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Trends in Contemporary Mathematics

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Trends in Contemporary Mathematics

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Preface

INdAM, the Istituto Nazionale di Alta Matematica, has been a leading Italian mathematics research institute ever since it was founded in 1939. Throughout its existence, one of its principal aims has been to invite leading scientists in order to present their research and to deliver high-level training, while at the same time interacting with the mathematical communities scattered around the country.

An important initiative in accomplishing this goal is the INdAM Day, an event conceived a decade ago by Corrado De Concini, then President of INdAM, with the intention of providing an insight into the state of the art in contemporary mathematics by means of four high-level expository lectures. Since the first INdAM Day was held on 18 June 2004 in Rome, each year speakers have been chosen by INdAM from among leading mathematicians around the world, and various Departments of Mathematics around Italy have hosted the initiative: Naples in 2005, Milan in 2006, Pisa in 2007, Padua in 2008, Turin in 2009, Catania in 2010, L'Aquila in 2011, Genoa in 2012, and Palermo in 2013.

To date, more than 40 mathematicians of international renown have delivered talks covering a wide spectrum of current trends in mathematics. These talks have not only been of obvious scientific interest but have also managed to prove the cultural relevance of mathematics. None of us on the INdAM staff would ever pretend to be capable of emulating Hilbert more than a century ago. As we could not possibly match his breadth of vision, we have simply reached a compromise in focusing on certain topics. These have not always been in areas in which we have extensive personal knowledge, but for whatever reason our selection seems to have repeatedly captured the attention of people over the years. We could never predict which areas of mathematics are likely to be fashionable, but our speakers have certainly succeeded in making predictions about mathematics as a whole.

We have also had some strokes of luck, as when Claire Voisin was awarded the Clay Research Award shortly before delivering her talk in Padua in 2008, or when Cédric Villani was awarded the Fields Medal at the ICM in Hyderabad just 2 months after giving his talk in Catania in 2010.

This volume presents a selection of these talks in order to leave a visible trace of the original efforts of INdAM.

Rome, Italy, 2014

Vincenzo Ancona
Elisabetta Strickland

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The Editors of this volume would like to thank Dott.ssa Elisabetta Esposito of INdAM, who contributed patiently to the birth of this collection of papers, a task which required special devotion in order to achieve the goals typical of a research institute.

Contents

Chapter 1
Interpolation and Comparison Methods in the Mean Field Spin Glass Model

Francesco Guerra

Abstract We give a short overview of the recent rigorous mathematical methods developed for the study of complex disordered systems, in particular spin glasses in the mean field Sherrington-Kirkpatrick formulation. We show that interpolation methods, and related comparison arguments, are very powerful tools in order to study these models. We consider the problem of the infinite volume limit for the free energy, Then we introduce the Parisi solution for the spin glass, based on the spontaneous breaking of replica symmetry, and characterized by a functional order parameter entering in a variational principle. We show how the validity of the Parisi representation can be rigorously established. Finally, we point out some perspective for future developments.

1.1 Introduction

In a famous paper on Physical Review Letters, more than 30 years ago, David Sherrington and Scott Kirkpatrick introduced a celebrated mean field model for spin glasses [1, 2], then considered to be a "solvable model".

The impact of this model on the theoretical physics research has been impressive. During the three decades after its introduction, hundreds and hundreds of papers have been devoted to the study of its properties, even through numerical methods.

Expanded version of an invited lecture delivered at the INdAM Day, Rome, June 18, 2004.

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The relevance of the model surely comes from the fact that it is able to represent successfully, at least at the level of the mean field approximation, some important features of the physical spin glass systems, of great interest for their peculiar properties.

Some dilute magnetic alloys called spin glasses (see [3] and [4] for extensive reviews) are extremely interesting systems from a physical point of view. Their peculiar feature is to exhibit a new magnetic phase, where magnetic moments are frozen into disordered equilibrium orientations, without any long-range order. Moreover, these materials have some very slowly relaxing modes, with consequent memory effects.

The Sherrington-Kirkpatrick (SK) model is a simplified mean field model, intended to capture some basic properties of spin glasses.

There is also an additional very important reason for the relevance of this model, and related ones. In fact, recently it has become progressively clear that disordered systems of the Sherrington-Kirkpatrick type, and their generalizations, seem to play a very important role for theoretical and practical applications to hard optimization problems, as it is shown for example by Marc Mézard, Giorgio Parisi and Riccardo Zecchina in [5].

It is interesting to remark that the original paper was entitled "Solvable Model of a Spin-Glass", while a previous draft, according to what reported by David Sherrington, contained even the stronger denomination "Exactly Solvable". However, it turned out that the very natural solution devised by the authors is valid only at high temperatures, or for large external magnetic fields. At low temperatures, the proposed solution exhibits a nonphysical drawback given by a negative entropy, as properly recognized by the authors in their very first paper.

It took a few years to find an acceptable solution. This was done by Giorgio Parisi in a series of papers, by marking a radical departure from the previous methods. In fact, a very deep method of "spontaneous replica symmetry breaking" was developed. As a consequence the physical content of the theory was encoded in a functional order parameter of new type, and a remarkable structure began to show up for the pure states of the theory, characterized by a kind of hierarchical, ultrametric organization. These very interesting developments, due to Giorgio Parisi, and his coworkers, are explained in a challenging way in the classical book [6]. Part of this structure will be recalled in the following.

It is important to remark that the Parisi solution is presented in the form of an ingenious and clever *Ansatz*. Until a few years ago it was not known whether this *Ansatz* would give the true solution for the model, in the so-called thermodynamic limit, when the size of the system becomes infinite, or it would be only a very good approximation to the true solution.

The general structures offered by the Parisi solution, and their possible generalizations for similar models, exhibit an extremely rich and interesting mathematical content. In a very significant way, Michel Talagrand inserted a strongly suggestive sentence in the title to his book [7]: "Spin glasses: a challenge for mathematicians".

As a matter of fact, the problem of giving a proper mathematical understanding of the spin glass structure is extremely difficult. In this talk, we would like to recall the main features of a very powerful method, yet extremely simple in its very essence, based on comparison and interpolation arguments on families of Gaussian random variables.

The method found its first simple application in [8], where it was shown that the Sherrington-Kirkpatrick replica symmetric approximate solution is a rigorous lower bound for the quenched free energy of the system, uniformly in the size, for any value of the temperature and the external magnetic field. Then, it was possible to reach a long awaited result [9]: the convergence of the free energy density in the thermodynamic limit.

Moreover, still by a generalized interpolation on families of Gaussian random variables, the first mentioned result, on the replica symmetric solution, was extended to give a rigorous proof that the expression given by the Parisi *Ansatz* is also a lower bound for the quenched free energy of the system, uniformly in the size [10]. The method gives not only the bound, but also the explicit form of the correction terms in the form of a sum rule. In a subsequent very important result, Michel Talagrand has been able to dominate these correction terms, showing that they vanish in the thermodynamic limit. This extraordinary achievement was firstly announced in a short note [11], containing only a synthetic sketch of the proof, and then presented with all details in a long paper in Annals of Mathematics [12].

The interpolation method is also at the basis of the far-reaching generalized variational principle proven by Michael Aizenman, Robert Sims and Shannon Starr in [13].

In this lecture, we will concentrate mostly on the main questions connected with the free energy. In particular, we will consider the subadditivity of the quenched free energy with respect to the system size, the existence of the infinite-volume limit, the broken replica symmetry sum rules and bounds, and the Parisi variational principle. Our treatment will be as simple as possible, by relying on the basic structural properties, and by describing methods of presumably very long lasting power.

The organization of the paper is as follows. In Sect. 1.2 we explain the basic features of the mean field spin glass models, by introducing all necessary definitions. In next Sect. 1.3 we give a simple application of the interpolation method to the mean-field spin glass model, by showing the sub-additivity of the quenched free energy with respect to the system size, and the existence of the infinite-volume limit [9].

Section 1.4 is devoted to a description of the main features of the Parisi representation for the free energy and to its rigorous establishment.

Section 1.5 is devoted to some results, which have been obtained after the talk given at INdAM, and to perspectives for further developments.

In conclusion, the author would like to thank the organizers of the first 2004 INdAM Day in Rome, in particular Corrado De Concini, for the kind invitation and exquisite hospitality.

1.2 Basic Definitions for the Mean Field Spin Glass Model

The generic configuration of the mean field spin glass model is defined through Ising spin variables $\sigma_i = \pm 1$, attached to each site $i = 1, 2, \ldots, N$.

But now there is also an external quenched disorder given by the $N(N-1)/2$ independent and identical distributed random variables J_{ij}, defined for each couple of sites. For the sake of simplicity, we assume each J_{ij} to be a centered unit Gaussian with averages $E(J_{ij}) = 0$, $E(J_{ij}^2) = 1$. By quenched disorder we mean that the J have a kind of stochastic external influence on the system, without participating to the thermal equilibrium.

Now the Hamiltonian of the model is given by the mean field expression

$$H_N(\sigma, J) = -\frac{1}{\sqrt{N}} \sum_{(i,j)} J_{ij}\sigma_i\sigma_j . \tag{1.1}$$

Here, the sum runs over all couples of sites. Notice that the term $\sqrt{N}$ is necessary in order to ensure a good thermodynamic behavior to the free energy, extensive in the system size N. For the sake of simplicity, we have considered only the case of zero external field. But the general case, with a magnetic external field, can be treated without any essential additional complication.

For a given inverse temperature β, let us now introduce the disorder-dependent partition function $Z_N(\beta, J)$ and the quenched average of the free energy per site $f_N(\beta)$, according to the definitions

$$Z_N(\beta, J) = \sum_{\sigma_1 \ldots \sigma_N} \exp(-\beta H_N(\sigma, J)), \tag{1.2}$$

$$-\beta f_N(\beta) = N^{-1} E \log Z_N(\beta, J). \tag{1.3}$$

Notice that in (1.3) the average E with respect to the external noise is made *after* the log is taken. This procedure is called quenched averaging. It represents the physical idea that the external noise does not participate in the thermal equilibrium. Only the σ_i variables are thermalized.

For the sake of simplicity, it is also convenient to write the partition function in the following equivalent form. First of all let us introduce a family of centered Gaussian random variables $\mathcal{K}(\sigma)$, indexed by the configurations σ, and characterized by the covariances

$$E\big(\mathcal{K}(\sigma)\mathcal{K}(\sigma')\big) = q^2(\sigma, \sigma'), \tag{1.4}$$

where $q(\sigma, \sigma')$ are the overlaps between two generic configurations, defined by

$$q(\sigma, \sigma') = N^{-1} \sum_i \sigma_i \sigma'_i, \tag{1.5}$$

with the obvious bounds $-1 \leq q(\sigma, \sigma') \leq 1$, and the normalization $q(\sigma, \sigma) = 1$. Then, starting from the definition (1.1), it is immediately seen that the partition function in (1.2) can be also written, by neglecting unessential constant terms, in the form

$$Z_N(\beta, \mathcal{K}) = \sum_{\sigma_1 \dots \sigma_N} \exp(\beta \sqrt{\frac{N}{2}} \mathcal{K}(\sigma)), \tag{1.6}$$

which will be the starting point of our treatment. Here the dependence of the partition function on the random variables $\mathcal{K}$ has been stressed in the notation.

According to the general well established strategy of statistical mechanics [14], firstly we consider the problem of the infinite volume limit.

1.3 The Thermodynamic Limit for the Free Energy

The proof of the convergence of the free energy per site in the thermodynamic limit was a result long awaited since decades. In [9] it was possible to give an unexpected very simple proof. Let us show the argument. Consider a system of size N and two smaller systems of sizes N_1 and N_2 respectively, with $N = N_1 + N_2$. Let us now compare

$$E \log Z_N(\beta, \mathcal{K}) = E \log \sum_{\sigma_1 \dots \sigma_N} \exp(\beta \sqrt{\frac{N}{2}} \mathcal{K}(\sigma)), \tag{1.7}$$

with

$$E \log \sum_{\sigma_1 \dots \sigma_N} \exp(\beta \sqrt{\frac{N_1}{2}} \mathcal{K}_1(\sigma^{(1)})) \exp(\beta \sqrt{\frac{N_2}{2}} \mathcal{K}_2(\sigma^{(2)})) =$$

$$E \log Z_{N_1}(\beta, \mathcal{K}_1) + E \log Z_{N_2}(\beta, \mathcal{K}_2), \tag{1.8}$$

where $\sigma^{(1)}$ are the $(\sigma_i,\ i = 1, \dots, N_1)$, and $\sigma^{(2)}$ are the $(\sigma_i,\ i = N_1 + 1, \dots, N)$. Covariances for $\mathcal{K}_1$ and $\mathcal{K}_2$ are expressed as in (1.4), but now the overlaps are replaced with the partial overlaps of the first and second block, q_1 and q_2 respectively, defined as

$$q_1(\sigma, \sigma') = N_1^{-1} \sum_{i=1}^{N_1} \sigma_i \sigma_i', \tag{1.9}$$

and analogously for the q_2 of the second block.

The key idea now is to build an interpolation scheme, between the large system and the two small systems. This is easily achieved by introducing the interpolation parameter $0 \leq t \leq 1$, and the interpolating auxiliary function $\phi(t)$, defined as

$$\phi(t) = E \log \sum_{\sigma_1 \ldots \sigma_N} \exp(\sqrt{t}\beta\sqrt{\frac{N}{2}}\mathcal{K} + \sqrt{1-t}\beta\sqrt{\frac{N_1}{2}}\mathcal{K}_1 + \sqrt{1-t}\beta\sqrt{\frac{N_2}{2}}\mathcal{K}_2). \tag{1.10}$$

Here, we have realized the families of random variables $\mathcal{K}, \mathcal{K}_1, \mathcal{K}_2$ as independent on the same probability space. The interpolation through the $\sqrt{t}$ and $\sqrt{1-t}$ assures a linear interpolation between the respective covariances. Obviously, we have

$$\phi(1) = E \log Z_N(\beta, \mathcal{K}),$$

while

$$\phi(0) = E \log Z_{N_1}(\beta, \mathcal{K}_1) + E \log Z_{N_2}(\beta, \mathcal{K}_2).$$

Now it is easy to calculate directly the t derivative of ϕ (see for example [15]), with the result

$$\frac{d}{dt}\phi(t) = \frac{\beta^2}{4}\frac{N_1 N_2}{N}\langle(q_1 - q_2)^2\rangle_t, \tag{1.11}$$

where $\langle\rangle_t$ is a quite complicated, but explicitly given, t dependent probability measure on the random variables (q_1, q_2) [15]. In this derivation we have exploited the simple connection between the global overlap and the block overlaps

$$Nq = N_1 q_1 + N_2 q_2. \tag{1.12}$$

Since in any case the square in (1.11) is positive, by integrating on t and by exploiting the recognized boundary values at $t = 0$ and $t = 1$, we reach the superadditivity property

$$E \log Z_N(\beta, \mathcal{K}) \geq E \log Z_{N_1}(\beta, \mathcal{K}_1) + E \log Z_{N_2}(\beta, \mathcal{K}_2), \tag{1.13}$$

firstly established in [9]. Of course, the corresponding free energies show a subadditive property, because of the minus sign involved in their definition.

From the superaddivity property, through standard methods [14], the existence of the limit follows in the form

$$\lim_{N\to\infty} N^{-1} E \log Z_N(\beta, \mathcal{K}) = \sup_N N^{-1} E \log Z_N(\beta, h, \mathcal{K}) \equiv -\beta f(\beta). \tag{1.14}$$

1.4 Comparison with the Parisi Representation for the Free Energy

We refer to the original paper [16], and to the extensive review given in [6], for the general motivations, and the derivation of the broken replica symmetry *Ansatz*, in the frame of the ingenious replica trick. Here we limit ourselves to a synthetic description of its general structure, independently from the replica trick. The deep motivation for the introduction of the Parisi trial functional is sketched in [17], in the frame of the cavity method (see also [18]).

First of all, let us introduce the convex space $\mathcal{X}$ of the functional order parameters x, as nondecreasing functions of the auxiliary variable q, both x and q taking values on the interval [0, 1], i.e.

$$\mathcal{X} \ni x : [0,1] \ni q \to x(q) \in [0,1]. \tag{1.15}$$

Notice that we call x the function, and $x(q)$ its values. We introduce a metric on $\mathcal{X}$ through the $L^1([0,1], dq)$ norm, where dq is the Lebesgue measure.

For our purposes, we will consider the case of piecewise constant functional order parameters, characterized by an integer K, and two sequences $q_0, q_1, \dots, q_K$, $m_1, m_2, \dots, m_K$ of numbers satisfying

$$0 = q_0 \le q_1 \le \dots \le q_{K-1} \le q_K = 1, \quad 0 \le m_1 \le m_2 \le \dots \le m_K \le 1, \tag{1.16}$$

such that

$$x(q) = m_1 \text{ for } 0 = q_0 \le q < q_1, \quad x(q) = m_2 \text{ for } q_1 \le q < q_2, \\ \dots, x(q) = m_K \text{ for } q_{K-1} \le q \le q_K. \tag{1.17}$$

In the following, we will find it convenient to define also $m_0 \equiv 0$, and $m_{K+1} \equiv 1$. The replica symmetric case of Sherrington and Kirkpatrick corresponds to

$$K = 2, \; q_1 = \bar{q}, \; m_1 = 0, \; m_2 = 1. \tag{1.18}$$

Let us now introduce the function f, with values $f(q, y; x, \beta)$, of the variables $q \in [0,1]$, $y \in R$, depending also on the functional order parameter x, and on the inverse temperature β, defined as the solution of the nonlinear antiparabolic equation

$$(\partial_q f)(q,y) + \frac{1}{2}(\partial_y^2 f)(q,y) + \frac{1}{2}x(q)(\partial_y f)^2(q,y) = 0, \tag{1.19}$$

with final condition

$$f(1,y) = \log\cosh(\beta y). \tag{1.20}$$

Here, we have stressed only the dependence of f on q and y.

It is very simple to integrate Eq. (1.19) when x is piecewise constant. In fact, consider $x(q) = m_a$, for $q_{a-1} \le q \le q_a$, firstly with $m_a > 0$. Then, it is immediately seen that the correct solution of Eq. (1.19) in this interval, with the right final boundary condition at $q = q_a$, is given by

$$f(q, y) = \frac{1}{m_a} \log \int \exp\big(m_a f(q_a, y + z\sqrt{q_a - q})\big)\, d\mu(z), \tag{1.21}$$

where $d\mu(z)$ is the centered unit Gaussian measure on the real line. On the other hand, if $m_a = 0$, then (1.19) loses the nonlinear part and the solution is given by

$$f(q, y) = \int f(q_a, y + z\sqrt{q_a - q})\, d\mu(z), \tag{1.22}$$

which can be seen also to follow from (1.21) in the limit $m_a \to 0$. Starting from the last interval K, and using (1.21) iteratively on each interval, we easily get the solution of (1.19) and (1.20), in the case of piecewise constant order parameter x, as in (1.17), through a chain of Gaussian integrations.

Now we introduce the following important definitions. The trial auxiliary function, associated to a given mean field spin glass system, as described in Sect. 1.3, depending on the functional order parameter x, is defined as

$$\log 2 + f(0, 0; x, \beta) - \frac{\beta^2}{2} \int_0^1 q\, x(q)\, dq. \tag{1.23}$$

Notice that in this expression the function f appears evaluated at $q = 0$, and $y = 0$.

The Parisi spontaneously broken replica symmetry expression for the free energy is given by the definition

$$-\beta f_P(\beta) \equiv \inf_x \Big(\log 2 + f(0, 0; x, \beta) - \frac{\beta^2}{2} \int_0^1 q\, x(q)\, dq\Big), \tag{1.24}$$

where the infimum is taken with respect to all functional order parameters x.

Notice that the infimum appears here, as compared to the supremum that would appear in a variational principle of the usual entropy type in statistical mechanics. Therefore, Parisi variational principle is really a new structure in statistical mechanics, that deserves careful study in itself.

In [10], by exploiting a suitable interpolation scheme, we have established a rigorous connection between the partition function of the mean field spin glass and the Parisi *Ansatz*. We skip all details and state only the final result, in the form of the sum rule

$$\log 2 + f(0, 0; x, \beta) - \frac{\beta^2}{2} \int_0^1 q\, x(q)\, dq =$$

$$N^{-1} E \log Z_N(\beta, \mathcal{K}) + \frac{\beta^2}{4} \langle (q_{12} - q_a)^2 \rangle, \tag{1.25}$$

where $\langle\rangle$ is an explicitly given but quite complicated measure average over the variables σ, σ', appearing in the two replica overlap q_{12}, and the variable $q_{.}$, taking the values q_a. The sum rule holds for any value of the order parameter x. One of the miracles occurring in the proof of this sum rule is that the second term appearing in the Parisi trial functional here comes for free from the completion of the square in the third term of the sum rule.

In any case, the third term, being the average of a square, is positive. Therefore we have the following important result.

Theorem 1.1. *For all values of the inverse temperature β, and for any functional order parameter x, the following bound holds*

$$N^{-1} E \log Z_N(\beta, \mathcal{K}) \leq \log 2 + f(0,0; x, \beta) - \frac{\beta^2}{2} \int_0^1 q\, x(q)\, dq,$$

uniformly in N. Consequently, we have also

$$N^{-1} E \log Z_N(\beta, \mathcal{K}) \leq \inf_x \big(\log 2 + f(0,h; x, \beta) - \frac{\beta^2}{2} \int_0^1 q\, x(q)\, dq\big),$$

uniformly in N.

This result can be understood also in the frame of the generalized variational principle established by Aizenman-Sims-Starr [13], as shown for example in [15], by exploiting the general structure of the Derrida-Ruelle-Parisi probabibility cascades.

Up to this point we have seen how to obtain upper bounds. The problem arises whether we can also get lower bounds, so as to shrink the thermodynamic limit to the value given by the $\inf_x$ in Theorem 1.1. After a short announcement in [11], Michel Talagrand wrote an extended paper [12], where the complete proof of the control of the lower bound is firmly established. We refer to the original paper for the complete details of this remarkable achievement. About the methods, here we only recall that the sum rule in [10], explained above, gives also the corrections to the bounds appearing in Theorem 1.1, albeit in a quite complicated form. Talagrand has been able to establish that these corrections do in fact vanish in the thermodynamic limit. In order to be able to reach this important result it is necessary to prove an extension of the broken replica symmetry bounds of Theorem 1.1 to the case where two replicas of the system are coupled together. This task has not been reached yet in its full generality, but the treatment given by Talagrand is sufficient to prove the vanishing of the correction terms in the infinite volume limit.

In conclusion, we can establish the following conclusive result about the expression of the free energy in the mean field spin glass.

Theorem 1.2. *For the mean field spin glass model we have*

$$\lim_{N\to\infty} N^{-1} E \log Z_N(\beta,\mathcal{K}) = \sup_N N^{-1} E \log Z_N(\beta,\mathcal{K}) \tag{1.26}$$

$$= \inf_x \bigl(\log 2 + f(0,0;x,\beta) - \frac{\beta^2}{2}\int_0^1 q\, x(q)\, dq\bigr). \tag{1.27}$$

1.5 Further Developments and Outlook

As we have seen, in these last few years there has been an impressive progress in the understanding of the mathematical structure of spin glass models, mainly due to the systematic exploitation of interpolation methods. However many important problems are still open. The most important one is the full understanding of the hierarchical ultrametric organization of the overlap distributions, as appears in Parisi theory, and the decomposition in pure states of the glassy phase, at low temperatures. An important step in this direction have been obtained through the establishment of the so called Ghirlanda-Guerra identities [19]. Based on these, Dmitry Panchenko [20] has been able to prove ultra-metricity of the overlap distribution, a very remarkable achievement of the last years.

Moreover, interpolation and comparison methods have been extended to other important disordered models, such as for example neural networks, bipartite models, multi-species models. Here the difficulty is that the positivity arguments, so essential in the application of the interpolation methods, do not seem to emerge naturally inside the structure of the theory. For recent results see [21–27].

Even for a class of simple mean field diluted ferromagnetic systems, the treatment of the infinite volume limit has not been reached yet, due to the lack of positivity arguments. Only the $\beta \to \infty$ limit is well understood [28].

For extensions to diluted spin glass models we refer for example to [29–31].

Finally, the problem of connecting properties of the short-range model with those arising in the mean field case is still almost completely open. For partial results, and different points of view, see [32–37].

Finally, we mention a pedagogically very useful complete review appeared [38], about the application of the interpolation methods, and the other methods of spin glass theory, to the simple case of the ferromagnetic mean field model.

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References

1. D. Sherrington, S. Kirkpatrick, Solvable model of a spin-glass. Phys. Rev. Lett. **35**, 1792–1796 (1975)
2. S. Kirkpatrick, D. Sherrington, Infinite-ranged models of spin-glasses. Phys. Rev. **B17**, 4384–4403 (1978)
3. P. Young (ed.), *Spin Glasses and Random Fields* (World Scientific, Singapore, 1987)
4. D.L. Stein, Disordered systems: mostly spin glasses, in *Lectures in the Sciences of Complexity*, ed. by D.L. Stein. (Addison-Wesley, New York, 1989)
5. M. Mézard, G. Parisi, R. Zecchina, Analytic and algorithmic solution of random satisfiability problems. Science **297**, 812 (2002)
6. M. Mézard, G. Parisi, M.A. Virasoro, *Spin Glass Theory and Beyond* (World Scientific, Singapore, 1987)
7. M. Talagrand, *Spin Glasses: A Challenge for Mathematicians. Mean Field Models and Cavity Method* (Springer, Berlin, 2003)
8. F. Guerra, Sum rules for the free energy in the mean field spin glass model. Fields Inst. Commun. **30**, 161 (2001)
9. F. Guerra, F.L. Toninelli, The thermodynamic limit in mean field spin glass models. Commun. Math. Phys. **230**, 71–79 (2002)
10. F. Guerra, Broken replica symmetry bounds in the mean field spin glass model. Commun. Math. Phys. **233**, 1–12 (2003)
11. M. Talagrand, The generalized Parisi formula. Comptes Rendus de l'Académie des Sciences, Paris **337**, 111–114 (2003)
12. M. Talagrand, The Parisi formula. Ann. Math. **163**, 221–263 (2006)
13. M. Aizenman, R. Sims, S. Starr, Extended variational principle for the Sherrington-Kirkpatrick spin-glass model. Phys. Rev. **B68**, 214403 (2003)
14. D. Ruelle, *Statistical Mechanics. Rigorous Results* (W.A. Benjamin Inc., New York, 1969)
15. F. Guerra, An introduction to mean field spin glass theory: methods and results, in *Mathematical Statistical Physics*, ed. by A. Bovier et al. (Elsevier, Oxford/Amsterdam, 2006), pp. 243–271
16. G. Parisi, A sequence of approximate solutions to the S-K model for spin glasses. J. Phys. **A13**, L-115 (1980)
17. F. Guerra, Fluctuations and thermodynamic variables in mean field spin glass models, in *Stochastic Processes, Physics and Geometry, II*, ed. by S. Albeverio, U. Cattaneo, D. Merlini (World Scientific, Singapore, 1995)
18. F. Guerra, About the cavity fields in mean field spin glass models, invited lecture at the international congress of mathematical physics, Lisboa, 2003, available on http://arxiv.org/abs/cond-mat/0307673
19. F. Guerra, S. Ghirlanda, General properties of overlap probability distributions in disordered spin systems. Towards Parisi ultrametricity. J. Phys. A-Math. Gen. **31**, 9149–9155 (1998)
20. D. Panchenko, *The Sherrington-Kirkpatrick Model* (Springer, New York, 2013)
21. A. Barra, F. Guerra, About the ergodic regime in the analogical Hopfield neural networks: moments of the partition function. J. Math. Phys. **50**, 125217 (2008)
22. A. Barra, F. Guerra, *Constraints for the order parameters in analogical neural networks*, Percorsi d'Ateneo, S. Vitolo Ed., Salerno (2008)
23. A. Barra, G. Genovese, F. Guerra, The replica symmetric approximation of the analogical neural network. J. Stat. Phys. **140**, 784–796 (2010)
24. A. Barra, G. Genovese, F. Guerra, Equilibrium statistical mechanics of bipartite spin systems. J. Phys. A: Math. Theor. **44**, 245002 (2011)
25. A. Barra, G. Genovese, F. Guerra, D. Tantari, How glassy are neural networks? J. Stat. Mech. **2012**, P07009 (2012)
26. A. Barra, P. Contucci, E. Mingione, D. Tantari, *Multi-species mean-field spin-glasses. Rigorous results*, arXiv:1307.5154

27. D. Panchenko, The free energy in a multi-species Sherrington-Kirkpatrick model. arXiv:1310.6679
28. L. De Sanctis, F. Guerra, Mean field dilute ferromagnet: high temperature and zero temperature behavior. J. Stat. Phys. **132**, 759–785 (2008)
29. S. Franz, M. Leone, Replica bounds for optimization problems and diluted spin systems. J. Stat. Phys. **111**, 535–564 (2003)
30. F. Guerra, F.L. Toninelli, The high temperature region of the Viana-Bray diluted spin glass model. J. Stat. Phys. **115**, 531–555 (2004)
31. D. Panchenko, M. Talagrand, Bounds for diluted mean-field spin glass models. Probab. Theory Relat. Fields **130**, 319–336 (2004)
32. F. Guerra, F.L. Toninelli, Some comments on the connection between disordered long range spin glass models and their mean field version. J. Phys. A: Math. Gen. **36**, 10987–10995 (2003)
33. S. Franz, F.L. Toninelli, The Kac limit for finite-range spin glasses. Phys. Rev. Lett. **92**, 030602 (2004)
34. S. Franz, F.L. Toninelli, Finite-range spin glasses in the Kac limit: free energy and local observables. J. Phys. A: Math. Gen. **37**, 7433 (2004)
35. E. Marinari, G. Parisi, J.J. Ruiz-Lorenzo, Numerical simulations of spin glass systems, in ed. by P. Young *Spin Glasses and Random Fields* (World Scientific, Singapore, 1987), pp. 59–98
36. E. Marinari, G. Parisi, F. Ricci-Tersenghi, J.J. Ruiz-Lorenzo, F. Zuliani, Replica symmetry breaking in short range spin glasses: a review of the theoretical foundations and of the numerical evidence. J. Stat. Phys. **98**, 973–1074 (2000)
37. C.M. Newman, D.L. Stein, Simplicity of state and overlap structure in finite-volume realistic spin glasses. Phys. Rev. E **57**, 1356–1366 (1998)
38. A. Barra, The mean field ising model trough interpolating techniques. J. Stat. Phys. **132**, 787–809 (2008)

Chapter 2
Integrability of Dirac Reduced Bi-Hamiltonian Equations

Alberto De Sole, Victor G. Kac, and Daniele Valeri

Abstract First, we give a brief review of the theory of the Lenard-Magri scheme for a non-local bi-Poisson structure and of the theory of Dirac reduction. These theories are used in the remainder of the paper to prove integrability of three hierarchies of bi-Hamiltonian PDE's, obtained by Dirac reduction from some generalized Drinfeld-Sokolov hierarchies.

2.1 Introduction

It has been demonstrated in a series of papers [1–5] that the framework of Poisson vertex algebras is extremely useful for the theory of Hamiltonian PDE's. For example, the theories of non-local Poisson structures [2], and of the infinite dimensional Dirac reduction [5], have been developed in this framework. Moreover, this languages turned out to be very convenient not only for the development of the general theory, but also for the study of concrete bi-Hamiltonian systems, like the generalized Drinfeld-Sokolov (DS) hierarchies, considered in [3, 4]. In these two papers we studied in more detail three integrable bi-Hamiltonian hierarchies: the homogeneous DS hierarchy, associated to a simple Lie algebra $\mathfrak{g}$, studied already in [6], and the generalized DS hierarchies attached to a minimal and to a short nilpotent

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element of $\mathfrak{g}$. We also considered the Dirac reductions of the last two hierarchies by elements of conformal weight 1. In the case of a "short" hierarchy we thus obtain Svinolupov's integrable hierarchy [7], constructing thereby (non-local) bi-Poisson structures for them. However, it is not at all clear (and probably false in general) that the equations obtained by Dirac reduction from integrable bi-Hamiltonian equations remain bi-Hamiltonian integrable. We were able to prove this in [5] for the reduced "minimal" hierarchy only in the first non-trivial case of $\mathfrak{g} = \mathfrak{sl}_3$.

In the present paper, using the theory of singular degree of a rational matrix pseudodifferential operator [8], we prove integrability of the reduced "minimal" and "short" hierarchies for arbitrary $\mathfrak{g}$. Furthermore, considering Dirac reduction of the homogeneous DS hierarchy, associated to a fixed regular element s in a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, we prove integrability of the following bi-Hamiltonian PDE, for all $a \in \mathfrak{h}$:

$$\frac{de_\alpha}{dt} = \frac{\alpha(a)}{\alpha(s)} e'_\alpha + \sum_{\beta \in \Delta \setminus \{-\alpha\}} \frac{\beta(a)}{\beta(s)} e_{-\beta} [e_\beta, e_\alpha] \ , \quad \alpha \in \Delta \, , \tag{2.1}$$

where Δ is the root system of $\mathfrak{g}$ and $\{e_\alpha\}_{\alpha \in \Delta}$ are root vectors such that $(e_\alpha | e_{-\alpha}) = 1$ with respect to an invariant non-degenerate bilinear form $(\cdot \,|\, \cdot)$ on $\mathfrak{g}$. Equation (2.1) is bi-Hamiltonian with respect to the following two compatible Poisson structures ($\alpha, \beta \in \Delta$):

$$(H_0)_{\alpha,\beta}(\partial) = \delta_{\alpha,-\beta} \beta(s) \, , \tag{2.2}$$

and

$$\begin{aligned} &(H_1)_{\alpha,\beta}(\partial) = [e_\beta, e_\alpha] - (\alpha|\beta) e_\alpha \partial^{-1} \circ e_\beta \quad \text{for } \beta \neq -\alpha \, , \\ &(H_1)_{\alpha,-\alpha}(\partial) = \partial + (\alpha|\alpha) e_\alpha \partial^{-1} \circ e_{-\alpha} \, . \end{aligned} \tag{2.3}$$

The corresponding first two conserved Hamiltonian densities are

$$h_0 = a \ , \quad h_1 = \frac{1}{2} \sum_{\alpha \in \Delta} \frac{\alpha(a)}{\alpha(s)} e_\alpha e_{-\alpha} \, . \tag{2.4}$$

The proof of integrability in all cases is based on the Lenard-Magri scheme of integrability for non-local bi-Poisson structures, developed in [2].

2.2 Non-local Poisson Structures and Hamiltonian Equations

2.2.1 Evolutionary Vector Fields, Frechet Derivatives and Variational Derivatives

Let $\mathcal{V}$ be the algebra of differential polynomials in ℓ variables: $\mathcal{V} = \mathbb{F}[u_i^{(n)} \,|\, i \in I, n \in \mathbb{Z}_+]$, where $I = \{1, \dots, \ell\}$, over a field $\mathbb{F}$ of characteristic zero. (In fact, most of the results hold in the generality of algebras of differential functions, as defined in [2].) It is a differential algebra with derivation defined by $\partial(u_i^{(n)}) = u_i^{(n+1)}$. We also let $\mathcal{K}$ be the field of fractions of $\mathcal{V}$ (it is still a differential algebra). We also denote by $\tilde{\mathcal{K}}$ the *linear closure* of $\mathcal{K}$, which is the smallest differential field extension of $\mathcal{K}$ containing solutions to any linear differential equation with coefficients in $\tilde{\mathcal{K}}$, and whose subfield of constants is $\overline{\mathbb{F}}$, the algebraic closure of $\mathbb{F}$, see e.g. [9].

For $P \in \mathcal{V}^\ell$ we have the associated *evolutionary vector field*

$$X_P = \sum_{i\in I, n\in\mathbb{Z}_+} (\partial^n P_i) \frac{\partial}{\partial u_i^{(n)}} \in \mathrm{Der}(\mathcal{V})\,.$$

This makes $\mathcal{V}^\ell$ into a Lie algebra, with Lie bracket $[X_P, X_Q] = X_{[P,Q]}$, given by

$$[P, Q] = X_P(Q) - X_Q(P) = D_Q(\partial)P - D_P(\partial)Q\,,$$

where $D_P(\partial)$ and $D_Q(\partial)$ denote the Frechet derivatives of $P, Q \in \mathcal{V}^\ell$.

In general, for $\theta = \left(\theta_\alpha\right)_{\alpha=1}^m \in \mathcal{V}^m$, the *Frechet derivative* $D_\theta(\partial) \in \mathrm{Mat}_{m\times\ell}\,\mathcal{V}[\partial]$ is defined by

$$D_\theta(\partial)_{\alpha i} = \sum_{n\in\mathbb{Z}_+} \frac{\partial\theta_\alpha}{\partial u_i^{(n)}}\partial^n \;\;,\;\; \alpha = 1,\dots,m\,,\; i = 1,\dots,\ell\,. \tag{2.5}$$

Its adjoint $D_\theta^*(\partial) \in \mathrm{Mat}_{\ell\times m}\,\mathcal{V}[\partial]$ is then given by

$$D_\theta^*(\partial)_{i\alpha} = \sum_{n\in\mathbb{Z}_+} (-\partial)^n \frac{\partial\theta_\alpha}{\partial u_i^{(n)}} \;\;,\;\; \alpha = 1,\dots,m\,,\; i = 1,\dots,\ell\,.$$

For $f \in \mathcal{V}$ its *variational derivative* is $\frac{\delta f}{\delta u} = \left(\frac{\delta f}{\delta u_i}\right)_{i\in I} \in \mathcal{V}^{\oplus\ell}$, where

$$\frac{\delta f}{\delta u_i} = \sum_{n\in\mathbb{Z}_+} (-\partial)^n \frac{\partial f}{\partial u_i^{(n)}}\,.$$

Given an element $\xi \in \mathcal{V}^{\oplus\ell}$, the equation $\xi = \frac{\delta h}{\delta u}$ can be solved for $h \in \mathcal{V}$ if and only if $D_\xi(\partial)$ is a self-adjoint operator: $D_\xi(\partial) = D_\xi^*(\partial)$ (see e.g. [1]).

2.2.2 Rational Matrix Pseudodifferential Operators

Consider the skewfield $\mathcal{K}((\partial^{-1}))$ of pseudodifferential operators with coefficients in $\mathcal{K}$, and the subalgebra $\mathcal{V}[\partial]$ of differential operators on $\mathcal{V}$.

The algebra $\mathcal{V}(\partial)$ of *rational* pseudodifferential operators consists of pseudodifferential operators $L(\partial) \in \mathcal{V}((\partial^{-1}))$ which admit a fractional decomposition $L(\partial) = A(\partial)B(\partial)^{-1}$, for some $A(\partial), B(\partial) \in \mathcal{V}[\partial]$, $B(\partial) \neq 0$. The algebra of *rational matrix pseudodifferential operators* is, by definition, $\mathrm{Mat}_{\ell\times\ell}\,\mathcal{V}(\partial)$ [9, 10].

A matrix differential operator $B(\partial) \in \mathrm{Mat}_{\ell\times\ell}\,\mathcal{V}[\partial]$ is called *non-degenerate* if it is invertible in $\mathrm{Mat}_{\ell\times\ell}\,\mathcal{K}((\partial^{-1}))$. Any matrix $H(\partial) \in \mathrm{Mat}_{\ell\times\ell}\,\mathcal{V}(\partial)$ can be written as a ratio of two matrix differential operators: $H(\partial) = A(\partial)B^{-1}(\partial)$, with $A(\partial), B(\partial) \in \mathrm{Mat}_{\ell\times\ell}\,\mathcal{V}[\partial]$, and $B(\partial)$ non-degenerate.

2.2.3 Singular Degree of a Rational Matrix Pseudodifferential Operator

The *Dieudonné determinant* of $A \in \mathrm{Mat}_{\ell\times\ell}\,\mathcal{K}((\partial^{-1}))$ is defined as follows. If A is degenerate, then $\det(A) = 0$. Otherwise, $\det(A)$ is a pair

$$\det(A) = (\det{}_1(A), \deg(A)) \in \mathcal{K} \times \mathbb{Z}\,,$$

where $\det_1(A)$ and $\deg(A)$ are defined by the following conditions:

(i) $\det_1(AB) = \det_1(A)\det_1(B)$ for all non-degenerate $A, B \in \mathrm{Mat}_{\ell\times\ell}\,\mathcal{K}((\partial^{-1}))$;
(ii) $\deg(AB) = \deg(A) + \deg(B)$ for all non-degenerate $A, B \in \mathrm{Mat}_{\ell\times\ell}\,\mathcal{K}((\partial^{-1}))$;
(iii) if A is upper triangular, with diagonal entries $A_i = a_i\partial^{d_i}$ +lower terms, $i = 1, \dots, \ell$, with $a_i \neq 0$, then

$$\det{}_1(H) = \prod_{i=1}^{\ell} a_i \;\;,\quad \deg(A) = \sum_{i=1}^{\ell} d_i\,.$$

For a non-degenerate $A \in \mathrm{Mat}_{\ell\times\ell}\,\mathcal{K}((\partial^{-1}))$, the integer $\deg(A)$ is called the *degree* of A. (It is a non-negative integer if A is a matrix differential operator.)

Let $H \in \mathrm{Mat}_{\ell\times\ell}\,\mathcal{V}(\partial)$ be a rational matrix pseudodifferential operator. The *singular degree* of H, denoted $\mathrm{sdeg}(H)$ [8], is, by definition, the minimal possible value of $\deg(B)$ among all fractional decomposition $H = AB^{-1}$, with $A, B \in \mathrm{Mat}_{\ell\times\ell}\,\mathcal{V}[\partial]$, and $B(\partial)$ non-degenerate.

Suppose that we have a rational expression for $H \in \mathrm{Mat}_{\ell\times\ell}\,\mathcal{V}(\partial)$ of the form

$$H = \sum_{\alpha\in\mathcal{A}} A_1^{\alpha}(B_1^{\alpha})^{-1} \dots A_n^{\alpha}(B_n^{\alpha})^{-1}\,, \tag{2.6}$$

with $A_i^{\alpha}, B_i^{\alpha} \in \mathrm{Mat}_{\ell\times\ell}\,\mathcal{K}[\partial]$ and B_i^{α} non-degenerate, for all $i \in \mathcal{I} = \{1, \dots, n\}, \alpha \in \mathcal{A}$ (a finite index set). It is not hard to show that $\mathrm{sdeg}(H) \leq \sum_{\alpha\in\mathcal{A}}\sum_{i=1}^{n}\deg(B_i^{\alpha})$, [8]. We say that the rational expression (2.6) is *minimal* if equality holds.

Theorem 2.1 ([8, Cor.4.11]). *The rational expression (2.6) is minimal if and only if both the following systems of differential equations in the variables* $\{F_i^{\alpha}\}_{\alpha\in\mathcal{A}, i\in\{1,\dots,n\}}$

$$\begin{cases} B_n^{\alpha} F_n^{\alpha} = 0\,,\ \alpha \in \mathcal{A} \\ A_i^{\alpha} F_i^{\alpha} = B_{i-1}^{\alpha} F_{i-1}^{\alpha}\,,\ 2 \le i \le n,\, \alpha \in \mathcal{A} \\ \sum_{\alpha\in\mathcal{A}} A_1^{\alpha} F_1^{\alpha} = 0 \end{cases} \tag{2.7}$$

and

$$\begin{cases} B_1^{\alpha *} F_1^{\alpha} = 0\,,\ \alpha \in \mathcal{A} \\ A_i^{\alpha *} F_{i-1}^{\alpha} = B_i^{\alpha *} F_i^{\alpha}\,,\ 2 \le i \le n,\, \alpha \in \mathcal{A} \\ \sum_{\alpha\in\mathcal{A}} F_n^{\alpha} = 0 \end{cases} \tag{2.8}$$

have only the zero solution over the linear closure $\tilde{\mathcal{K}}$ of $\mathcal{K}$.

2.2.4 Association Relation

Given $H(\partial) \in \mathrm{Mat}_{\ell\times\ell}\,\mathcal{V}(\partial)$, we say that $\xi \in \mathcal{V}^{\oplus l}$ and $P \in \mathcal{V}^{\ell}$ are H-*associated*, and denote it by

$$\xi \overset{H}{\longleftrightarrow} P\,, \tag{2.9}$$

if there exist a fractional decomposition $H = AB^{-1}$ with $A, B \in \mathrm{Mat}_{\ell\times\ell}\,\mathcal{V}[\partial]$ and B non-degenerate, and an element $F \in \mathcal{K}^{\ell}$, such that $\xi = BF,\ P = AF$ [2].

Theorem 2.2 ([8, Thm4.12]). *Let* (2.6) *be a minimal rational expression for $H(\partial) \in \mathrm{Mat}_{\ell\times\ell}\,\mathcal{V}(\partial)$. Then, $\xi \overset{H}{\longleftrightarrow} P$ if and only the system of differential equations*

$$\begin{cases} B_n^{\alpha} F_n^{\alpha} = \xi\,,\ \alpha \in \mathcal{A} \\ A_i^{\alpha} F_i^{\alpha} = B_{i-1}^{\alpha} F_{i-1}^{\alpha}\,,\ 2 \le i \le n,\, \alpha \in \mathcal{A} \\ \sum_{\alpha\in\mathcal{A}} A_1^{\alpha} F_1^{\alpha} = P \end{cases} \tag{2.10}$$

has a solution $\{F_i^{\alpha}\}_{\alpha\in\mathcal{A},i\in\{1,\dots,n\}}$ over $\mathcal{K}$.

2.2.5 Non-local Poisson Structures

To a matrix pseudodifferential operator $H = \left(H_{ij}(\partial)\right)_{i,j\in I} \in \mathrm{Mat}_{\ell\times\ell}\,\mathcal{V}((\partial^{-1}))$ we associate a map, called λ-bracket, $\{\cdot\,{}_\lambda\,\cdot\}_H : \mathcal{V}\times\mathcal{V} \to \mathcal{V}((\lambda^{-1}))$, given by the following *Master Formula* (see [2]):

$$\{f_\lambda g\}_H = \sum_{\substack{i,j\in I\\ m,n\in\mathbb{Z}_+}} \frac{\partial g}{\partial u_j^{(n)}}(\lambda+\partial)^n H_{ji}(\lambda+\partial)(-\lambda-\partial)^m \frac{\partial f}{\partial u_i^{(m)}} \in \mathcal{V}((\lambda^{-1}))\,. \tag{2.11}$$

In particular,

$$H_{ji}(\partial) = \{u_i\,{}_\partial u_j\}_{H\to}\,. \tag{2.12}$$

(The arrow means that we move ∂ to the right.)

The following facts are proved in [1] and [2]. For arbitrary H, the λ-bracket (2.11) satisfies the following sesquilinearity conditions:

(i) $\{\partial f_\lambda g\} = -\lambda\{f_\lambda g\}$,
(ii) $\{f_\lambda \partial g\} = (\lambda + \partial)\{f_\lambda g\}$,

and left and right Leibniz rules ($f, g, h \in \mathcal{V}$):

(iii) $\{f_\lambda gh\} = \{f_\lambda g\}h + \{f_\lambda h\}g$,
(iv) $\{fg_\lambda h\} = \{f_{\lambda+\partial}h\}g + \{g_{\lambda+\partial}h\}f$.

Here and further an expression $\{f_{\lambda+\partial}h\}_\to g$ is interpreted as follows: if $\{f_\lambda h\} = \sum_{n=-\infty}^{N} c_n\lambda^n$, then $\{f_{\lambda+\partial}h\}_\to g = \sum_{n=-\infty}^{N} c_n(\lambda+\partial)^n g$, where we expand $(\lambda+\partial)^n$ in non-negative powers of ∂.

Skewadjointness of H is equivalent to the following skewsymmetry condition

(v) $\{f_\lambda g\} = -\{g_{-\lambda-\partial}f\}$.

The RHS of the skewsymmetry condition should be interpreted as follows: we move $-\lambda-\partial$ to the left and we expand its powers in non-negative powers of ∂, acting on the coefficients on the λ-bracket.

Let $\mathcal{V}_{\lambda,\mu} := \mathcal{V}[[\lambda^{-1}, \mu^{-1}, (\lambda+\mu)^{-1}]][\lambda, \mu]$, i.e. the quotient of the $\mathbb{F}[\lambda, \mu, \nu]$-module $\mathcal{V}[[\lambda^{-1}, \mu^{-1}, \nu^{-1}]][\lambda, \mu, \nu]$ by the submodule $(\nu-\lambda-\mu)\mathcal{V}[[\lambda^{-1}, \mu^{-1}, \nu^{-1}]][\lambda, \mu, \nu]$. We have the natural embedding $\iota_{\mu,\lambda} : \mathcal{V}_{\lambda,\mu} \hookrightarrow V((\lambda^{-1}))((\mu^{-1}))$ defined by expanding the negative powers of $\nu = \lambda + \mu$ by geometric series in the domain $|\mu| > |\lambda|$. In general, if H is an arbitrary matrix pseudodifferential operator, we have $\{f_\lambda\{g_\mu h\}\} \in \mathcal{V}((\lambda^{-1}))((\mu^{-1}))$ for all $f, g, h \in \mathcal{V}$. If H is a rational matrix pseudodifferential operator, we have the following admissibility condition ($f, g, h \in \mathcal{V}$):

(vi) $\{f_\lambda\{g_\mu h\}\} \in \mathcal{V}_{\lambda,\mu}$,

where we identify the space $\mathcal{V}_{\lambda,\mu}$ with its image in $\mathcal{V}((\lambda^{-1}))((\mu^{-1}))$ via the embedding $\iota_{\mu,\lambda}$.

Definition 2.1. A *non-local Poisson structure* on $\mathcal{V}$ is a skewadjoint rational matrix pseudodifferential operator H with coefficients in $\mathcal{V}$, satisfying the following Jacobi identity ($f, g, h \in \mathcal{V}$):

(vii) $\{f_\lambda\{g_\mu h\}\} - \{g_\mu\{f_\lambda h\}\} = \{\{f_\lambda g\}_{\lambda+\mu}h\}$,

where the equality is understood in the space $\mathcal{V}_{\lambda,\mu}$.

(Note that, if skewsymmetry (v) and admissibility (vi) hold, then all three terms of Jacobi identity lie in the image of $\mathcal{V}_{\lambda,\mu}$ via the appropriate embedding $\iota_{\mu,\lambda}$, $\iota_{\lambda,\mu}$ or

$\iota_{\lambda+\mu,\lambda}$.) Note that Jacobi identity (vii) holds for all $f, g, h \in \mathcal{V}$ if and only if it holds for any triple of generators u_i, u_j, u_k [1,2].

Two non-local Poisson structures $H_0, H_1 \in \mathrm{Mat}_{\ell\times\ell}\,\mathcal{V}(\partial)$ on $\mathcal{V}$ are said to be *compatible* if any their linear combination (or, equivalently, their sum) is a non-local Poisson structure. In this case we say that (H_0, H_1) form a *bi-Poisson structure* on $\mathcal{V}$.

Definition 2.2. A *non-local Poisson vertex algebra* is, by definition, a differential algebra $\mathcal{V}$ endowed with a λ-bracket $\{\cdot\,{}_\lambda\,\cdot\} : \mathcal{V}\times\mathcal{V} \to \mathcal{V}((\lambda^{-1}))$ satisfying conditions (i)–(vii).

We shall often drop the term "non-local", so when we will refer to Poisson structures and λ-brackets we will always mean *non-local PVA*'s and *non-local λ-brackets*. (This, of course, includes the local case as well.)

2.2.6 Hamiltonian Equations and Integrability

Recall that we have a non-degenerate pairing $(\cdot\,|\,\cdot) : \mathcal{V}^\ell \times \mathcal{V}^\ell \to \mathcal{V}/\partial\mathcal{V}$ given by $(P|\xi) = \int P\cdot\xi$ (see e.g. [1]). Let $H \in \mathrm{Mat}_{\ell\times\ell}\,\mathcal{V}(\partial)$ be a non-local Poisson structure. An evolution equation on the variables $u = \big(u_i\big)_{i\in I}$,

$$\frac{du}{dt} = P\,, \tag{2.13}$$

is called *Hamiltonian* with respect to the Poisson structure H and the Hamiltonian functional $\int h \in \mathcal{V}/\partial\mathcal{V}$ if (cf. Sect. 2.2.4)

$$\frac{\delta h}{\delta u} \overset{H}{\longleftrightarrow} P\,.$$

Equation (2.13) is called *bi-Hamiltonian* if there are two compatible non-local Poisson structures H_0 and H_1, and two local functionals $\int h_0, \int h_1 \in \mathcal{V}/\partial\mathcal{V}$, such that

$$\frac{\delta h_0}{\delta u} \overset{H_1}{\longleftrightarrow} P \quad\text{and}\quad \frac{\delta h_1}{\delta u} \overset{H_0}{\longleftrightarrow} P\,. \tag{2.14}$$

An *integral of motion* for the Hamiltonian equation (2.13) is a local functional $\int f \in \mathcal{V}/\partial\mathcal{V}$ which is constant in time, i.e. such that $(P|\frac{\delta f}{\delta u}) = 0$. The usual requirement for *integrability* is to have sequences $\{\int h_n\}_{n\in\mathbb{Z}_+} \subset \mathcal{V}/\partial\mathcal{V}$ and $\{P_n\}_{n\in\mathbb{Z}_+} \subset \mathcal{V}^\ell$, starting with $\int h_0 = \int h$ and $P_0 = P$, such that

(C1) $\frac{\delta h_n}{\delta u} \overset{H}{\longleftrightarrow} P_n$ for every $n \in \mathbb{Z}_+$.
(C2) $[P_m, P_n] = 0$ for all $m, n \in \mathbb{Z}_+$.