clear on the subject (see below).

The main assumptions of the molecular model are:

1. The molecules are treated as material points subjected to opposing forces directed along their joining line (central forces assumption).
2. The force between two molecules decreases rapidly starting from a distance, small but much larger than the normal distance between two molecules, called *ray of molecular action*.
3. The molecules have all the same mass and the force between any two molecules is provided by the same function $f(r)$ of their distance r.
4. The relative displacements of the molecules are 'small'.
5. The function $f(r)$ which expresses the force between two molecules is regular in r, and then can be differentiated.
6. The motion of the molecules is defined by a smooth vector field in the continuum where the system of molecules are imagined to be embedded.

The first three assumptions are physical, the remaining are of mathematical character, introduced clearly to simplify the treatment.

[13] pp. 227–252.

1.1.1.1 The Components of Stress

In his work of 1828 [29] Cauchy adopted a variant of Poisson's definition of stress. The difference was that he considered the force of the molecules m in A (Fig. 1.2b) on the molecules $\mathfrak{m}$ in B instead of the force of the molecules m' in A′.[14]

Consider the cylinder B of Fig. 1.2b having an infinitesimal base ω on a plane perpendicular to the unit vector n, located in the half space A′. Let $\mathfrak{m}$ be an assigned molecule inside the cylinder and m the molecules located in the half-space A on the same side of n. The force exerted on $\mathfrak{m}$ by all the molecules m is characterized by the three components [29][15]:

$$\sum \pm \mathfrak{m} m \cos\alpha \; f(r); \qquad \sum \pm \mathfrak{m} m \cos\beta \; f(r); \qquad \sum \pm \mathfrak{m} m \cos\gamma \; f(r), \tag{1.3}$$

where $f(r)$ is the force between m and $\mathfrak{m}$, α, β, γ are the direction cosines of the radius vector r connecting m—that is the components of the unit vector parallel to r—and $\mathfrak{m}$, with respect to an arbitrary coordinate system and the sum is extended to all the molecules m of the half space A opposite to the cylinder, or rather to all those in the sphere of molecular action (the sphere defined by the radius of molecular action) of $\mathfrak{m}$. To obtain the force exerted on the cylinder and, according to Poisson, the pressure on the surface ω, the summations of the relation (1.3) should be extended to all the molecules $\mathfrak{m}$ of the cylinder and divided by ω. Since all the molecules are assumed to be equal, this summation was made explicit in a simple way by Cauchy, who after some steps obtained the components for the stress on the faces orthogonal to the coordinate axes. For instance those on the face orthogonal to x are given by Cauchy [29][16]:

$$\begin{cases} A = \Delta \sum \pm m \cos^2\alpha \; f(r) \\ F = \Delta \sum \pm m \cos\alpha \cos\beta \; f(r) \\ E = \Delta \sum \pm m \cos\alpha \cos\gamma \; f(r), \end{cases} \tag{1.4}$$

with Δ the specific mass of the body, supposed locally homogeneous.

Cauchy had already introduced the symbols for the stress components in the work of 1827 [26];[17] they will be adopted by other scholars long before the indexed notations was established (see below). Full symbols and correspondences with modern notations are given in the following list and shown in Fig. 1.3:

[14] Actually Cauchy introduced various slightly different definitions of stress. In a memoir of 1845 [34] he adopted the definition considered also by Saint Venant and Jean-Marie Constant Duhamel according to which the "stress (la pression) on a very small area (ω is defined) as the resultant of the actions of all the molecules located on the one side over all the molecules located on the other side whose directions cross this element" [141], p. 24.

[15] p. 257.

[16] p. 257, Eq. (1.13).

[17] pp. 60–81.

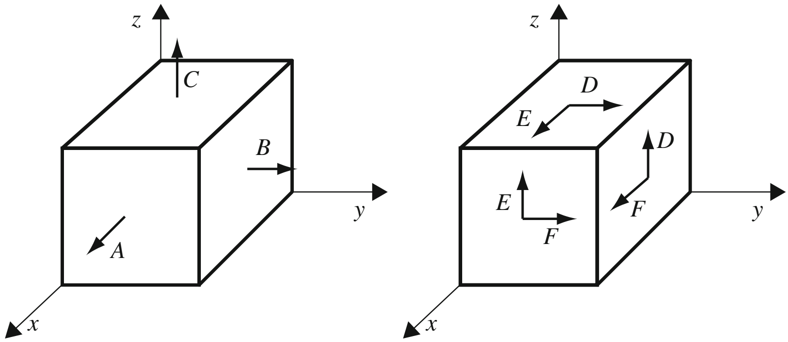

Fig. 1.3 The components of the stress tensor according to Cauchy

$$\begin{matrix} A(\equiv \sigma_x) & F(\equiv \tau_{yx}) & E(\equiv \tau_{zx}) \\ F(\equiv \tau_{xy}) & B(\equiv \sigma_y) & D(\equiv \tau_{zy}) \\ E(\equiv \tau_{xz}) & D(\equiv \tau_{yz}) & C(\equiv \sigma_z). \end{matrix} \tag{1.5}$$

1.1.1.2 The Component of Strains and the Constitutive Relationships

In the modern theories of continuum mechanics, the components of the stress and strain are defined independently first, then the function connecting them, which is precisely the constitutive law, is introduced.

In the classical molecular theory the historical path was different. The definition of the strain passed in the background and implicitly stemmed from the attempt to establish the link between stresses and displacements, as soon as the latter are approximated with their infinitesimal values. This approach was certainly influenced by the work of Navier in 1821 [114] which had the aim of finding the differential equations for displacement components in an elastic body, without any examination of the internal forces.

To obtain the relations that link the components of the stresses to those of the strains, Cauchy rewrote the relations analogous to (1.4), taking into account the displacement with components $\xi,\ \eta,\ \zeta$ of the molecules from their initial position. Cauchy indicated with $\Delta a,\ \Delta b,\ \Delta c$ the components of the distance r between two molecules in the undeformed state and with $\Delta x,\ \Delta y,\ \Delta z$ those of the distance in the deformed state, resulting in the relations:

$$\Delta x = \Delta a + \Delta\xi,\ \Delta y = \Delta b + \Delta\,\eta,\ \Delta z = \Delta c + \Delta\,\zeta\,. \tag{1.6}$$

The new distance among molecules was defined by Cauchy by means of its percentage variation ϵ as $(1+\epsilon)r$.

The components of stress in the deformed configuration were obtained by replacing in the relation (1.4) the new expressions of forces and distances [29][18]:

$$\begin{cases} A = \frac{\rho}{2}\sum\left\{\pm m\frac{f[r(1+\epsilon)]}{r(1+\epsilon)}\Delta x^2\right\};\ D = \frac{\rho}{2}\sum\left\{\pm m\frac{f[r(1+\epsilon)]}{r(1+\epsilon)}\Delta y\Delta z\right\} \\ B = \frac{\rho}{2}\sum\left\{\pm m\frac{f[r(1+\epsilon)]}{r(1+\epsilon)}\Delta y^2\right\};\ E = \frac{\rho}{2}\sum\left\{\pm m\frac{f[r(1+\epsilon)]}{r(1+\epsilon)}\Delta z\Delta x\right\} \\ C = \frac{\rho}{2}\sum\left\{\pm m\frac{f[r(1+\epsilon)]}{r(1+\epsilon)}\Delta z^2\right\};\ F = \frac{\rho}{2}\sum\left\{\pm m\frac{f[r(1+\epsilon)]}{r(1+\epsilon)}\Delta x\Delta y\right\}, \end{cases} \tag{1.7}$$

where ρ is the mass density in the deformed configuration, different in general from the mass density Δ in the undeformed configuration, and the sum is extended to all the molecules contained inside the sphere of molecular action of $\mathfrak{m}$, both in the half space containing the infinitesimal cylinder and the opposite one. That justifies the factor 1/2.

To obtain relations suitable for algebraic manipulation and thus for simplification, Cauchy [30] introduced the assumption of small displacements, which allowed him to derive linearized relations in ϵ; and a linear elastic relationship between stresses and strains:

$$\begin{aligned} \frac{f[r(1+\epsilon)]}{r(1+\epsilon)} &\approx \frac{f(r)}{r} + \frac{rf'(r) - f(r)}{r}\epsilon, \\ \epsilon &= \frac{1}{r}\left(\cos\alpha\,\Delta\xi + \cos\beta\,\Delta\eta + \cos\gamma\,\Delta\zeta\right). \end{aligned} \tag{1.8}$$

Having chosen a reference molecule $\mathfrak{m}$, the one at the center of the elementary surface ω of the cylinder, for instance, Cauchy linearized the variation of the components of the displacements interior to the sphere of action of $\mathfrak{m}$ with respect to the spatial variables. This is possible because of the small distance among the molecules inside the molecular sphere of action:

$$\begin{aligned} \frac{\Delta\xi}{r} &= \frac{\partial\xi}{\partial a}\cos\alpha + \frac{\partial\xi}{\partial b}\cos\beta + \frac{\partial\xi}{\partial c}\cos\gamma, \\ \frac{\Delta\eta}{r} &= \frac{\partial\eta}{\partial a}\cos\alpha + \frac{\partial\eta}{\partial b}\cos\beta + \frac{\partial\eta}{\partial c}\cos\gamma, \\ \frac{\Delta\zeta}{r} &= \frac{\partial\zeta}{\partial a}\cos\alpha + \frac{\partial\zeta}{\partial b}\cos\beta + \frac{\partial\zeta}{\partial c}\cos\gamma \end{aligned} \tag{1.9}$$

where the derivatives are evaluated at $\mathfrak{m}$.

By replacing in (1.7) the linearized expressions of f and ϵ, simplifying and neglecting the higher order infinitesimals in $\Delta\xi$, $\Delta\eta$, $\Delta\zeta$, Cauchy derived the relations referred to in Fig. 1.4, which express the constitutive relationship. They give the expression of the components of stress A, B, C, D, E, F versus the nine components

[18] p. 260, Eq. (1.18).

$$
\left\{
\begin{array}{l}
A = \rho \mathrm{S}\left[\pm \frac{m\rho}{2}\cos^2\alpha f(r)\right] \\
+2\rho\left\{\frac{\partial\xi}{\partial a}\mathrm{S}\left[\pm\frac{m\rho}{2}\cos^2\alpha f(r)\right] + \frac{\partial\xi}{\partial b}\mathrm{S}\left[\pm\frac{m\rho}{2}\cos\alpha\cos\beta f(r)\right] + \frac{\partial\xi}{\partial c}\mathrm{S}\left[\pm\frac{m\rho}{2}\cos\alpha\cos\gamma f(r)\right]\right\} \\
+\rho\left\{
\begin{array}{l}
\frac{\partial\xi}{\partial a}\mathrm{S}\left[\frac{m\rho}{2}\cos^4\alpha f(r)\right] + \frac{\partial\xi}{\partial b}\mathrm{S}\left[\frac{m\rho}{2}\cos^3\alpha\cos\beta f(r)\right] + \frac{\partial\xi}{\partial c}\mathrm{S}\left[\frac{m\rho}{2}\cos^3\alpha\cos\gamma f(r)\right] \\
\frac{\partial\eta}{\partial a}\mathrm{S}\left[\frac{m\rho}{2}\cos^3\alpha\cos\beta f(r)\right] + \frac{\partial\eta}{\partial b}\mathrm{S}\left[\frac{m\rho}{2}\cos^2\alpha\cos^2\beta f(r)\right] + \frac{\partial\eta}{\partial c}\mathrm{S}\left[\frac{m\rho}{2}\cos^2\alpha\cos\beta\cos\gamma f(r)\right] \\
\frac{\partial\zeta}{\partial a}\mathrm{S}\left[\frac{m\rho}{2}\cos^3\alpha\cos\gamma f(r)\right] + \frac{\partial\zeta}{\partial b}\mathrm{S}\left[\frac{m\rho}{2}\cos^2\alpha\cos\beta\cos\gamma f(r)\right] + \frac{\partial\zeta}{\partial c}\mathrm{S}\left[\pm\frac{m\rho}{2}\cos^2\alpha\cos^2\gamma f(r)\right]
\end{array}
\right\}
\end{array}
\right.
$$

$$
\left\{
\begin{array}{l}
B = \rho \mathrm{S}\left[\pm \frac{m\rho}{2}\cos^2\beta f(r)\right] \\
+2\rho\left\{\frac{\partial\eta}{\partial a}\mathrm{S}\left[\pm\frac{m\rho}{2}\cos\alpha\cos\beta f(r)\right] + \frac{\partial\eta}{\partial b}\mathrm{S}\left[\pm\frac{m\rho}{2}\cos^2\beta f(r)\right] + \frac{\partial\eta}{\partial c}\mathrm{S}\left[\pm\frac{m\rho}{2}\cos\alpha\cos\gamma f(r)\right]\right\} \\
+\rho\left\{
\begin{array}{l}
\frac{\partial\xi}{\partial a}\mathrm{S}\left[\frac{m\rho}{2}\cos^2\alpha\cos^2\beta f(r)\right] + \frac{\partial\xi}{\partial b}\mathrm{S}\left[\frac{m\rho}{2}\cos\alpha\cos^3\beta f(r)\right] + \frac{\partial\xi}{\partial c}\mathrm{S}\left[\frac{m\rho}{2}\cos\alpha\cos^2\beta\cos\gamma f(r)\right] \\
\frac{\partial\eta}{\partial a}\mathrm{S}\left[\frac{m\rho}{2}\cos\alpha\cos^3\beta f(r)\right] + \frac{\partial\eta}{\partial b}\mathrm{S}\left[\frac{m\rho}{2}\cos^4\beta f(r)\right] + \frac{\partial\eta}{\partial c}\mathrm{S}\left[\frac{m\rho}{2}\cos^3\beta\cos\gamma f(r)\right] \\
\frac{\partial\zeta}{\partial a}\mathrm{S}\left[\frac{m\rho}{2}\cos\alpha\cos^2\beta\cos\gamma f(r)\right] + \frac{\partial\zeta}{\partial b}\mathrm{S}\left[\frac{m\rho}{2}\cos^3\beta\cos\gamma f(r)\right] + \frac{\partial\zeta}{\partial c}\mathrm{S}\left[\frac{m\rho}{2}\cos^2\beta\cos^2\gamma f(r)\right]
\end{array}
\right\}
\end{array}
\right.
$$

$$
\left\{
\begin{array}{l}
C = \rho \mathrm{S}\left[\pm \frac{m\rho}{2}\cos^2\gamma f(r)\right] \\
+2\rho\left\{\frac{\partial\xi}{\partial a}\mathrm{S}\left[\pm\frac{m\rho}{2}\cos\alpha\cos\gamma f(r)\right] + \frac{\partial\xi}{\partial b}\mathrm{S}\left[\pm\frac{m\rho}{2}\cos\beta\cos\gamma f(r)\right] + \frac{\partial\xi}{\partial c}\mathrm{S}\left[\pm\frac{m\rho}{2}\cos^2\gamma f(r)\right]\right\} \\
+\rho\left\{
\begin{array}{l}
\frac{\partial\xi}{\partial a}\mathrm{S}\left[\frac{m\rho}{2}\cos^2\alpha\cos^2\beta f(r)\right] + \frac{\partial\xi}{\partial a}\mathrm{S}\left[\frac{m\rho}{2}\cos\alpha\cos\beta\cos^2\gamma f(r)\right] + \frac{\partial\xi}{\partial a}\mathrm{S}\left[\frac{m\rho}{2}\cos\alpha\cos^3\gamma f(r)\right] \\
\frac{\partial\eta}{\partial a}\mathrm{S}\left[\frac{m\rho}{2}\cos\alpha\cos\beta\cos^2\gamma f(r)\right] + \frac{\partial\eta}{\partial b}\mathrm{S}\left[\frac{m\rho}{2}\cos^2\beta\cos^2\gamma f(r)\right] + \frac{\partial\eta}{\partial c}\mathrm{S}\left[\frac{m\rho}{2}\cos\beta\cos^3\gamma f(r)\right] \\
\frac{\partial\zeta}{\partial a}\mathrm{S}\left[\frac{m\rho}{2}\cos\alpha\cos^3\gamma f(r)\right] + \frac{\partial\zeta}{\partial b}\mathrm{S}\left[\frac{m\rho}{2}\cos\beta\cos^3\gamma f(r)\right] + \frac{\partial\zeta}{\partial c}\mathrm{S}\left[\frac{m\rho}{2}\cos^4\gamma f(r)\right]
\end{array}
\right\}.
\end{array}
\right.
$$

Fig. 1.4 Components of stress in Cauchy's molecular model [29, p. 263]

of the displacement gradient $\partial\xi/\partial a$, $\partial\xi/\partial b$, $\partial\xi/\partial c$, $\partial\eta/\partial a$, $\partial\eta/\partial b$, $\partial\eta/\partial c$, $\partial\zeta/\partial a$, $\partial\zeta/\partial b$, $\partial\zeta/\partial c$, that implicitly define the components of the strains.

The stress components are related to those of the strain by 21 distinct coefficients, defined by the summation extended to all the molecules inside the sphere of action of the point-molecule in which one wants to calculate the stress, which multiply the derivatives of the components of the displacement at the same point (in the tables the symbol S stands for summation). The exception is the first term, which contains

$$
\begin{cases}
D = \rho \mathrm{S}\left[\pm \frac{m\rho}{2}\cos\beta\cos\gamma f(r)\right] \\
+\rho\left\{
\begin{aligned}
&\frac{\partial\eta}{\partial a}\mathrm{S}\left[\pm\frac{m\rho}{2}\cos\alpha\cos\gamma f(r)\right]+\frac{\partial\eta}{\partial b}\mathrm{S}\left[\pm\frac{m\rho}{2}\cos\beta\cos\gamma f(r)\right]+\frac{\partial\eta}{\partial c}\mathrm{S}\left[\pm\frac{m\rho}{2}\cos^2\gamma f(r)\right]\\
&\frac{\partial\xi}{\partial a}\mathrm{S}\left[\pm\frac{m\rho}{2}\cos\alpha\cos\beta f(r)\right]+\frac{\partial\xi}{\partial b}\mathrm{S}\left[\pm\frac{m\rho}{2}\cos^2\beta f(r)\right]+\frac{\partial\xi}{\partial c}\mathrm{S}\left[\pm\frac{m\rho}{2}\cos\beta\cos\gamma f(r)\right]
\end{aligned}\right\} \\
+\rho\left\{
\begin{aligned}
&\frac{\partial\xi}{\partial a}\mathrm{S}\left[\frac{m\rho}{2}\cos^2\alpha\cos\beta\cos\gamma f(r)\right]+\frac{\partial\xi}{\partial b}\mathrm{S}\left[\frac{m\rho}{2}\cos\alpha\cos^2\beta\cos\gamma f(r)\right]+\frac{\partial\xi}{\partial c}\mathrm{S}\left[\frac{m\rho}{2}\cos\alpha\cos\beta\cos^2\gamma f(r)\right]\\
&\frac{\partial\eta}{\partial a}\mathrm{S}\left[\frac{m\rho}{2}\cos\alpha\cos^2\beta\cos\gamma f(r)\right]+\frac{\partial\eta}{\partial b}\mathrm{S}\left[\frac{m\rho}{2}\cos^3\beta\cos\gamma f(r)\right]+\frac{\partial\eta}{\partial c}\mathrm{S}\left[\frac{m\rho}{2}\cos^2\alpha\cos\beta\cos\gamma f(r)\right]\\
&\frac{\partial\zeta}{\partial a}\mathrm{S}\left[\frac{m\rho}{2}\cos^2\beta\cos^2\gamma f(r)\right]+\frac{\partial\zeta}{\partial b}\mathrm{S}\left[\frac{m\rho}{2}\cos\alpha\cos\beta\cos^2\gamma f(r)\right]+\frac{\partial\zeta}{\partial c}\mathrm{S}\left[\pm\frac{m\rho}{2}\cos\alpha\cos^3\gamma f(r)\right]
\end{aligned}\right\}
\end{cases}
$$

$$
\begin{cases}
E = \rho \mathrm{S}\left[\pm \frac{m\rho}{2}\cos\alpha\cos\gamma f(r)\right] \\
+\rho\left\{
\begin{aligned}
&\frac{\partial\eta}{\partial a}\mathrm{S}\left[\pm\frac{m\rho}{2}\cos^2\alpha f(r)\right]+\frac{\partial\eta}{\partial b}\mathrm{S}\left[\pm\frac{m\rho}{2}\cos\alpha\cos\beta f(r)\right]+\frac{\partial\eta}{\partial c}\mathrm{S}\left[\pm\frac{m\rho}{2}\cos\alpha\cos\gamma f(r)\right]\\
&\frac{\partial\xi}{\partial a}\mathrm{S}\left[\pm\frac{m\rho}{2}\cos\alpha\cos\gamma f(r)\right]+\frac{\partial\xi}{\partial b}\mathrm{S}\left[\pm\frac{m\rho}{2}\cos\beta\cos\gamma f(r)\right]+\frac{\partial\xi}{\partial c}\mathrm{S}\left[\pm\frac{m\rho}{2}\cos^2\gamma f(r)\right]
\end{aligned}\right\} \\
+\rho\left\{
\begin{aligned}
&\frac{\partial\xi}{\partial a}\mathrm{S}\left[\frac{m\rho}{2}\cos^3\alpha\cos\gamma f(r)\right]+\frac{\partial\xi}{\partial b}\mathrm{S}\left[\frac{m\rho}{2}\cos^2\alpha\cos\beta\cos\gamma f(r)\right]+\frac{\partial\xi}{\partial c}\mathrm{S}\left[\frac{m\rho}{2}\cos^2\alpha\cos^2\gamma f(r)\right]\\
&\frac{\partial\eta}{\partial a}\mathrm{S}\left[\frac{m\rho}{2}\cos^2\alpha\cos\beta\cos\gamma f(r)\right]+\frac{\partial\eta}{\partial b}\mathrm{S}\left[\frac{m\rho}{2}\cos\alpha\cos^2\beta\cos\gamma f(r)\right]+\frac{\partial\eta}{\partial c}\mathrm{S}\left[\frac{m\rho}{2}\cos\alpha\cos\beta\cos^2\gamma f(r)\right]\\
&\frac{\partial\zeta}{\partial a}\mathrm{S}\left[\frac{m\rho}{2}\cos^2\alpha\cos\beta\cos\gamma f(r)\right]+\frac{\partial\zeta}{\partial b}\mathrm{S}\left[\frac{m\rho}{2}\cos\alpha\cos\beta\cos^2\gamma f(r)\right]+\frac{\partial\zeta}{\partial c}\mathrm{S}\left[\pm\frac{m\rho}{2}\cos\alpha\cos^3\gamma f(r)\right]
\end{aligned}\right\}
\end{cases}
$$

$$
\begin{cases}
F = \rho \mathrm{S}\left[\pm \frac{m\rho}{2}\cos\alpha\cos\beta f(r)\right] \\
+\rho\left\{
\begin{aligned}
&\frac{\partial\eta}{\partial a}\mathrm{S}\left[\pm\frac{m\rho}{2}\cos\alpha\cos\beta f(r)\right]+\frac{\partial\eta}{\partial b}\mathrm{S}\left[\pm\frac{m\rho}{2}\cos^2\beta f(r)\right]+\frac{\partial\eta}{\partial c}\mathrm{S}\left[\pm\frac{m\rho}{2}\cos\beta\cos\gamma f(r)\right]\\
&\frac{\partial\xi}{\partial a}\mathrm{S}\left[\pm\frac{m\rho}{2}\cos^2\alpha f(r)\right]+\frac{\partial\xi}{\partial b}\mathrm{S}\left[\pm\frac{m\rho}{2}\cos\alpha\cos\beta f(r)\right]+\frac{\partial\xi}{\partial c}\mathrm{S}\left[\pm\frac{m\rho}{2}\cos\alpha\cos\gamma f(r)\right]
\end{aligned}\right\} \\
+\rho\left\{
\begin{aligned}
&\frac{\partial\xi}{\partial a}\mathrm{S}\left[\frac{m\rho}{2}\cos^3\alpha\cos\beta f(r)\right]+\frac{\partial\xi}{\partial b}\mathrm{S}\left[\frac{m\rho}{2}\cos^2\alpha\cos^2\beta f(r)\right]+\frac{\partial\xi}{\partial c}\mathrm{S}\left[\frac{m\rho}{2}\cos^2\alpha\cos\beta\cos\gamma f(r)\right]\\
&\frac{\partial\eta}{\partial a}\mathrm{S}\left[\frac{m\rho}{2}\cos^2\alpha\cos^2\beta f(r)\right]+\frac{\partial\eta}{\partial b}\mathrm{S}\left[\frac{m\rho}{2}\cos\alpha\cos^3\beta f(r)\right]+\frac{\partial\eta}{\partial c}\mathrm{S}\left[\frac{m\rho}{2}\cos\alpha\cos^2\beta\cos\gamma f(r)\right]\\
&\frac{\partial\zeta}{\partial a}\mathrm{S}\left[\frac{m\rho}{2}\cos^2\alpha\cos^2\beta\cos\gamma f(r)\right]+\frac{\partial\zeta}{\partial b}\mathrm{S}\left[\frac{m\rho}{2}\cos\alpha\cos^2\beta\cos\gamma f(r)\right]+\frac{\partial\zeta}{\partial c}\mathrm{S}\left[\frac{m\rho}{2}\cos\alpha\cos\beta\cos^2\gamma f(r)\right]
\end{aligned}\right\}.
\end{cases}
$$

Fig. 1.4 (continued)

no derivatives of displacement. Cauchy noted that if the primitive undeformed state is equilibrated with zero external forces (in modern terms, a natural state) six of the coefficients between the components of the stress and the derivatives of the displacement cancel. In fact for the undeformed state one must assume $\epsilon = 0$ and the components of the stress A, B, C, D, E, F reduce to the first elements of Fig. 1.4. In the absence of external forces, they must vanish, with all the sums which contain quadratic terms in the direction cosines. That also implies the vanishing of the terms in the second row of Fig. 1.4 that depend on displacements. Therefore, the non-zero coefficients are only those of the third row, characterized by terms of fourth order in the direction cosines, that are 15 in number, equal to the combinations with repetition of three objects ($\cos\alpha$, $\cos\beta$, $\cos\gamma$) of class 4 (the order of the product of the cosines).

Figure 1.4, in addition to enabling a control over the number of coefficients, shows a certain symmetry. The coefficients of the derivatives associated with the variables of displacement and position are equal; for example, the coefficients of $\partial\xi/\partial b$ and $\partial\eta/\partial a$ are equal; the same holds for $\partial\xi/\partial c$ and $\partial\zeta/\partial a$, etc. A modern reader can thus state that the components of the tension are expressed as a function of the six components of infinitesimally small deformation, arriving at a constitutive stress–strain relationship characterized by 15 coefficients only.

Cauchy did not report these considerations; he was not interested in a theory of constitutive relationships, he just wanted to get the stress as a function of the displacement derivatives in order to write the equations of equilibrium and motion for a system of material points in terms of displacement, as done by Navier. The partition of the elastic problem of continuum in stress analysis (equilibrium), strain analysis (compatibility) and the imposition of the constitutive relationship will be fully developed only with Lamé [86] and Saint Venant [143]. Cauchy also did not care about the number of constants that he had found for more general elastic models, in particular whether they are 15 or 21, although in a work of 1829 he gave a name to each coefficient and exposed them in the proper order [32].[19] According to Augustus Edward Hough Love [93], Rudolph Julius Emmanuel Clausius was among the first to highlight the particular number, 15, of the constants of the molecular model.[20] In fact already Poisson [127] had 'counted' the coefficients of the constitutive relationship in the form of infinitesimal strain versus stress, observing that those required are in general 36 and only as a result of the classical model hypotheses is the number reduced to 15.[21] Cauchy took the following further assumptions of material symmetry:

1. The body has three orthogonal planes of symmetry (orthotropy): the coefficients with at least one odd exponent of direction cosines vanish (the sums which express them cancel); the number of distinct coefficients is reduced to six.
2. The body has three planes of symmetry and the arrangement of the molecules is identical in the three orthogonal directions to these plans (complete orthotropy): in the expression of the coefficients one can exchange β with α, α with γ, etc.; the number of distinct coefficients goes down to two.
3. The body has the same arrangement of molecules around the point where the stress is to be evaluated (isotropy): with a complicated reasoning, perhaps not flawless, Cauchy showed that there is only a distinct coefficient.

[19] pp. 162–173.

[20] p. 9.

[21] pp. 83–85.

1.1.2 Internal Criticisms Toward the Classical Molecular Model

The molecular model by Navier, Cauchy and Poisson was accepted by the scientific international community, especially in France, because of the simplicity of the theory and the physical basis universally shared. However its conclusions were slightly but inexorably falsified by the experimental evidence. Thus it clearly appeared, with the advance of precision in measuring instruments, that to characterize isotropic linear elastic materials two constants were needed and not only one as suggested by the molecular model.[22]

A first attempt to adapt the classical molecular model to the experimental results consisted in relaxing some of the basic assumptions. Poisson was among the first, in a memoir read before the Académie des sciences de Paris in 1829 [127], to formulate the hypothesis of non-point molecules and crystalline arrangement; the idea of central forces depending only on the mutual distance between (the centers of) the molecules was thus released:

> It is assumed that, in a body of this nature, the molecules are uniformly distributed and attract or repel unevenly from their different sides. For this reason it is no longer possible, in calculating the force exerted by one part of a body to another, to consider the mutual force of two molecules as a simple function of the distance between them […]. In the case of a homogeneous body that is in its natural state, where it is not subjected to any external force, we can consider it as an assembly of molecules of the same nature and the same *shape* whose homologous sections are parallel to each other [127].[23] (A.1.5)

According to Poisson, in cristalline bodies the relations among the elastic constants that reduce their number to 15, obtained in his preceding works and in those by Cauchy, are no longer valid:

> The components P, Q,&c., thus being reduced to six different forces, and the value of each force may contain six particular coefficients, it follows that the general equations of equilibrium, and consequently those of the movement, contain thirty-six coefficients which may not reduce to a lesser number without limiting the generality of the question [127].[24] (A.1.6)

On the other hand, in non-crystalline bodies, with weak or irregular crystallization, even if the molecules are no longer considered punctiform, everything remains as if the forces were central. This is due to a compensation of causes:

> It follows that if we consider two parts A and B of a body that are not crystallized, which extend insensitively but which, however, include a great number of molecules, and we want to determine the total action of A on B, we can assume in this calculation that the mutual action of two molecules m and m' is reduced, as in the case of fluids, to a force R directed along the line joining their centers of gravity M and M', whose intensity will depend on the distance MM'. Indeed, whatever the action, it can be replaced by a similar force, which is the

[22] See the results found by Guillaume Wertheim (1815–1861) [158, pp. 581–610]. The greater the accuracy and reliability of the experimental results the more the theoretical predictions of Cauchy and Poisson were disclaimed, though it was not clear why [80, pp. 481–503].

[23] p. 69. Our translation.

[24] p. 85. Our translation.

> average of the actions of all points of m' on all of m, and we combine it with another force R', or, if necessary with two other forces R' and R'', dependent on the relative arrangement of the two molecules. However, because this disposition by hypothesis has not assumed any kind of regularity in A and B, and the number of molecules of A and B is extremely large and nearly infinite, one concludes that all the forces R' and R'' will compensate without altering the total action of A on B, which will not depend, therefore, but on the forces R. It should moreover be added that for the same increase in the distance, the intensity of the forces R' and R'' increases faster in general than that of the forces R; which will still contribute to make disappear the influence of the first forces on the mutual action of A on B [127].[25] (A.1.7)

Cauchy also expressed doubts about the validity of the classical molecular model in some memoirs of 1839 [35][26] and in a review of 1851 of some of Wertheim's memoirs about the experimental determination of elastic constants [36]. Cauchy stated that the molecules in crystalline bodies should not be considered as point-like but as very small particles composed of atoms. Since in crystals there is a regular arrangement of molecules, the elastic moduli are periodic functions of spatial variables; assertions taken later by Adhémar J.C. Barré de Saint Venant [116].[27] In order to obtain a constitutive relation with uniform coefficients, Cauchy expanded the number of elastic moduli, finally reaching only two in the case of isotropic materials.[28]

Gabriel Lamé [86, 87] in his works on the theory of elasticity raised a number of questions on the issue . For instance, much of the twentieth lesson of the *Leçons sur les coordonnées curvilignes et leurs diverses applications* of 1859 [87] was dedicated to concerns about the real nature of molecules, to the assumption about the exact mutual actions, to what is a reasonable form of the law of the intermolecular actions, to what is the direction of the latter. In his 1852 monograph on the mathematical theory of elasticity, *Leçons sur la théorie mathématique de l'élasticité des corps solides*, Lamé [86] first obtained the linear elastic constitutive relations for point molecules and intermolecular central forces. Moreover, assuming that each component of the stress is a linear function of all the components of the strain, the linear elasticity in general is described by 36 coefficients. Also assuming isotropy (*élasticité constante*), considerations about invariance with rotations reduce the number of coefficients to two, denoted by λ and μ:

> By this method of reduction, it is obtained finally for N_i, T_i, in the case of homogeneous solids and constant elasticity, the values [...] containing two coefficients, λ and μ. When with the method indicated at the end of the third lesson, we find $\lambda = \mu$, it remains a single coefficient only. We will not accept this relationship, which is necessarily based on the assumption of continuity of the material in the solid media. The results of Wertheim's experiments show clearly that ratio λ to μ is not the unity, but neither seem to assign to this ratio another immovable value. We retain the two coefficients λ and μ, leaving undetermined their ratio [86].[29] (A.1.8)

[25] pp. 7–8. Our translation.

[26] s. 2, vol XI, pp. 11–27; 51–74; 134–172.

[27] Appendix V, p. 689.

[28] A detailed reconstruction of Cauchy's topics is shown in [116], Appendix V, pp. 691–706.

[29] pp. 51–52. Our translation.

With arguments similar to those of Poisson in 1829 [127], Lamé showed that even for crystalline bodies, the relation with 36 constants [86][30] holds good and identified the error of Cauchy's and Poisson's treatment in the assumption of the uniformity of matter, which allows the symmetry considerations that would otherwise be ineligible:

> This is the method followed by Navier and other geometers to obtain the general equations of elasticity in solid bodies. But obviously this method implies the continuity of matter, an unacceptable hypothesis. Poisson believes to overcome this difficulty, [...] but [...], in reality, he simply substitutes the sign Σ to the sign $\int$ [...]. The method we have followed [...] whose origins lie in the work of Cauchy, seems at the basis of any objection [...] [86].[31] (A.1.9)

Although the results of the molecular theory of elasticity were clearly considered unsatisfactory even by the followers of the French school of mechanics, it was not the case for the validity of the molecular approach. One of the main proponents of this approach was Saint Venant; his ideas on the matter, besides in publications to his name, are contained in the enormous amount of notes, comments and appendices to the *Theorie der Elasticität fester Körper* by Alfred Clebsch, translated into French [42], and to the *Résumé des leçons donnés a l'école des pontes et chaussées* by Navier [116] where Saint Venant said:

> The elasticity of solid bodies, as well as of fluids, [...], all their mechanical properties prove that the molecules, or the last particles composing them, exert on each other actions [which are] repulsive [and] infinitely growing for the smallest possible mutual distances, and becoming attractive for considerable distances, but relatively inappreciable when such distances, of which they [the molecular actions] are functions, assume a sensible value [116].[32] (A.1.10)

For crystalline bodies the classical molecular model seemed not to be valid:

> I do not yet refuse to recognize that the molecules whose various settings make up the texture of the solids and whose small change of distance produce noticeable strains called ∂, g are not the *atoms* constituting matter, but are unknown groups. I accordingly recognize, thinking that the actions between atoms are governed by laws of intensity depending on the distances only where they operate, it is not certain that the *resultant* actions and the actions of the molecules must exactly follow the same law of the distances from their centers of gravity. We also consider that the groups, changing distances, can change orientation [42].[33] (A.1.11)

But, added Saint Venant, this is only an ideal situation, because the ordinary bodies are not crystals and also the thermal motions produce a chaotic situation that on average leads to a law of force at a distance of molecules substantially of the same type as that which there is between the atoms. Saint Venant made the six components of the tension to depend linearly on the six strain components, yet resulting in an elastic relationship in terms of 36 coefficients. However he continued to admit the validity of the equalities known as Cauchy-Poisson relations (see note 69 of Chap. 1), which for isotropic bodies leads to a single constant:

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[30] pp. 36–37.

[31] p. 38. Our translation.

[32] pp. 542–543. Our translation.

[33] p. 759. Our translation.

> The 36 coefficients [...] are not independent of each other, and it is easy to see that there are 21 equalities among them [116].[34] (A.1.12)

In fact, the proof that these relations are valid considers variations of the intermolecular distance that are the same under an extension in a given direction and an appropriate angular distortion [116].[35] If the intermolecular force is central and depends only on the variation of the distance between the centers of the molecules, the force between the molecules and consequently the stress, is equal. Thus, there are similarities between the elastic constants, which reduce the number from 36 to 15, in particular, for isotropic bodies, Saint Venant found a single constant:

> The thirty-six coefficients [...] reduce to two [...] and one may even say to one only [...] in the same way as the thirty six coefficients are reducible to fifteen [116].[36] (A.1.13)

Saint Venant knew very well that these conclusions were contradicted by experiments, and since he did not find evident defects in the molecular theory of elasticity, preferred to accept that there are no isotropic bodies in nature:

> Yet experiences [...] and the simple consideration on the way cooling and solidification take place in bodies, prove that isotropy is quite rare [...]. So, instead of using, in place of the equations [...] with one coefficient only, the formulas [...] with two coefficients [...], which hold, like these others, only for perfectly isotropic bodies, it will be convenient to use as many times as possible the formulas [...] relative to the more general case of different elasticity in two or three directions [116].[37] (A.1.14)

In some works in the *Journal de mathématiques pures et appliquèes*, from 1863 to 1868 [145–147],[38] Saint Venant introduced the concept of *amorphous bodies* (corps amorphes) to define the properties acquired by bodies that were initially isotropic as a result of geological processes. In this state, the mechanical properties are characterized by three coefficients and not just two as in the case of isotropic bodies.

Saint Venant spent more than 200 pages of notes and appendices to Navier's lessons in order to present experimental results and attempts to explain the paradox, showing a wide knowledge of the literature of his time (among others, he quoted Savart, Wertheim, Hodgkinson, Regnault, Oersted, Green, Clebsch, Kirchhoff, Rankine, William Thomson). In the end, however, the question remained, because there was no agreement between the approaches of Saint Venant's contemporaries. Although it was clear that two elastic constants were necessary, where was the flaw in a theory attractive and apparently founded as Navier's, Cauchy's and Poisson's?

The debate between the scholars of mechanics was strengthened, from different points of view, by the works of Augustin Cauchy, George Green and Auguste Bravais, who gave life to different schools of elasticity in England and Germany.

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[34] p. 556. Our translation.

[35] pp. 556–560.

[36] p. 582. Our translation.

[37] p. 583. Our translation.

[38] In the order: pp. 353–430; 297–350; 242–254.

1.1.3 Substitutes for the Classical Molecular Model

The molecular model was not the only model with which engineers, physicists and mathematicians tried to represent the behavior of elastic bodies. On September 30th, 1822, 1 year after Navier's memoir, Cauchy [25] presented to the Académie des sciences de Paris a memoir that dealt with the study of elasticity according to a continuist approach largely unchanged since then. That of Cauchy was a purely phenomenological approach, in line with the positivistic tendencies that had developed among French scientists.[39]

The matter was modeled as a mathematical continuum without any assumption of physical nature. It was assumed that the different parts of matter exchange forces and become deformed. The relations between internal forces and deformations had a general nature and the number of elastic constants that defined the problem was simply determined by counting the components of stress and strain. In its most complete version, Cauchy's continuous model led to a stress–strain relationship defined by 36 coefficients.

A different approach was that of Green (1793–1841), who in a work of 1839 [75] also followed a phenomenological point of view assuming a three dimensional continuum to model matter, uninterested even in the concept of internal forces. Green, however, recurred to a mechanical principle, that of the existence of a potential of the internal forces, which somehow gave some theoretical force to his arguments.

1.1.3.1 Cauchy's Phenomenological Approach

Of the presentation before the Académie des sciences de Paris in 1822, there is an excerpt published in 1823 [25],[40] where the *principle of stress* is formulated.[41] Over any oriented and regular surface separating a body into two parts there is a regular vector field that expresses the actions between the two parts:

> If in an elastic or non-elastic solid body a small invariable volume element, terminated by any faces at will, is made [imagined] rigid, this small element will experience on its different sides, and at each point of each of them, a determined pressure or tension. This pressure or tension is similar to the pressure a fluid exerts against a part of the envelope of a solid body, with the only difference that the pressure exerted by a fluid at rest, against the surface of a solid body, is directed perpendicularly to the surface inwards from the outside, and in each point independent of the inclination of the surface relative to the coordinate planes, while the pressure or tension exerted at a given point of a solid body against a very small element of surface through the point can be directed perpendicularly or obliquely to the surface, sometimes from outside to inside, if there is condensation, sometimes from within

[39] For a discussion of the positivistic conceptions of French science in the first half of the 19th century, see [124].

[40] It seems that on September 30th 1822, Cauchy notified the Académie of his researches neither delivering a public reading, nor depositing a manuscript; see [3] p. 97. In [154] it is stated that Cauchy, as a matter of fact, presented his memoir.

[41] Cauchy used *tension* or *pressure* for traction and compression respectively.